Mathematics and Statistics 11(5): 820-826, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110508

Aspects of Algebraic Structure of Rough Sets†

S. Sangeetha, Shakeela Sathish*

Department of Mathematics, SRM Institute of Science and Technology, Ramapuram Campus, India

Received April 5, 2023; Revised August 4, 2023; Accepted August 23, 2023

Cite This Paper in the following Citation Styles

***(a)**: [1] S. Sangeetha, Shakeela Sathish, "Aspects of Algebraic Structure of Rough Sets†," Mathematics and Statistics, Vol.11, No.5, pp. 820-826, 2023. DOI: 10.13189/ms.2023.110508*

***(b)**: S. Sangeetha, Shakeela Sathish (2023). Aspects of Algebraic Structure of Rough Sets†. Mathematics and Statistics, 11(5), 820-826. DOI: 10.13189/ms.2023.110508*

Abstract Rough sets are extensions of classical sets characterized by vagueness and imprecision. The main idea of rough set theory is to use incomplete information to approximate the concept of imprecision or uncertainty, or to treat ambiguous phenomena and problems based on observation and measurement. In Pawlak rough set model, equivalence relations are a key concept, and equivalence classes are the foundations for lower and upper approximations.

Developing an algebraic structure for rough sets will allow us to study set theoretic properties in detail. Several researchers studied rough sets from an algebraic perspective and a number of structures have been developed in recent years, including rough semigroups, rough groups, rough rings, rough modules, and rough vector spaces. The purpose of this study is to demonstrate the usefulness of rough set theory in group theory. There have been several papers investigating the roughness in algebraic structures by substituting an algebraic structure for the universe set. In this paper, rough groups are defined using upper and lower approximations of rough sets from a finite universe instead of considering the whole universe. Here we have considered a finite universe Λ along with a relation χ which classifies the universe into equivalence classes. We have identified all rough sets with respect to this relation. The upper and lower approximated sets have been taken separately and these form a rough group equivalence relation (χ_{ro_g}) and it partitions the group $(2^\Lambda, \triangle)$ into equivalence classes. In this paper, the rough group approximation space $(2^\Lambda, \chi_{ro_g})$ has been defined along with upper and lower approximations and properties of subsets of 2^Λ with respect to rough group equivalence relations have been illustrated.

Keywords Rough Group, Rough Group Approximation Space, Rough Group Equivalence Relation

1 Introduction

There are numerous mathematical concepts that are delivered through the use of set theory, which is used as a core method in the entire field of mathematics. Pawlak introduced the concept of rough sets [1]. There has been an increase in interest in this newly emerging theory in recent years. Since its introduction, the rough set theory has been continued to develop as a tool for classifying.

The rough set has been evaluated algebraically by a variety of experts to date. The topics of interest range from pure theory, such as topological and algebraic foundations, to applications as discussed in [2], [3], & [4]. The concept of rough sets has been approached algebraically by Bonikowski [5], Iwinski [6] & Pomykala [7]. A rough subring is defined by Davvaz [8] when rough set theory and ring theory are considered. The rough group has been evaluated by N. Kuroki and Wang [9] in order to approximate the upper and lower bounds of any subset of a group in terms of its normal subgroup. In addition, topological rough groups were defined and their properties were examined in [10]. A generalized rough set can be viewed in two different ways according to Radwan et al [11]. Based on a family of dominance relationships, Salama et al [12] gave properties of different types of rough approximations. A topological approach was given by Al-Shami [13] to generate new rough set models. Also using E-neighborhoods, Al-Shami[14] provided new rough approximations. A topological approach to rough approximations based on closure operators was developed by El-Bably et al [15]. Through ideals, Guler et al [16] provided rough approximations based on different topologies. The concept of generalized rough approximation spaces based on maximal neighbourhoods and ideals is discussed by Hosny et al [17]. Several types of rough sets based on coverings were provided by Nawar et al [18]. Based on j-neighborhood space and j-adhesion neighborhood space, Atef et al [19] compared six types of rough approximations. Using J-Nearly Concepts via Ideals Hosny [20] gave a topological approach for rough sets. Pradeep Shende et al [21] presented a novel concept of uncertainty optimization through multi-granular rough sets. A rough set with uncertainty optimization based on incomplete information systems was introduced by Arvind et al [22].

In particular, Biswas [23] introduced rough groups and rough subgroups. Miao et al [24] modified the approach by proving the group axioms to the upper approximation of a set. Wang [25] examined the relationship between the normal series of a group and its rough approximations in order to determine the properties of rough groups.

In this paper, modified approach to rough groups is provided followed by a rough group approximation space based on rough group equivalences.

The rough set theory has been briefly reviewed in section 2. In section 3, we explored rough groups from a different perspective. We have introduced the rough group approximation space in section 4 and given the upper and lower approximations of any set based on rough group equivalence relations. The significance of this work is presented in results and discussion section 5.

2 Basics of Rough Set Theory

Definition 2.1 *[1]*
Approximation space is composed of a finite set $univ_1$ $(\neq \phi)$ *and* $"\iota_1"$*, an equivalence relation on* $univ_1$ *and it is represented by* $(univ_1, "\iota_1")$*.*

Definition 2.2 *[1]*
A family of subsets $E = \{E_1, E_2, E_3 \ldots\ldots E_n\}$ *of* $univ_1$ *are said to be a classification of* $univ_1$ *if*

- $E_1 \cup E_2 \cup \ldots\ldots \cup E_n = univ_1$
- $E_i \cap E_j = \phi$*, for* $i \neq j$

Definition 2.3 *[1]*
Let $(univ_1, "\iota_1")$ *be an approximation space and for any* $k \in univ_1$ *the set* $[k]_{"\iota_1"}$ *is called the equivalence class induced by* $"\iota_1"$*.*

Definition 2.4 *[1]*
Consider $K = (univ_1, "\iota_1")$*, an approximation space and A be any subset of* $univ_1$ *then*

- $univ_1{}^A = \{a_i \mid [a_i]_{"\iota_1"} \cap A \neq \phi\}$
- $univ_{1A} = \{a_i \mid [a_i]_{"\iota_1"} \subseteq A\}$
- $BN_A = univ_1{}^A - univ_{1A}$

are called approximations of upper, lower & boundary regions of A in relation to χ *respectively and if the boundary of the set A is not empty, it is said to be rough, otherwise it is said to be crisp.*

If $A, B \subseteq univ_1$, then the following results are due to [1]

- $univ_{1A} \subseteq A \subseteq univ_1{}^A$
- $univ_{1univ_1} = univ_1{}^{univ_1} = univ_1$
- $univ_{1A\cap B} = univ_{1A} \cap univ_{1B}$
- $univ_{1A\cup B} \supseteq univ_{1A} \cup univ_{1B}$
- $univ_1{}^{A\cup B} = univ_1{}^A \cup univ_1{}^B$
- $univ_1{}^{A\cap B} \subseteq univ_1{}^A \cap univ_1{}^B$
- If $A \subseteq B$ then $univ_{1A} \subseteq univ_{1B}$ & $univ_1{}^A \subseteq univ_1{}^B$

3 Rough Groups

Definition 3.1 *: **Group**[26]*
Groups are non-empty sets with binary operations $*$ *that satisfy closure, associativity, identity, and inverse properties under* $*$*.*

Definition 3.2 *: **Power Set**[26]*
Collection of all possible subsets of G forms a Power set represented by 2^G *which forms an abelian group along with operation* $\triangle$*.*

Definition 3.3 *[1]*
(U, R_1)*, an approximation space.* R_1*, an equivalence relation partitions U into classes of equivalence. Let* $W(\neq \phi) \subseteq U$*.*
$R_1{}^W = \{w \mid [w]_{R_1} \cap W \neq \phi\}$*, which is upper approximation of* W
$R_{1W} = \{w \mid [w]_{R_1} \subseteq W\}$ *which is lower approximation of W*
if $R^W - R_W \neq \phi$
then $W = (R^W, R_W)$ *is a rough set otherwise crisp*

Definition 3.4
(U, R)*, an approximation space which consists of a finite set U of n elements.* $(2^U, \triangle)$ *forms an abelian group and* $R(U)$*, a collection of all rough sets in U is said to be a rough group if* $\overline{R(U)} \cup \underline{R(U)}$ *with respect to the binary operation* $\triangle$ *forms subgroup of* $(2^U, \triangle)$

Theorem 3.1
Let $R(U)$ *represents all possible rough sets in a space* (U, R)*.* $\overline{R(U)} \cup \underline{R(U)}$ *with respect to the binary operation* $\triangle$ *forms subgroup of* $(2^U, \triangle)$ *and hence* $R(U)$ *is said to be rough group.*

Proof 1
Let $R(U)$ *be the set of all rough sets of* (U, R)*.*
$\overline{R(U)} = R^U = \{\overline{W} | W \in R(U)\}$
$\underline{R(U)} = R_U = \{\underline{W} | W \in R(U)\}$
Let us denote $R(ro_g) = R^U \cup R_U$

To prove $R(ro_g)$*, subgroup of* 2^U*.*
$R(ro_g)$ *is non empty since* ϕ *is always a subset of any set and it will be in* R_U

Let $W_1, W_2 \in R(ro_g)$
Claim : $W_1 \triangle W_2 = (W_1 \cup W_2) - (W_1 \cap W_2) \in R(ro_g)$
$\because W_1, W_2 \in R(ro_g)$
$W_1, W_2 \in R^U \cup R_U$

$W_1, W_2 \in R^U$ *or* $W_1, W_2 \in R_U$

Case 1

If $W_1, W_2 \in R^U$

$W_1 = \overline{B_1}$ & $W_2 = \overline{B_2}$ *where* $B_1, B_2 \in R(U)$

$W_1 \triangle W_2 = (\overline{B_1} \cup \overline{B_2}) - (\overline{B_1} \cap \overline{B_2})$

$\overline{B_1} \cup \overline{B_2} = \{[u]_R | [u]_R \cap \overline{B_1} \cup \overline{B_2} \neq \phi\}$

$= \{[u]_R | [u]_R \cap \overline{B_1} \cup [u]_R \cap \overline{B_2} \neq \phi\} \implies \overline{B_1} \cup \overline{B_2} \in \overline{R(U)}$

$\overline{B_1} \cap \overline{B_2} = \{[u]_R | [u]_R \cap \overline{B_1} \cap \overline{B_2} \neq \phi\}$

$= \{[u]_R | [u]_R \cap \overline{B_1} \cap [u]_R \cap \overline{B_2} \neq \phi\} \implies \overline{B_1} \cap \overline{B_2} \in \overline{R(U)}$

Assuming $W_2 \subseteq W_1$

$(\overline{B_1} \cup \overline{B_2}) - (\overline{B_1} \cap \overline{B_2}) = \{[u]_R \cap \overline{B_1} \neq \phi$ *or* $[u]_R \cap \overline{B_2} \neq \phi\} - \{[u]_R \cap \overline{B_1} \neq \phi$ *and* $[x]_R \cap \overline{B_2} \neq \phi\}$

$= \{[u]_R \cap \overline{B_1} \neq \phi$ *or* $[u]_R \cap \overline{B_2} \neq \phi\} = \overline{B_1} \cup \overline{B_2} \in \overline{R(U)}$

$\implies W_1 \triangle W_2 \in R^U$

Case 2

If $W_1, W_2 \in R_U$

$W_1 = \underline{B_1}$ & $W_2 = \underline{B_2}$ *where* $B_1, B_2 \in R(U)$

$W_1 \triangle W_2 = (\underline{B_1} \cup \underline{B_2}) - (\underline{B_1} \cap \underline{B_2})$

$\underline{B_1} \cup \underline{B_2} \supseteq \underline{(B_1 \cup B_2)} \in \underline{R(U)}$

$\underline{B_1} \cap \underline{B_2} = \underline{(B_1 \cap B_2)}$

Assume $B_2 \subseteq B_1$

$\underline{(B_1 \cap B_2)} \in \underline{R(U)}$

$\underline{B_1} \cup \underline{B_2} = \{[u]_R | [u]_R \subseteq \underline{B_1} \cup \underline{B_2} \implies [u]_R \cap \underline{B_1} \cup \underline{B_2} = [u]_R\}$

$\underline{B_1} \cap \underline{B_2} = \{[u]_R | [u]_R \subseteq \underline{B_1} \cap \underline{B_2} \implies [u]_R \cap \underline{B_1} \cap \underline{B_1} = [u]_R\}$

$(\underline{B_1} \cup \underline{B_2}) - (\underline{B_1} \cap \underline{B_2}) = [u]_R \cap \underline{B_1} \cup \underline{B_2} \in \underline{R(U)}$

$W_1 \triangle W_2 \in R_U$

Hence if $W_1, W_2 \in R^U \cup R_U$, *then* $W_1 \triangle W_2 \in R^U \cup R_U$.
Also all elements poss self inverse. Hence $R^U \cup R_U$ *is a subgroup of* 2^U *and hence* $R(U)$ *is said to be Rough group.*

Example 3.1

Let $U_1 = \{1, 2, 3\}$ *&* R*, an equivalence relation on* U_1.

$U_1/R = \{\{1, 2\}, \{3\}\}$

$2^{U_1} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \phi\}$

Among the eight subsets of Universe set U the following subsets are rough sets $X_1 = \{1\}, X_2 = \{2\}, X_5 = \{1, 3\}, X_6 = \{2, 3\}$ *with respect to R. Since*

- $R^{X_1} = \{1, 2\}$ & $R_{X_1} = \{\}$
- $R^{X_2} = \{1, 2\}$ & $R_{X_2} = \{\}$
- $R^{X_5} = \{1, 2, 3\}$ & $R_{X_5} = \{3\}$
- $R^{X_6} = \{1, 2, 3\}$ & $R_{X_6} = \{3\}$

$$R(U) = \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}$$

$$\overline{R(U)} = \{\{1, 2\}, \{1, 2, 3\}\}$$

$$\underline{R(U)} = \{\{\}, \{3\}\}$$

$$R(ro_g) = \overline{R(U)} \cup \underline{R(U)} = \{\{\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$$

	{}	{3}	{1, 2}	{1, 2, 3}
{}	{}	{3}	{1,2}	{1,2,3}
{3}	{3}	{}	{1,2,3}	{1,2}
{1, 2}	{1,2}	{1,2,3}	{}	{3}
{1, 2, 3}	{1,2,3}	{1,2}	{3}	{}

Hence

$R(ro_g)$ *is a subgroup of* 2^U *and hence* $R(U)$ *is a Rough Group.*

Theorem 3.2 *If* $"\partial_a"(ro_g)$, $"\partial_b"(ro_g)$ *are rough groups, then* $"\partial_a"(ro_g) \cap "\partial_b"(ro_g)$ *is also rough group. (*uni_1 *is finite universe)*

Proof

Let $x, y \in "\partial_a"(ro_g) \cap "\partial_b"(ro_g)$

Since $\phi \subseteq$ *any set. so* $\phi \in \underline{"\partial_a"(uni_1)} \cap \underline{"\partial_b"(uni_1)}$

$\implies \phi \in "\partial_a"(ro_g) \cap "\partial_b"(ro_g)$

$\implies "\partial_a"(ro_g) \cap "\partial_b"(ro_g) \neq \phi$

Since $"\partial_a"(ro_g)$ & $"\partial_b"(ro_g)$ *are subgroups of* 2^{uni_1}

$\implies x \triangle y \in "\partial_a"(ro_g) \cap "\partial_b"(ro_g)$

Hence the result.

Proposition 3.1

Let $"\partial_a"_{ro_g}$ & $"\partial_b"_{ro_g}$ *be two rough groups then* $\overline{"\partial_a"_{ro_g} \cap "\partial_b"_{ro_g}} \subseteq "\partial_a"_{ro_g} \cap "\partial_b"_{ro_g}$ *(where* $"\partial_a"$ & $"\partial_b"$ *are two equivalence relations,* uni_1 *is the universe)*

Proof

$$\begin{aligned}
\overline{"\partial_a"_{ro_g} \cap "\partial_b"_{ro_g}} &= \overline{[\overline{"\partial_a"(uni_1)} \cup \underline{"\partial_a"(uni_1)}] \cap [\overline{"\partial_b"(uni_1)} \cup \underline{"\partial_b"(uni_1)}]} \\
&= \overline{[\overline{"\partial_a"(uni_1)} \cap \overline{"\partial_b"(uni_1)}] \cup [\underline{"\partial_a"(uni_1)} \cap \underline{"\partial_b"(uni_1)}]} \\
&= \overline{[\overline{"\partial_a"(\kappa_1)} \cap \overline{"\partial_b"(uni_1}]} \cup \overline{[\underline{"\partial_a"(uni_1)} \cap \underline{"\partial_b"(uni_1)}]} \\
&\subseteq [\overline{\overline{"\partial_a"(uni_1)}} \cap \overline{\overline{"\partial_b"(uni_1)}}] \cup [\overline{\underline{"\partial_a"(uni_1)}} \cap \overline{\underline{"\partial_b"(uni_1)}}] \\
&= [\overline{"\partial_a"(uni_1)} \cap \overline{"\partial_b"(uni_1)}] \cup [\underline{"\partial_a"(uni_1)} \cap \underline{"\partial_b"(uni_1)}] \\
&= [\overline{"\partial_a"(uni_1)} \cup \underline{"\partial_a"(uni_1)}] \cap [\overline{"\partial_b"(uni_1)} \cup \underline{"\partial_b"(uni_1)}] \\
&\subseteq "\partial_a"ro_g \cap "\partial_b"ro_g, \text{ hence upper approximation of intesection of rough groups is contained in its intersection}
\end{aligned}$$

B

Proposition 3.2 *Let* $"\partial_a"_{ro_g}$ & $"\partial_b"_{ro_g}$ *be two rough groups then* $\overline{"\partial_a"_{ro_g} \cup "\partial_b"_{ro_g}} = "\partial_a"_{ro_g} \cup "\partial_b"_{ro_g}$

Proof

$$\begin{aligned}
\overline{"\partial_a"_{r_g} \cup "\partial_b"_{r_g}} &= \overline{[\overline{"\partial_a"(uni_1)} \cup \underline{"\partial_a"(uni_1)}] \cup [\overline{"\partial_b"(uni_1)} \cup \underline{"\partial_b"(uni_1)}]} \\
&= \overline{[\overline{"\partial_a"(uni_1)} \cup \overline{"\partial_b"(uni_1)}] \cup [\underline{"\partial_a"(uni_1)} \cup \underline{"\partial_b"(uni_1)}]} \\
&= \overline{[\overline{"\partial_a"(uni_1)} \cup \overline{"\partial_b"(uni_1)}]} \cup \overline{[\underline{"\partial_a"(uni_1)} \cup \underline{"\partial_b"(uni_1)}]} \\
&= [\overline{\overline{"\partial_a"(uni_1)}} \cup \overline{\overline{"\partial_b"(uni_1)}}] \cup [\overline{\underline{"\partial_a"(uni_1)}} \cup \overline{\underline{"\partial_b"(uni_1)}}] \\
&= [\overline{"\partial_a"(uni_1)} \cup \overline{"\partial_b"(uni_1)}] \cup [\underline{"\partial_a"(uni_1)} \cup \underline{"\partial_b"(uni_1)}] \\
&= [\overline{"\partial_a"(uni_1)} \cup \underline{"\partial_a"(uni_1)}] \cup [\overline{"\partial_b"(uni_1)} \cup \underline{"\partial_b"(uni_1)}] \\
&= "\partial_a"_{ro_g} \cup "\partial_b"_{ro_g}, \textit{ hence upper approximation of union of rough groups is equal to its union}
\end{aligned}$$

B

Proposition 3.3 *Let* $"\partial_a"_{ro_g}$ & $"\partial_b"_{ro_g}$ *are two rough groups then* $\underline{"\partial_a"_{ro_g} \cap "\partial_b"_{r_g}} = "\partial_a"_{ro_g} \cap "\partial_b"_{ro_g}$

Proof:

$$\begin{aligned}
\underline{"\partial_a"_{ro_g} \cap "\partial_b"_{ro_g}} &= \underline{[\overline{"\partial_a"(uni_1)} \cup \underline{"\partial_a"(uni_1)}] \cap [\overline{"\partial_b"(uni_1)} \cup \underline{"\partial_b"(uni_1)}]} \\
&= \underline{[\overline{"\partial_a"(uni_1)} \cap \overline{"\partial_b"(uni_1)}] \cup [\underline{"\partial_a"(uni_1)} \cap \underline{"\partial_b"(uni_1)}]} \\
&= \underline{[\overline{"\partial_a"(uni_1)} \cap \overline{"\partial_b"(uni_1)}]} \cup \underline{[\underline{"\partial_a"(uni_1)} \cap \underline{"\partial_b"(uni_1)}]} \\
&= [\underline{\overline{"\partial_a"(uni_1)}} \cap \underline{\overline{"\partial_b"(uni_1)}}] \cup [\underline{\underline{"\partial_a"(uni_1)}} \cap \underline{\underline{"\partial_b"(uni_1)}}] \\
&= [\overline{"\partial_a"(uni_1)} \cap \overline{"\partial_b"(uni_1)}] \cup [\underline{"\partial_a"(uni_1)} \cap \underline{"\partial_b"(uni_1)}] \\
&= [\overline{"\partial_a"(uni_1)} \cup \underline{"\partial_a"(uni_1)}] \cap [\overline{"\partial_b"(uni_1)} \cup \underline{"\partial_b"(uni_1)}] \\
&= "\partial_a"_{ro_g} \cap "\partial_b"_{ro_g}, \textit{ hence lower approximation of intesection of rough groups is equal to its intersection}
\end{aligned}$$

4 Rough Group Approximation Space

Definition 4.1

Rough Groups Relation

Let $(uni_1, "\gamma_1")$ *be approximation space &* $G = (2^{uni_1}, \triangle)$ *be a group & an equivalence relation* $"\underline{\gamma_1"(ro_g)}$ *partitions* G *into equivalence classes* $\{"\gamma_1"(uni_1), \overline{"\gamma_1"(uni_1)}, \underline{"\gamma_1"(uni_1)}\}$. *Then the space* $(2^{uni_1}, \triangle)$ *is called rough group approximation* *space. For $A \subseteq 2^{uni_1}$, the upper & lower approximations are given by*

$$"\gamma_1"(ro_g)^A = \{A_1 \in 2^{uni_1} | [A_1]_{"\gamma_1"(ro_g)} \cap A \neq \phi\}$$

$$"\gamma_1"(ro_g)_A = \{A_1 \in 2^{uni_1} | [A_1]_{"\gamma_1"(ro_g)} \subseteq A\}$$

Proposition 4.1
Let $A, B \subseteq 2^{uni_1}$ be nonempty and $"\gamma_1"(ro_g)$ be a rough group equivalence relation then

1. $"\gamma_1"(ro_g)_A \subseteq A \subseteq "\gamma_1"(ro_g)^A$
2. $"\gamma_1"(ro_g)^{A\cup B} = "\gamma_1"(ro_g)^A \cup "\gamma_1"(ro_g)^B$
3. $"\gamma_1"(ro_g)_{A\cap B} = "\gamma_1"(ro_g)_A \cap "\gamma_1"(ro_g)_B$
4. $A \subseteq B \implies "\gamma_1"(ro_g)_A \subseteq "\gamma_1"(ro_g)_B$
5. $A \subseteq B \implies "\gamma_1"(ro_g)^A \subseteq "\gamma_1"(ro_g)^B$
6. $"\gamma_1"(ro_g)_{A\cup B} \supseteq "\gamma_1"(ro_g)_A \cup "\gamma_1"(ro_g)_B$
7. $"\gamma_1"(ro_g)^{A\cap B} \subseteq "\gamma_1"(ro_g)^A \cap "\gamma_1"(ro_g)^B$
8. *$"\gamma_1"(ro_g)$ and $"\gamma_1"(ro_g)$ are equivalence relations then* $"\gamma_1"(ro_g) \subseteq "\gamma_1"(ro_g) \implies "\gamma_1"(ro_g)^A \subseteq "\gamma_1"(ro_g)^A$

Proof:

1. Let $X_1 \in "\gamma_1"(ro_g)_A$
   $\implies X_1 \in [Y_1]_{"\gamma_1"(ro_g)}$
   also $[Y_1]_{"\gamma_1"(ro_g)} \subseteq A \implies X_1 \in A$

   $X_1 \triangle \phi \subseteq X\underline{"\gamma_1"(uni_1)}$ $(\because \phi \in \underline{"\gamma_1"(uni_1)})$

   $X_1 \in X_1\underline{"\gamma_1"(uni_1)} \cap A$

   $\implies X_1 \in "\gamma_1"(ro_g)^A$

2. To prove
   $"\gamma_1"(ro_g)^{A_1\cup B_1} = "\gamma_1"(ro_g)^{A_1} \cup "\gamma_1"(ro_g)^{B_1}$

   $X \in "\gamma_1"(ro_g)^{A_1\cup B_1} \Leftrightarrow X \in [Y]_{"\gamma_1"(ro_g)} \cap (A_1 \cup B_2)$
   $\Leftrightarrow X \in ["\gamma_1"(uni_1) \cup \overline{("\gamma_1"(uni_1)}$
   $\cup \underline{("\gamma_1"(uni_1))}] \cap (A_1 \cup B_1)$
   $\Leftrightarrow X \in ["\gamma_1"(uni_1) \cap A_1]$
   $\cup\ X \in [\overline{"\gamma_1"(uni_1)} \cup \underline{"\gamma_1"(uni_1)} \cap B]$
   $\Leftrightarrow X \in "\gamma_1"(ro_g)^{A_1}$ or $X \in "\gamma_1"(ro_g)^{B_1}$
   $\Leftrightarrow X \in "\gamma_1"(ro_g)^{A_1} \cup X \in "\gamma_1"(ro_g)^{B_1}$
   $\Leftrightarrow X \in "\gamma_1"(ro_g)^{A_1} \cup "\gamma_1"(ro_g)^{B_1}$

3. To Prove $"\gamma_1"(ro_g)_{A\cap B} = "\gamma_1"(ro_g)_A \cap "\gamma_1"(ro_g)_B$

   $X \in "\gamma_1"(ro_g)_{(A\cap B)} \Leftrightarrow X \in [Y]_{"\gamma_1"(ro_g)} \subseteq (A \cap B)$
   $\Leftrightarrow X \in ["\gamma_1"(uni_1) \cup \overline{("\gamma_1"(uni_1)}$
   $\cup \underline{("\gamma_1"(uni_1))}] \subseteq (A \cap B)$
   $\Leftrightarrow X \in ["\gamma_1"(uni_1) \cup \overline{("\gamma_1"(uni_1)}$
   $\cup \underline{("\gamma_1"(uni_1))}]$
   $\subseteq A \cap ["\gamma_1"(uni_1)$
   $\cup \overline{("\gamma_1"(uni_1)} \cup \underline{("\gamma_1"(uni_1))}] \subseteq B)$
   $\Leftrightarrow X \in "\gamma_1"(ro_g)_A \cap "\gamma_1"(ro_g)_B$

4. To prove $A \subseteq B \implies "\gamma_1"(ro_g)_A \subseteq "\gamma_1"(ro_g)_B$
   Since $A \cap B = A$
   $"\gamma_1"(ro_g)_A = "\gamma_1"(ro_g)_{(A\cap B)} = "\gamma_1"(ro_g)_A \cap "\gamma_1"(ro_g)_B$
   $\implies "\gamma_1"(ro_g)_A \subseteq "\gamma_1"(ro_g)_B$

5. To prove
   $A \subseteq B \implies "\gamma_1"(ro_g)^A \subseteq "\gamma_1"(ro_g)^B$
   Since $A \cup B = B$
   $"\gamma_1"(ro_g)^B = "\gamma_1"(ro_g)^{(A\cup B)} = "\gamma_1"(ro_g)^A \cup "\gamma_1"(ro_g)^B$
   $\implies "\gamma_1"(ro_g)^A \subseteq "\gamma_1"(ro_g)^B$

6. To prove
   $"\gamma_1"(ro_g)_{(A_1\cup B_1)} \supseteq "\gamma_1"(ro_g)_{A_1} \cup "\gamma_1"(ro_g)_{B_1}$
   $A_1 \cup B_1 \supseteq A_1$ and B_1
   $"\gamma_1"(ro_g)_{A_1} \subseteq "\gamma_1"(ro_g)_{(A_1\cup B_1)}$
   $"\gamma_1"(ro_g)_{B_1} \subseteq "\gamma_1"(ro_g)_{(A_1\cup B_1)}$
   Hence $"\gamma_1"(ro_g)_{(A_1\cup B_1)} \supseteq "\gamma_1"(ro_g)_{A_1} \cup "\gamma_1"(ro_g)_{B_1}$

7. To prove
   $"\gamma_1"(ro_g)^{A_1\cap B_1} \subseteq "\gamma_1"(ro_g)^{A_1} \cap "\gamma_1"(ro_g)^{B_1}$
   since $A_1 \supseteq A_1 \cap B_1$ & $A_1 \cap B_1 \subseteq B_1$
   then $"\gamma_1"(ro_g)^{A_1\cap B_1} \subseteq "\gamma_1"(ro_g)^{A_1} \cap "\gamma_1"(ro_g)^{B_1}$

8. Let $\eth 1_{ro_g}$ & $\eth 2_{ro_g}$ be two rough group relations on $(2^{\kappa_1}, \triangle)$
   To prove $\eth 1_{ro_g} \subseteq \eth 2_{ro_g} \implies \eth 1^A_{ro_g} \subseteq \eth 2^A_{ro_g}$
   $\eth 1_{ro_g} \subseteq \eth 2_{ro_g} \implies \eth 1(\kappa_1) \cup \overline{\eth 1(\kappa_1)} \cup \underline{\eth 1(\kappa_1)} \subseteq \eth 2(\kappa_1) \cup \overline{\eth 2(\kappa_1)} \cup \underline{\eth 2(\kappa_1)}$
   $\implies \eth 1^A_{ro_g} \subseteq \eth 2^A_{ro_g}$

Proposition 4.2 *Let $"\gamma_1(ro_g)"$ and $"\gamma_2"(ro_g)$ be rough group equivalence relations on $G = (2^{uni_1}, \triangle)$ then* $("\gamma_1"(ro_g) \cap "\gamma_2"(ro_g))^A = ("\gamma_1"(ro_g))^A \cap ("\gamma_2"(ro_g))^A$

Proof

$$
\begin{aligned}
X_1 \in (''\gamma_1''(ro_g) \cap ''\gamma_2''(ro_g))^A &\Leftrightarrow X_1 \in [y_1]_{(''\gamma_1''(ro_g) \cap ''\gamma_2''(ro_g))} \cap A \\
&\Leftrightarrow X_1 \in [''\gamma_1''(uni_1) \cap ''\gamma_1''(uni_1)] \cup [\overline{''\gamma_1''(uni_1)} \cap \overline{''\gamma_2''(uni_1)}] \cup [\underline{''\gamma_2''''(uni_1)} \cap \underline{''\gamma_2''(uni_1)}] \cap A \\
&\Leftrightarrow X_1 \in [''\gamma_1''(uni_1) \cup \overline{''\gamma_1''(uni_1)} \cup \underline{''\gamma_1''(uni_1)}] \cap [''\gamma_2''(uni_1) \cup \overline{''\gamma_2''(uni_1)} \cup \underline{''\gamma_2''(uni_1)}] \cap A \\
&\Leftrightarrow X_1 \in [[''\gamma_1''(uni_1) \cup \overline{''\gamma_1''(uni_1)} \cup \underline{''\gamma_1''(uni_1)}] \cap A] \cap [[''\gamma_2''(uni_1) \cup \overline{''\gamma_2''(uni_1)} \cup \underline{''\gamma_2''(uni_1)}] \\
&\cap A] \\
&\Leftrightarrow X_1 \in (''\gamma_1''(ro_g))^A \cap (''\gamma_2''(ro_g))^A
\end{aligned}
$$

Hence the upper approximation of any set with respect to intersection of rough group equivalence relation is equal to intersection of upper approximation of the set.

B

Proposition 4.3
Let $''\gamma_1(ro_g)''$ *and* $''\gamma_2''(ro_g)$ *be rough group equivalence relations on* $G = (2^{\kappa_1}, \triangle)$ *then* $(''\gamma_1''(ro_g) \cap ''\gamma_2''(ro_g))_A = (''\gamma_1''(ro_g))_A \cap (''\gamma_2''(ro_g))_A$

Proof:

$$
\begin{aligned}
X_1 \in (''\gamma_1''(ro_g) \cap ''\gamma_2''(ro_g))_A &\Leftrightarrow X_1 \in [y_1]_{(''\gamma_1''(ro_g) \cap ''\gamma_2''(ro_g))} \subseteq A \\
&\Leftrightarrow X_1 \in \{[''\gamma_1''(uni_1) \cap ''\gamma_2''(uni_1)] \cup [\overline{''\gamma_1''(uni_1)} \cap \overline{''\gamma_2''(uni_1)}] \cup [\underline{''\gamma_1''(uni_1)} \cap \underline{''\gamma_2''(uni_1)}]\} \\
&\subseteq A \\
&\Leftrightarrow X_1 \in \{[''\gamma_1''(uni_1) \cap ''\gamma_2''(uni_1)] \subseteq A\} \cap \{[\overline{''\gamma_1''(uni_1)} \cap \overline{''\gamma_2''(uni_1)}] \subseteq A\} \\
&\cap \{[\underline{''\gamma_2''(uni_1)} \cap \underline{''\gamma_2''(uni_1)}] \subseteq A\} \\
&\Leftrightarrow X_1 \in \{[''\gamma_1''(uni_1) \cup \overline{''\gamma_1''(uni_1)} \cup \underline{''\gamma_1''(uni_1)}] \subseteq A\} \cap \{[''\gamma_2''(uni_1) \\
&\cup \overline{''\gamma_2''(uni_1)} \cup \underline{''\gamma_2''(uni_1)}] \subseteq A\} \\
&\Leftrightarrow X_1 \in (''\gamma_1''(ro_g))_A \cap (''\gamma_2''(ro_g))_A
\end{aligned}
$$

Hence the lower approximation of any set with respect to intersection of .rough group equivalence relation is equal to intersection of lower approximation of the set.

5 Results and Discussion

To develop rough set theory, it is essential to look at its algebraic structure. By considering upper and lower approximations of rough sets, we defined rough groups in a more extended sense than previous approaches such as considering the upper approximation of any subset in a finite universe and demonstrating closure, associativity, and identity in the upper approximation, but the inverse exists in the set itself [23] also considering abstract groups as universe set and its normal subgroups as equivalence relation [9]. An exploration of the expansive properties of rough groups based on the rough group equivalence relation has been presented in this paper.

6 Conclusions

The algebraic aspects of rough set theory have been integral to the development of rough set theory concept as algebraic structures allow the detailed study of set theoretic properties. Rough groups are introduced in this paper using both upper and lower approximations to rough sets within a finite universe. Furthermore, rough groups have been shown to have expansive properties such as upper approximation of intersection of rough groups is contained in its intersection while lower approximation of intersection of rough groups is equal to its intersection. Also upper approximation of union of rough groups is equal to it union. Based on rough group equivalence, we have defined a rough group approximation space and derived the upper and lower approximations of any set. More studies will be conducted in the future to examine rough group properties in greater detail. A similar extension can be made to other algebraic structures as well.

REFERENCES

[1] Zdzislaw Pawlak, Rough Sets,International Journal of Computer & Information Sciences, 1982.

[2] Pawlak Z, Rough sets – Theoretical aspects of reasoning about Data. Kluwer Academic Publishers, 1991.

[3] Yao, Y.Y., On generalizing Pawlak approximation operators, Lecture Notes in Artificial Intelligence, pp.298-307, 1998

[4] Miao, D.Q., Wang, J, An information representation of concepts and operations in rough Sets. Journal of Software, vol. 10(2), pp. 113-116, 1999.

[5] Z. Bonikowski, Algebraic Structures of Rough Sets, Rough Sets, Fuzzy Sets and Knowledge Discovery - Proceedings of the International Workshop on Rough Sets and Knowledge Discovery RSKD'1993 Springer-Verlag, pp. 242-247, 1994.

[6] J. Iwinski, Algebraic Approach to Rough Sets, Bull Polish Acad Sci Math, Vol. 35, pp. 673-683, 1987.

[7] J. Pomykala and J. A. Pomykala, The Stone Algebra of Rough Sets, Bull Polish Acad Sci Math, vol. 36, pp. 495-508, 1988.

[8] B. Davvaz, Roughness in rings, Inform. Sci. vol.164 pp. 147–163, 2004.

[9] N. Kuroki & P. P. Wang, The Lower and Upper Approximations in a Fuzzy Group, Information Sciences, vol. 90, pp. 203-220, 1996.

[10] N. Bagirmaz, I. Icen, A.F. Ozcan, Topological rough groups, Topol. Algebra Appl. vol 4, pp. 31–38, 2016.

[11] Radwan Abu-Gdairi, Mostafa A. El-Gayar, Mostafa K. El-Bably and Kamel K. Fleifel, Two Different Views for Generalized Rough Sets with Applications, Mathematics, vol. 9,2275, 2021.

[12] A.S. Salama, Essam El-Seidy, A.K. Salah, Properties of different types of rough approximations defined by a family of dominance relations, International Journal of Fuzzy Logic and Intelligent Systems, Vol. 22, pp. 193–201, 2022.

[13] T.M. Al-shami, Topological approach to generate new rough set models, Complex Intell. Syst. Vol. 8, pp.4101–4113, 2022.

[14] Al-shami, T.M, Fu, W.Q, Abo-Tabl, E.A., New rough approximations based on E-neighborhoods. Complexity, Vol.2021, 2021.

[15] El-Bably, M.K, Fleifel K.K, Embaby O.A, Topological approaches to rough approximations based on closure operators. Granul.Comput.pp. 1–14, 2021.

[16] A.C¸.Guler, E.D. Yildirim, O.B. Ozbakir, Rough approximations based on different topologies via ideals, Turk. J. Math., vol. 46, pp.1177–1192,2022.

[17] M. Hosny, Tareq M. Al-shami, Abdelwaheb Mhemdi, Novel approaches of generalized rough approximation spaces inspired by maximal neighbourhoods and ideals, Alexandria Engineering Journal, vol. 69, 497–520, 2023.

[18] A. S. Nawar, M. K. El-Bably, and A. A. El-Atik, Certain types of coverings based rough sets with application, Journal of Intelligent and Fuzzy Systems, Vol. 39, no.3, pp.3085–3098, 2020.

[19] M. Atef, A. M. Khalil, S.G. Li, A. A. Azzam, and A. A. El Atik, Comparison of six types of rough approximations based on j-neighborhood space and j-adhesion neighborhood space, Journal of Intelligent and Fuzzy Systems, vol. 39, no. 3, pp. 515–4531, 2020.

[20] M. Hosny, Topological approach for rough sets by using J-nearly concepts via ideals, Filomat Vol. 34, pp. 273–286, 2020.

[21] Pradeep Shende, Arvind Kumar Sinha, A Novel Concept of Uncertainty Optimization Based Multi-Granular Rough Set and Its Application, Mathematics and Statistics, Vol.9, No.4, pp. 608-616, 2021. DOI: 10.13189/ms.2021.090420.

[22] Arvind Kumar Sinha, Pradeep Shende, Uncertainty Optimization-Based Rough Set for Incomplete Information Systems, Mathematics and Statistics, Vol.10, No.4, pp. 759-772, 2022. DOI: 10.13189/ms.2022.100407.

[23] R. Biswas & S. Nanda, Rough groups and Rough Subgroups, Bull Polish Acad Sci Math, vol. 42, pp. 251–254, 1994.

[24] Duoqian Miao, Suqing Han, Daoguo Li, and Lijun Sun, Rough group, rough subgroup and their properties, Springer-Verlag Berlin Heidelberg, 2005.

[25] Changzhong Wang, Degang Chen, Qinghua Hu, On rough approximations of groups, Int. J. Mach. Learn. & Cyber, 2012.

[26] M. Artin, *Algebra*, Prentice Hall of India, N.Delhi, 2004.