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Generalization of Riemann-Liouville Fractional Operators in Bicomplex Space and Applications

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Abstract In this article, we generalize the Riemann-Liouville fractional differential and integral operators that can be applied to the functions of a bicomplex variable. For this purpose, we consider the bicomplex Cauchy integral formula and some contours in bicomplex space. We elaborate these operators through some examples. Also, we contemplate some significant properties of these operators which include a discussion of bicomplex analytical behavior of generalized bicomplex functions through Pochhammer contours, the law of exponents, generalized Leibniz rule along with a depiction of the region of convergence, and generalized chain rule for Riemann-Liouville fractional operators of bicomplex order. We give an application of our work in the construction of fractional Maxwell's type equations in vacuum and source-free domains equipped with the Riemann-Liouville derivative operator. For this, we define bicomplex *grad*, *div*, and *curl* operator with the help of these newly defined operators. The advantage of this fractional construction of Maxwell's equation is that it may be used to build fractional non-local electronics in bicomplex space. By considering bicomplex vector fields for the respective domains, we reduce the number of these fractional Maxwell's type equations by half, which makes it easier to extract electric and magnetic fields from the bicomplex vector fields.

Keywords Idempotent Representation, Bicomplex Gamma and Beta Functions, Functions of Bicomplex Variable, Riemann-Liouville Operators of Bicomplex Order

1 Introduction

There has been a separate development of theory in both bicomplex analysis and fractional calculus but to present a concept in these two fields together is a new effort in itself. In this work, the study of Riemann-Liouville fractional operators has been presented in the context of bicomplex analysis. In 1892, *Segre* defined bicomplex numbers as special generalization of complex numbers concern with four dimensional space. The theory of bicomplex number emerged with an important utilitarian mathematical tool [1, 2]. An analogy of several complex phenomena was explored in [3–7]. An investigation of the Schrödinger equation and its solution by an analytical method in the framework of bicomplex numbers were found in [8]. In [9], the author proposed bicomplex Fibonacci quaternions with some significant formulae and inequalities. Various integral transforms in bicomplex sense were proposed with their applications in [10, 11] etc.

In [12], the authors worked on reconstruction of high probability bicomplex sparse signal. from a reduced number of bicomplex random samples. With various types of standard metric, a construction of zero mean curvature complex surfaces in bicomplex numbers was given in [13]. In [14], a classification of singularities of bicomplex holomorphic functions with residue theorem for bicomplex holomorphic functions was described. A bicomplex-valued twin-multistate Hopfield neural network (BTMHNN) was introduced in [15] for reduction of the number of weight parameters.

Fractional calculus originated from a question asked by *L'Hôpital* to *Leibniz* through a letter in which it was asked that how to define the $(1/2)^{th}$ derivative of a function. Later various mathematicians such as *Euler, Laplace, Fourier, Abel, Liouville, Riemann,* and *Laurent* etc. made significant efforts in initial development of fractional calculus. A number of fractional operators have been formed viz. Riemann-Liouville, Liouville, Caputo, Prabhakar, Atangana-Baleanu etc till date. A generalization of Leibniz rule for fractional operators and its various aspects was proposed in [16, 17].

Various fractional operators concern with special functions were developed in [18]. In [19], an important description of fractional calculus concerning vector calculus was given. Cauchy's integral formula via the modified Riemann-Liouville derivative of fractional order was introduced in [20].

In [21], a complex analysis approach to Atangana-Baleanu operator was given. A new sigmoidal fractional derivative was introduced in [22]. In order to applications, fractional approach of Maxwell's equations in curved spacetime, and fractional electromagnetic waves in plasma and dielectric media with Caputo generalized fractional derivative were explored in [23–25]. A generalization of the Riemann-Liouville, Caputo, and modified Caputo fractional differential equations studied through linear first order Riemann-Liouville fractional differential equations was given in [26]. In [27], the author introduced fractional integration and differentiation on arbitrary nonempty closed sets, which were provided by Cauchy's formula for repeated integration on time scales.

With the intention to solve the corresponding nonlinear weakly singular Volterra-Fredholm integral equation, an investigation of the spectral collocation method for the Riemann–Liouville fractional terminal value problems was proposed in [28]. A description of integral and derivative operators of tempered fractional calculus with their analytic properties connected with the classical Riemann-Liouville fractional calculus was developed in [29]. In [30], The author proposed Scarpi's ideas within recent theory of general fractional derivatives and integrals.

The Atangana–Baleanu fractional model [31], proposed in 2016, is based on replacing the power function kernel by a non-singular function that has strong connections with fractional calculus [32, 33]. The development of the paper is as follows: In Section 2, we give an overview of the fundamental theory of bicomplex analysis and fractional calculus necessary for our work. Section 3 is devoted to the Riemann-Liouville fractional operators of bicomplex order, which can be applied to functions of bicomplex variable. Section 4 introduces some important properties of these operators. In Section 5, we find applications by constructing bicomplex fractional Maxwell's type equation using bicomplex vector fields in various domains. Last section concludes the paper.

2 Preliminaries

2.1 Bicomplex Numbers

We follow the work done in [34] to introduce the existing literature of the subject. The set of bicomplex numbers is defined as

$$\begin{aligned}\mathbb{C}_2 = \{w \ : \ w &= z_1 + jz_2 \\ &= x_1 + iy_1 + jx_2 + ijy_2\},\end{aligned}$$

where $x_m, y_m \in \mathbb{R}$; $m = 1, 2$ and $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$ with $i^2 = j^2 = -1, (ij)^2 = 1$, and $ij = ji$. z_1 and z_2 are said to be bi-real and bi-imaginary parts of w, respectively. The *idempotent representation*, which is unique for every bicomplex number, can be seen as

$$w = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2 = w_1e_1 + w_2e_2. \quad (1)$$

Here $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ which satisfy the identities $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 - e_2 = ij, e_1 + e_2 = 1$, and $e_1e_2 = e_2e_1 = 0$. The previous two equations show that e_1 and e_2 are zero divisors in $\mathbb{C}_2$ which conclude that $\mathbb{C}_2$ is not a division algebra. Using this fact, *null cone* in $\mathbb{C}_2$, containing all zero divisors, can be defined as

$$\mathcal{O}_2 = \{\lambda(1 \pm ij) \ : \ \lambda \in \mathbb{C}(i) \setminus \{0\}\}.$$

Conjugates of w w.r.t. i, j, and ij [34, p. 8] are as follows

$$\begin{aligned}\bar{w} &= \bar{w_2}e_1 + \bar{w_1}e_2,\\ w^\dagger &= w_2e_1 + w_1e_2,\\ w^* &= \bar{w_1}e_1 + \bar{w_2}e_2,\end{aligned}$$

respectively.

For every $w \in \mathbb{C}_2$, another representation of (1) can be considered as

$$w = \mathcal{P}_1(w)e_1 + \mathcal{P}_2(w)e_2, \quad (2)$$

where the mappings $\mathcal{P}_1, \mathcal{P}_2 : \mathbb{C}_2 \to \mathbb{C}(i)$ are projections of $\mathbb{C}_2$ onto $\mathbb{C}(i)$ and defined as

$$\mathcal{P}_1(z_1 + jz_2) = z_1 - iz_2 \quad (3)$$
$$\mathcal{P}_2(z_1 + jz_2) = z_1 + iz_2, \quad (4)$$

respectively, where $\mathbb{C}(i)$ is the set of complex numbers with imaginary unit i.

The idempotent representations (1) and (2) play a significant role in the reduction the number of computations with bicomplex numbers and this reduction is due to the identities $e_1e_2 = e_2e_1 = 0$. Thus, for any given $w = w_1e_1 + w_2e_2$ and $\xi = \xi_1e_1 + \xi_2e_2$ in $\mathbb{C}_2$, we have

$$\begin{aligned}w + \xi &= (w_1 + \xi_1)e_1 + (w_2 + \xi_2)e_2\\ w\xi &= w_1\xi_1e_1 + w_2\xi_2e_2\\ w^n &= w_1^ne_1 + w_2^ne_2.\end{aligned}$$

The bicomplex exponential function then can be written as

$$e^w = e^{w_1}e_1 + e^{w_2}e_2.$$

One can go through [35] to establish a better understanding with the subject.

Let $X = (X_1, X_2)$ be a domain in $\mathbb{C}_2$, then using projections (3) and (4) the corresponding domains in the complex plane are given as

$$X_1 = \mathcal{P}_1(X) = \{w_1 = z_1 - iz_2 : z_1, z_2 \in \mathbb{C}(i)\} \quad (5)$$
$$X_2 = \mathcal{P}_2(X) = \{w_2 = z_1 + iz_2 : z_1, z_2 \in \mathbb{C}(i)\}. \quad (6)$$

Combining (5) and (6), X can be rewritten as

$$X = \{(w_1, w_2) : w_1 \in X_1, w_2 \in X_2\} = X_1 \times X_2.$$

The *Euclidean norm* in $\mathbb{R}^4$ for $w = z_1 + jz_2 \in \mathbb{C}_2$, is defined as

$$\|w\|_2 = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2},$$

where we use '$|\cdot|$' to denotes Euclidean norm in complex space. It is easy to prove that

$$\|\xi w\|_2 \leq \sqrt{2}\|\xi\|_2\|w\|_2.$$

A *bicomplex sequence* w_n and its convergence can be defined in the same way as these were defined in the complex space.

A function $f : X \to \mathbb{C}_2$ such that $f = f_1 + jf_2$, is said to be *differentiable* at $w_0 \in \mathbb{C}_2$ if the limit

$$\lim_{\substack{w \to w_0 \\ w - w_0 \notin \mathcal{O}_2}} \frac{f(w) - f(w_0)}{w - w_0}$$
$$\text{or } \lim_{\substack{\triangle w \to 0 \\ \triangle w \notin \mathcal{O}_2}} \frac{f(w + \triangle w) - f(w)}{\triangle w}$$

exists finitely and we write

$$f'(w_0) = \lim_{\substack{w \to w_0 \\ w - w_0 \notin \mathcal{O}_2}} \frac{f(w) - f(w_0)}{w - w_0}$$
$$\text{or } f'(w_0) = \lim_{\substack{\triangle w \to 0 \\ \triangle w \notin \mathcal{O}_2}} \frac{f(w + \triangle w) - f(w)}{\triangle w}.$$

A bicomplex holomorphic function is the function defined on a non-empty open set, say $X \subseteq \mathbb{C}_2$, and has a derivative at each point of X. Equivalently, we say that the complex functions f_1 and f_2 are holomorphic in the variables z_1 and z_2 with $w = z_1 + jz_2$ and satisfy the bicomplex Cauchy-Riemann system

$$\frac{\partial f}{\partial w^*} = \frac{\partial f}{\partial w^\dagger} = \frac{\partial f}{\partial \bar{w}} = 0.$$

It can be verified that the bicomplex derivative of $f = f_1 + jf_2$, exists if and only if the complex functions-pair (f_1, f_2) satisfies the following bicomplex Cauchy–Riemann equations:

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \quad \text{and} \quad \frac{\partial f_1}{\partial z_2} = -\frac{\partial f_2}{\partial z_1}. \tag{7}$$

As a conclusion, if a bicomplex function satisfies the above conditions, described in (7), then it is said to be a bicomplex differentiable or bicomplex holomorphic function. The partial derivatives

$$\frac{\partial f}{\partial w} = \frac{1}{2}\left(\frac{\partial f}{\partial z_1} - j\frac{\partial f}{\partial z_1}\right),$$
$$\frac{\partial f}{\partial w^\dagger} = \frac{1}{2}\left(\frac{\partial f}{\partial z_1} + j\frac{\partial f}{\partial z_1}\right)$$

results in the *Cauchy-Riemann equation* $\frac{\partial f}{\partial w^\dagger} = 0$ in bicomplex form. If $f : X \to \mathbb{C}_2$ be a holomorphic function then

$$f(w) = f_1(w_1)e_1 + f_2(w_2)e_2, \tag{8}$$

where $w \in X$, $w_1 \in X_1$, $w_2 \in X_2$, and f_1 and f_2 are holomorphic functions of complex variables w_1 and w_2, respectively, in X_1 and X_2, respectively.

For $w = z_1 + jz_2 \equiv (z_1, z_2)$, we consider a bicomplex function

$$f(w) = f(z_1, z_2) =$$
$$f_1(z_1, z_2) + jf_2(z_1, z_2) \equiv (f_1(z_1, z_2), f_2(z_1, z_2))$$

and let $\Upsilon = (\gamma_1, \gamma_2)$ be a four dimensional piecewise continuously differentiable curve in a set $S \subseteq \mathbb{C}_2$. Then the bicomplex integration of bicomplex function f is defined as a line integral, that is evaluated along the curve Υ in $\mathbb{C}_2$ and defined as

$$\int_\Upsilon f(w)dw, \quad dw = (dz_1, dz_2). \tag{9}$$

" Consider the parametric form of Υ
$\Upsilon : w(t) \equiv (z_1(t), z_2(t))$, or $w(t) = w_1e_1 + w_2e_2$, where $r \leq t \leq s$. Then (9) can be rewritten as

$$\int_\Upsilon f(w)dw = \int_r^s f(w(t))w'(t)dt.$$

Here, it may be possible that $w'(t)$ is discontinuous at some points. We can consider Υ as a curve made up of two component curves γ_1 and γ_2 in $\mathbb{C}$ i.e.

$$\Upsilon = (\gamma_1, \gamma_2).$$

Along with the curve Υ, (9) can be given as

$$\int_\Upsilon f(w)dw = \left\{\int_{\gamma_1} f(w_1)dw_1\right\} e_1$$
$$+ \left\{\int_{\gamma_2} f(w_2)dw_2\right\} e_2.$$

For more details one can go through [34, 36].

2.2 Fractional Calculus

In fractional calculus, we are interested in the extension of the basic calculus operators of differentiation and integration and generalizing the order of these operators to the case where the order is allowed to be any real or complex number, not just an integer. In this context, the authors defined Riemann-Liouville operators of bicomplex order which can be applied to bicomplex valued functions of real variable [37]. In [38], fractional calculus in the context of bicomplex space was discussed on the basis of Cauchy-Riemann operators. By the time many different definitions of fractional calculus have been developed and each one has its own advantages and disadvantages. One of the most natural and commonly used models of fractional calculus is the Riemann–Liouville one. The Riemann-Liouville differential and integral operators of bicomplex order [37] are defined as

Definition 2.1. *Let $w = z_1 + jz_2 \in \mathbb{C}_2$ with $Re(z_1) > |Im(z_2)|$ and f defined in (8) be piecewise continuous on $J' = (0, \infty)$ and integrable on any finite subinterval of $J = [0, \infty)$. Then for $t > 0$*

$$ {}_0D_t^{-w} f(t) = \frac{1}{\Gamma_2(w)} \int_0^t f(x)(t-x)^{w-1} dx. $$

We take $\mathcal{C}$ to denote the class of functions defined in the Definition 2.1.

Definition 2.2. *Let f be a function of class $\mathcal{C}$ and let $w \in \mathbb{C}_2$ with $Re(z_1) > 0$. Let $m = \lfloor Re(z_1) \rfloor + 1$. Then the Riemann-Liouville fractional derivative of f of order w is*

$$ \begin{aligned} {}_0D_t^{w} f(t) &= {}_0D_t{}^m {}_0D_t{}^{-(m-w)} f(t) \\ &= \frac{1}{\Gamma_2(m-w)} \frac{d^m}{dt^m} \int_0^t f(x)(t-x)^{m-w-1} dx. \end{aligned} $$

Riemann-Liouville "differentiation and integration of purely bi-imaginary order is defined as

$$ {}_0D_t^{jx_2} f(t) = \frac{1}{\Gamma_2(1-jx_2)} \frac{d}{dt} \int_0^t f(x)(t-x)^{-jx_2} dx, $$

and

$$ {}_0D_t^{-jx_2} f = \frac{d}{dt} D^{-1-jx_2} f. $$

Therefore,

$$ {}_0D_t^{-jx_2} f(t) = \frac{1}{\Gamma_2(1+jx_2)} \frac{d}{dt} \int_0^t f(x)(t-x)^{jx_2} dx." $$

Riemann-Liouville fractional derivative of bicomplex order can be defined as the left-inverses of the fractional integral. Riemann-Liouville fractional operators of real or complex order can be taken as a particular case of fractional operators of bicomplex order. A detailed theory of fractional calculus can be found in [39,40]. [41] deals with the geometric and physical explanations of fractional calculus.

Many alternative definitions for fractional derivatives and integrals have been presented. Some of them are equal to the Riemann-Liouville definition, while others are defined by minor changes on the Riemann-Liouville formula.

3 Riemann-Liouville fractional operators of bicomplex order Applied on Functions of Bicomplex Variable

This section deals with Riemann-Liouville fractional differentiation and integration of functions of bicomplex variable.

Let $a = a_1 + ja_2 \in X$ and r_1 and r_2 be positive constants in $\mathbb{R}$. The closed discus $\bar{D}(a; r_1, r_2)$ with center a and radii r_1 and r_2 is defined as follows:

$$ \bar{D}(a; r_1, r_2) = \{w : |w_1 - o_1| \le r_1;\ |w_2 - o_2| \le r_2\}, $$

where $o_1 = a_1 - ia_2$ and $o_2 = a_1 + ia_2$. We consider X as an open set and $\bar{D}(a; r_1, r_2) \subset X$ for sufficiently small r_1 and r_2.

Let $\xi = \delta_1 + j\delta_2 = (\delta_1 - i\delta_2)e_1 + (\delta_1 + i\delta_2)e_2 = \xi_1 e_1 + \xi_2 e_2$ be a fixed point in the open discus $D(a; r_1, r_2)$, then $\mathcal{P}_1(\xi) = \xi_1$ is in the open circle c_1 with center o_1 and radius r_1 and $\mathcal{P}_2(\xi) = \xi_2$ is in the open circle c_2 with center o_2 and radius r_2, see Fig. 1.

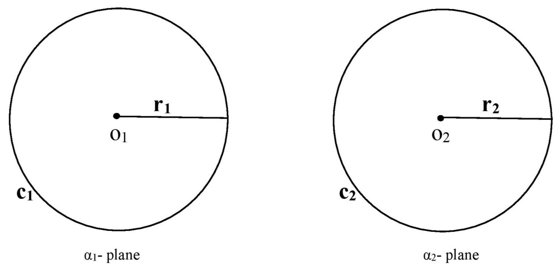

Figure 1. Contours for Riemann-Liouville operators

The circles c_1 and c_2 together with their interiors are in X_1 and X_2, respectively. Then c_1 and c_2 determine a curve c in X which lies in the boundary of $D(a; r_1, r_2)$.

In this consideration, the Cauchy integral formula in bicomplex space [36, p. 277] is defined as:

Theorem 3.1. *If $f : X \to \mathbb{C}_2$ is a bicomplex holomorphic function and $D(a; r_1, r_2)$ and c, are the discus and the curve, respectively, just described above and if $\xi = \delta_1 + j\delta_2$ is a point in $D(a; r_1, r_2)$, then*

$$ \begin{aligned} f(\xi) &= \frac{1}{2\pi i} \int_c \frac{f(\tau) d\tau}{\tau - \xi} \qquad (10) \\ \Rightarrow f(\delta_1 + j\delta_2) &= \frac{1}{2\pi i} \int_c \frac{f(\tau_1 + j\tau_2) d(\tau_1 + j\tau_2)}{(\tau_1 + j\tau_2) - (\delta_1 + j\delta_2)}. \end{aligned} $$

Based on integral (10), we define the Riemann-Liouville fractional derivative of bicomplex order of functions of bicomplex variable as

$$ D_\xi^w f(\xi) = \frac{1}{\Gamma_2(-w)} \int_0^\xi f(\alpha)(\xi - \alpha)^{-w-1} d\alpha, \qquad (11) $$

where $w = z_1 + jz_2$ such that $\text{Re}(z_1) < |\text{Im}(z_2)|$ and f is bicomplex holomorphic function. The above integral (11) can be justified by the following theorem:

Theorem 3.2. *Let $f : X \to \mathbb{C}_2$ be a bicomplex holomorphic function. Then, for $w = z_1 + jz_2$*

$$ D_\xi^w f(\xi) = \frac{1}{\Gamma_2(-w)} \int_0^\xi f(\alpha)(\xi - \alpha)^{-w-1} d\alpha, \quad Re(z_1) < |Im(z_2)|. $$

Proof. Separating (10) into its idempotent components [36, p. 277], we have

$$ \begin{aligned} f_1(\xi_1)e_1 &+ f_2(\xi_2)e_2 \\ &= \frac{1}{2\pi i} \left\{ \int_{c_1} \frac{f_1(\tau_1 - i\tau_2) d(\tau_1 - i\tau_2)}{(\tau_1 - i\tau_2) - (\delta_1 - i\delta_2)} \right\} e_1 \\ &+ \frac{1}{2\pi i} \left\{ \int_{c_2} \frac{f_2(\tau_1 + i\tau_2) d(\tau_1 + i\tau_2)}{(\tau_1 + i\tau_2) - (\delta_1 + i\delta_2)} \right\} e_2, \qquad (12) \end{aligned} $$

where c_1 and c_2 are described in Fig. 1. Let us consider one component of (12)

$$f_1(\xi_1) = \frac{1}{2\pi i}\left\{\int_{c_1} \frac{f_1(\tau_1 - i\tau_2)d(\tau_1 - i\tau_2)}{(\tau_1 - i\tau_2) - (\delta_1 - i\delta_2)}\right\} = \frac{1}{2\pi i}\left\{\int_{c_1} \frac{f_1(\alpha_1)d(\alpha_1)}{\alpha_1 - \xi_1}\right\}, \quad (13)$$

where $\alpha_1 = \tau_1 - i\tau_2$. Differentiating (13) $n-$times, we get

$$D_\xi^n f_1(\xi_1) = \frac{n!}{2\pi i}\left\{\int_{c_1} \frac{f_1(\alpha_1)d(\alpha_1)}{(\alpha_1 - \xi_1)^{n+1}}\right\}. \quad (14)$$

If n is an arbitrary number say w_1, we may replace $n!$ by $\Gamma(w_1+1)$ in (14) which gives

$$D_{\xi_1}^{w_1} f_1(\xi_1) = \frac{\Gamma(w_1+1)}{2\pi i}\left\{\int_{c_1} \frac{f_1(\alpha_1)d(\alpha_1)}{(\alpha_1 - \xi_1)^{w_1+1}}\right\}. \quad (15)$$

(15) shows that ξ_1 is branch point of integrand. Let the contour of integration in (15) be $c_1(0,\xi_1)$ see Fig. 2. Now, the end points determine the value of the integral (15). Thus we write

$$D_{\xi_1}^{w_1} f_1(\xi_1) = \frac{\Gamma(w_1+1)}{2\pi i}\left\{\int_{c_1(0,\xi_1)} f_1(\alpha_1)(\alpha_1 - \xi_1)^{-w_1-1} d\alpha_1\right\}. \quad (16)$$

We deform $c_1(0,\xi_1)$ into the union of three contours, which are described in Fig. 2, i.e.

$$c_1(0,\xi_1) = c_{11} \cup c_{12} \cup c_{13},$$

where

$c_{11} :=$ is a line segment from 0 to ξ_1,
$c_{12} :=$ is a small circle centered at $\alpha_1 = \xi_1$ and radius q_1,
$c_{13} :=$ is c_{11} traversed in the opposite direction.

Thus

$$\int_{c_1(0,\xi_1)} = \int_{c_{11}} + \int_{c_{12}} + \int_{c_{13}}.$$

Calculating the value of integrals [18, p. 252] on c_{11}, c_{12}, and c_{13}, we have

$$\int_{c_1(0,\xi_1)} f_1(\alpha_1)(\alpha_1 - \xi_1)^{-w_1-1} d\alpha_1 = -(e^{iw_1\pi} - e^{-iw_1\pi})\int_0^{\xi_1} f_1(\alpha_1)(\xi_1 - \alpha_1)^{-w_1-1} d\alpha_1 = -2i\sin(w_1\pi)\int_0^{\xi_1} f_1(\alpha_1)(\xi_1 - \alpha_1)^{-w_1-1} d\alpha_1 \quad (17)$$

provided, if under the condition on w_1, the integral over c_{12} tends to zero as the radius q_1 of the contour approaches zero. Substituting values from (17) into (16), we have

$$D_{\xi_1}^{w_1} f_1(\xi_1) = -\frac{\Gamma(w_1+1)}{2\pi i} 2i\sin(w_1\pi) \int_0^{\xi_1} f_1(\alpha_1)(\xi_1 - \alpha_1)^{-w_1-1} d\alpha_1.$$

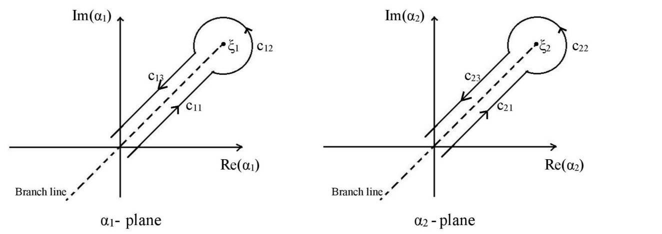

Figure 2. Contours for derivation of Riemann-Liouville differential operator

Using the Euler's reflection formula $\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin \pi\nu}$, $\nu \in \mathbb{C}$, we obtain

$$-\frac{\Gamma(w_1+1)}{2\pi i} 2i\sin(w_1\pi) = \frac{\Gamma(w_1+1)}{\Gamma(w_1)\Gamma(1-w_1)} = \frac{1}{\Gamma(-w_1)}.$$

Therefore,

$$D_{\xi_1}^{w_1} f_1(\xi_1) = \frac{1}{\Gamma(-w_1)}\int_0^{\xi_1} f_1(\alpha_1)(\xi_1 - \alpha_1)^{-w_1-1} d\alpha_1. \quad (18)$$

In a similar way, we can find

$$D_{\xi_2}^{w_2} f_2(\xi_2) = \frac{\Gamma(w_2+1)}{2\pi i}\left\{\int_{c_2(0,\xi_2)} f_2(\alpha_2)(\alpha_2 - \xi_2)^{-w_2-1} d\alpha_2\right\},$$

where $\alpha_2 = \tau_1 + i\tau_2$. Again, we deform $c_2(0,\xi_2)$ into the union of three contours, which are described in Fig. 2, i.e.

$$c_2(0,\xi_2) = c_{21} \cup c_{22} \cup c_{23},$$

where,

$c_{21} :=$ is a line segment from 0 to ξ_2,
$c_{22} :=$ is a small circle centered at $\alpha_2 = \xi_2$ and radius q_2,
$c_{23} :=$ is c_{21} traversed in the opposite direction.

Thus,

$$\int_{c_2(0,\xi_2)} = \int_{c_{21}} + \int_{c_{22}} + \int_{c_{23}}.$$

Proceeding the same way as above, we reach

$$D_{\xi_2}^{w_2} f_2(\xi_2) = \frac{1}{\Gamma(-w_2)}\int_0^{\xi_2} f_2(\alpha_2)(\xi_2 - \alpha_2)^{-w_2-1} d\alpha_2. \quad (19)$$

By combining (18) and (19) as idempotent components, we get

$$\left\{D_{\xi_1}^{w_1} f_1(\xi_1)\right\} e_1 + \left\{D_{\xi_2}^{w_2} f_2(\xi_2)\right\} e_2$$
$$= \left\{\frac{1}{\Gamma(-w_1)} \int_0^{\xi_1} f_1(\alpha_1)(\xi_1 - \alpha_1)^{-w_1-1} d\alpha_1\right\} e_1$$
$$+ \left\{\frac{1}{\Gamma(-w_2)} \int_0^{\xi_2} f_2(\alpha_2)(\xi_2-\alpha_2)^{-w_2-1} d\alpha_2\right\} e_2$$
$$\Rightarrow \left\{e_1 D_{\xi_1}^{w_1} + e_2 D_{\xi_2}^{w_2}\right\} \{f_1(\xi_1)e_1 + f_2(\xi_2)e_2\}$$
$$= \frac{1}{\Gamma_2(-w_1e_1 - w_2e_2)} \times \int_0^{\xi_1e_1+\xi_2e_2} \{f_1(\alpha_1)e_1 + f_2(\alpha_2)e_2\} \{(\xi_1e_1+\xi_2e_2) -(\alpha_1e_1+\alpha_2e_2)\}^{-w_1e_1-w_2e_2-1} d(\alpha_1e_1+\alpha_2e_2) \quad (20)$$
$$\Rightarrow \left\{e_1 D_{\xi_1}^{w_1} + e_2 D_{\xi_2}^{w_2}\right\} \{f(\xi)\}$$
$$= \frac{1}{\Gamma_2(-w)} \int_0^{\xi} f(\alpha)(\xi - \alpha)^{-w-1} d\alpha. \quad (21)$$

The right-hand-side of (21) allows us to write

$$D_\xi^w \equiv e_1 D_{\xi_1}^{w_1} + e_2 D_{\xi_2}^{w_2} \quad (22)$$

which gives

$$D_\xi^w f(\xi) = \frac{1}{\Gamma_2(-w)} \int_0^{\xi} f(\alpha)(\xi - \alpha)^{-w-1} d\alpha.$$

□

Remark 3.3. *In our whole discussion* Γ *represents the complex gamma function and* Γ_2 *represents the bicomplex gamma function [6].*

Just replacing w by $-w$, we can get Riemann-Liouville fractional integration of functions of bicomplex variables as follows:

Theorem 3.4. *Let* $f : X \to \mathbb{C}_2$ *be a bicomplex holomorphic function. Then for* $w = z_1 + jz_2$

$$D_\xi^{-w} f(\xi) = \frac{1}{\Gamma_2(w)} \int_0^{\xi} f(\alpha)(\xi - \alpha)^{w-1} d\alpha, \quad Re(z_1) > |Im(z_2)|. \quad (23)$$

On the basis of (22), we write

$$D_\xi^{-w} \equiv e_1 D_{\xi_1}^{-w_1} + e_2 D_{\xi_2}^{-w_2}. \quad (24)$$

Since a bicomplex number attains several forms of representations [34, p. 7], accordingly the condition of existence of integration and differentiation can be found in Table 1 of [37]. Now, we discuss some illustrative examples to elaborate these operators.

Example 3.5. *Find the Riemann-Liouville fractional integration and differentiation of bicomplex order of* $f(\xi) = \xi^u$*, where* $\xi = \delta_1 + j\delta_2 = \xi_1e_1 + \xi_2e_2 \in \mathbb{C}_2$ *and* $u > -1$.

Solution. **Differentiation:** Let $f(\xi) = \xi^u$, where $u > -1$ and $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with Re$(z_1) <$ |Im(z_2)|. Then using (22), we have

$$D_\xi^w \xi^u = \left\{e_1 D_{\xi_1}^{w_1} + e_2 D_{\xi_2}^{w_2}\right\} \{\xi_1^u e_1 + \xi_2^u e_2\}$$
$$= \{D_{\xi_1}^{w_1} \xi_1^u\} e_1 + \{D_{\xi_2}^{w_2} \xi_2^u\} e_2,$$

where

$$D_{\xi_1}^{w_1} \xi_1^u = \frac{1}{\Gamma(-w_1)} \int_0^{\xi_1} \alpha_1^u (\xi_1 - \alpha_1)^{-w_1-1} d\alpha_1$$
$$= \frac{B(u+1, -w_1)}{\Gamma(-w_1)} \xi_1^{u-w_1}$$
$$= \frac{\Gamma(u+1)}{\Gamma(u - w_1 + 1)} \xi_1^{u-w_1}; \quad (u+1) > 0,$$

and

$$D_{\xi_2}^{w_2} \xi_2^u = \frac{\Gamma(u+1)}{\Gamma(u - w_2 + 1)} \xi_2^{u-w_2}; \quad (u+1) > 0.$$

Hence,

$$D_\xi^w \xi^u = \left\{\frac{\Gamma(u+1)}{\Gamma(u - w_1 + 1)} \xi_1^{u-w_1}\right\} e_1 + \left\{\frac{\Gamma(u+1)}{\Gamma(u - w_2 + 1)} \xi_2^{u-w_2}\right\} e_2$$
$$= \frac{\Gamma_2(u+1)}{\Gamma_2(u - w + 1)} \xi^{u-w}, \quad (25)$$

where the bicomplex gamma function [6] has been used.

Integration: Let $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with Re$(z_1) >$ |Im(z_2)|. On putting $f(\xi) = \xi^u$ in (24), we have

$$D_\xi^{-w} \xi^u = \left\{e_1 D_{\xi_1}^{-w_1} + e_2 D_{\xi_2}^{-w_2}\right\} \{\xi_1^u e_1 + \xi_2^u e_2\}$$
$$= \{D_{\xi_1}^{-w_1} \xi_1^u\} e_1 + \{D_{\xi_2}^{-w_2} \xi_2^u\} e_2. \quad (26)$$

Firstly, we deal with $D_{\xi_1}^{-w_1} \xi_1^u$, we have

$$D_{\xi_1}^{-w_1} \xi_1^u = \frac{1}{\Gamma(w_1)} \int_0^{\xi_1} \alpha_1^u (\xi_1 - \alpha_1)^{w_1-1} d\alpha_1 \quad (27)$$
$$= \frac{B(u+1, w_1)}{\Gamma(w_1)} \xi_1^{u+w_1}$$
$$= \frac{\Gamma(u+1)}{\Gamma(u + w_1 + 1)} \xi_1^{u+w_1}; \quad (u+1) > 0, \quad (28)$$

where B denotes the complex beta function. In a similar way , we can find $D_{\xi_2}^{-w_2} \xi_2^u$ as

$$D_{\xi_2}^{-w_2} \xi_2^u = \frac{\Gamma(u+1)}{\Gamma(u + w_2 + 1)} \xi_2^{u+w_2}; \quad (u+1) > 0. \quad (29)$$

A substitution of values from (28) and (29) into (26), we get

$$D_\xi^{-w} \xi^u = \left\{\frac{\Gamma(u+1)}{\Gamma(u + w_1 + 1)} \xi_1^{u+w_1}\right\} e_1 + \left\{\frac{\Gamma(u+1)}{\Gamma(u + w_2 + 1)} \xi_2^{u+w_2}\right\} e_2$$
$$= \frac{\Gamma_2(u+1)}{\Gamma_2(u + w + 1)} \xi^{u+w}, \quad (30)$$

where the properties of the bicomplex gamma function has been used.

Remark 3.6. *We can take* $u = \varrho_1 + j\varrho_2 = u_1e_1 + u_2e_2 \in \mathbb{C}_2$ *such that* $\varrho_1 \neq -\frac{1}{2}(m+n)$ *and* $\varrho_2 \neq -\frac{i}{2}(m-n);\ m, n \in \mathbb{N} \cup \{0\}$, *this will not affect the above results obtained in (25) and (30).*

In the further discussion we take $u \in \mathbb{C}_2$. By applying some simple calculations, for $\text{Re}(z_1) < |\text{Im}(z_2)|$, $\xi \in \mathbb{C}_2 \setminus \mathcal{O}_2$, $\varrho_1 \neq -\frac{1}{2}(m+n)$, and $\varrho_2 \neq -\frac{i}{2}(m-n);\ m, n \in \mathbb{N} \cup \{0\}$, we can prove that

$$D_\xi^w \xi^u \log \xi = \frac{\Gamma_2(u+1)\xi^{u-w}}{\Gamma_2(u-w+1)} \left[\log \xi + \Psi(u+1) - \Psi(u-w+1)\right],$$

where Ψ is the bicomplex digammma function [6]. In Table 1, we calculated Riemann-Liouville fractional integration and differentiation of some functions in bicomplex space where Υ^* denotes the bicomplex incomplete gamma function (see Appendix).

4 Some Properties of Riemann-Liouville Operators

This section discusses some significant properties of Riemann-Liouville fractional operators viz. analytic behavior of some generalized functions of bicomplex variables, the law of exponent, Leibniz rule, and chain rule etc.

4.1 Analytic Behavior of Some Generalized Functions

By the time, $D_z^\alpha F(z)$, where $\alpha, z \in \mathbb{C}$ attained various representations given by *Euler* (1731), *Riemann-Liouville*, *Grunwald* (1867), and *Letnikov* (1868) etc. with some restrictions which shows that no single representation is obviously superior in all applications. Nevertheless, when the functions are differentiated, all representations provide the same result. Many publications have shown that the Pochhammer contour integral is frequently the most efficient for establishing a general theorem on fractional differentiation. A complete discussion of it for the complex variable can be found in [42].

Now, we are interested in finding fractional derivatives and discussing its analytic behavior. Let us consider the function of the type

$$F(\xi) = \xi^u (\log \xi)^\delta f(\xi)(\xi - \alpha)^{-w-1},$$

where δ is either 0 or 1, $\xi, u \in \mathbb{C}_2$, and $f(\xi)$ is a bicomplex holomorphic function. We begin by examining

$$\int_P F(\alpha) d\alpha = \left\{ \int_{P_1} \alpha_1^{u_1} (\log \alpha_1)^\delta f_1(\alpha_1)(\xi_1 - \alpha_1)^{-w_1-1} \right\} e_1 + \left\{ \int_{P_2} \alpha_2^{u_2} (\log \alpha_2)^\delta f_2(\alpha_2)(\xi_2 - \alpha_2)^{-w_2-1} \right\} e_2, \quad (31)$$

where P_1 and P_2 are complex Pochhammer contours demonstrated in Fig. 3 with their components $C_{ij};\ i = 1, 2;\ j = 1, 2, 3, 4$ in α_1-complex plane and α_2-complex plane, respectively. The value of integration on each P_i can be evaluated with the assistance of

$$\int_{P_1} = \int_{C_{11}} + \int_{C_{12}} + \int_{C_{13}} + \int_{C_{14}}$$
$$\int_{P_2} = \int_{C_{21}} + \int_{C_{22}} + \int_{C_{23}} + \int_{C_{24}}.$$

The contour P can be seen as Pochhammer contour formed

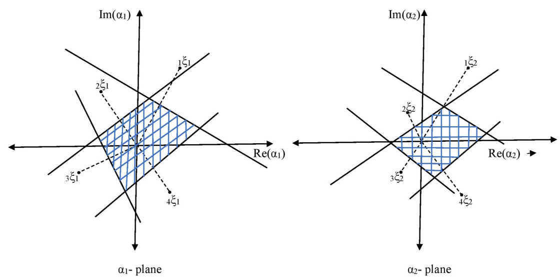

Figure 3. Components of the Pochhammer contours

with the help of two complex Pochhammer contours P_1 and P_2 and can be written as $P = (P_1, P_2)$. We call it Pochhammer contour in bicomplex space. Fig. 3 shows l_{r1} : branch line for $(\xi_r - \alpha_r)^{-w_r-1}$ and l_{r2} : branch line for $\alpha_r^{u_r}(\log \alpha_r)^\delta$ for $r = 1, 2$ of each integrand of (31) passing through the one point without intersecting the contours at any other point. In the demonstration of complex counter part it was described that after completely traversing all components of each P_i, the respective integrand of (31) returns to the value $F_i(a_i),\ i = 1, 2$ with which it started. With the appropriate conventions (as for complex space [42, p. 330]), taking [42, eq. 2.9] into account, we have

$$D_{\xi_1}^{w_1} \xi_1^{u_1} (\log \xi_1)^\delta f_1(\xi_1) = \frac{e^{-\pi u_1 i}\Gamma(w_1+1)}{4\pi \sin(\pi u_1)} \int_{P_1} \alpha_1^{u_1} (\log \alpha_1)^\delta f_1(\alpha_1)(\xi_1 - \alpha_1)^{-w_1-1} d\alpha_1 - \frac{\delta \Gamma(w_1+1)}{4 \sin^2 \pi u_1} \int_{P_1} \alpha_1^{u_1} f_1(\alpha_1)(\xi_1 - \alpha_1)^{-w_1-1} d\alpha_1, \quad (32)$$

and

$$D_{\xi_2}^{w_2} \xi_2^{u_2} (\log \xi_2)^\delta f_2(\xi_2) = \frac{e^{-\pi u_2 i}\Gamma(w_2+1)}{4\pi \sin(\pi u_2)} \int_{P_2} \alpha_2^{u_2} (\log \alpha_2)^\delta f_2(\alpha_2)(\xi_1 - \alpha_2)^{-w_2-1} d\alpha_2 - \frac{\delta \Gamma(w_2+1)}{4 \sin^2 \pi u_2} \int_{P_2} \alpha_2^{u_2} f_2(\alpha_2)(\xi_2 - \alpha_2)^{-w_2-1} d\alpha_2. \quad (33)$$

Taking the sum of (32) and (33) multiplied by e_1 and e_2, re-

Table 1. Riemann-Liouville differentiation and integration of bicomplex order

S.N.	Function	Differentiation	Integration
1	$\xi^u \log\xi$ $\xi \in X \setminus \mathcal{O}_2, \xi \neq 0$	$\frac{\Gamma_2(u+1)}{\Gamma_2(u-w+1)}\xi^{u-w}[\log\xi + \Psi(u+1) - \Psi(u-w+1)]$	$\frac{\Gamma_2(u+1)\xi^{w+u}}{\Gamma_2(u+w+1)}[\log\xi + \Psi(u+1) - \Psi(u+w+1)]$
2	$e^{a\xi}$, $a \in \mathbb{C}_2$	$D_\xi^w e^{a\xi} = \xi^{-w} e^{a\xi} \Upsilon^*(-w, a\xi) = E_\xi(-w, a)$	$D_\xi^{-w} e^{a\xi} = E_\xi(w, a)$
3	$\cos a\xi$, $a \in \mathbb{C}_2$	$D_\xi^w \cos a\xi = C_\xi(-w, a) = \frac{1}{\Gamma_2(-w)} \int_0^\xi \alpha^{(-w-1)} \cos a(\xi-\alpha) d\alpha$	$D_\xi^{-w} \cos a\xi = C_\xi(w, a)$
4	$\sin a\xi$, $a \in \mathbb{C}_2$	$D_\xi^w \sin a\xi = S_\xi(-w, a) = \frac{1}{\Gamma_2(-w)} \int_0^\xi \alpha^{(-w-1)} \sin a(\xi-\alpha) d\alpha$	$D_\xi^{-w} \sin a\xi = S_\xi(w, a)$.

spectively and applying some simple calculations, we attain

$$D_\xi^w \xi^u (\log\xi)^\delta f(\xi) = \frac{e^{-\pi u i}\Gamma_2(w+1)}{4\pi\sin(\pi u)} \int_P \alpha^u (\log\alpha)^\delta f(\alpha)(\xi-\alpha)^{-w-1} d\alpha - \frac{\delta\Gamma_2(w+1)}{4\sin^2\pi u} \int_P \alpha^u f(\alpha)(\xi-\alpha)^{-w-1} d\alpha, \quad (34)$$

for $\delta = 0$ or 1.

Before we get to the main point of this section, let's look at the analytic behaviour of a simple example we computed before, which is

$$D_\xi^w \xi^u = \frac{\Gamma_2(u+1)}{\Gamma_2(u-w+1)} \xi^{u-w}.$$

Some ordinary observations for $\xi \in X \setminus \mathcal{O}_2$ and $\xi \neq 0$ are as follows:

(i) If non-zero u is such that $\varrho_1 \neq -\frac{1}{2}(m+n)$ and $\varrho_2 \neq -\frac{i}{2}(m-n)$; $m, n \in \mathbb{N} \cup \{0\}$, then$D_\xi^w \xi^u$ is an entire function of w.

(ii) The function being differentiated has no singularity as $\xi = 0$, but its fractional derivative has a branch point singularity at $\xi = 0$ for $w \neq u$, such that $\varrho_1 - z_1 \neq -\frac{1}{2}(m+n)$ and $\varrho_2 - z_2 \neq -\frac{i}{2}(m-n)$; $m, n \in \mathbb{N} \cup \{0\}$.

(iii) If w is such that $z_1 \neq \pm\frac{1}{2}(m+n)$ and $z_2 \neq \pm\frac{i}{2}(m-n)$; $m, n \in \mathbb{N} \cup \{0\}$, then $D_\xi^w \xi^u$ has a simple pole at the non-zero u such that $\varrho_1 = -\frac{1}{2}(m+n)$ and $\varrho_2 = -\frac{i}{2}(m-n)$; $\forall\, m, n \in \mathbb{N} \cup \{0\}$.

(iv) If w is such that $z_1 = \frac{1}{2}(m+n)$ and $z_2 = \frac{i}{2}(m-n)$; $m, n \in \mathbb{N} \cup \{0\}$, then $D_\xi^w \xi^u$ is an entire function of u.

(v) If w is such that $z_1 = -\frac{1}{2}(m+n)$ and $z_2 = -\frac{i}{2}(m-n)$; $m, n \in \mathbb{N} \cup \{0\}$, then $D_\xi^w \xi^u$ has a simple pole at the non-zero u such that $\varrho_1 = -\frac{1}{2}(m+n)$ and $\varrho_2 = -\frac{i}{2}(m-n)$.

These observations are readily reflected in the following Propositions. Let us consider the two cases for (34).

Case - I: $\delta = 0$.

In this case (34) becomes

$$D_\xi^w \xi^u f(\xi) = \frac{e^{-\pi u i}\Gamma_2(w+1)}{4\pi\sin(\pi u)} \int_P \alpha^u f(\alpha)(\xi-\alpha)^{-w-1} d\alpha. \quad (35)$$

The further calculation with the use of the method of integration by part for each idempotent component described in (32) and (33) gives us

$$D_\xi^w \xi^u f(\xi) = \frac{e^{-\pi u i}\Gamma_2(w+n+1)}{4\pi\sin(\pi u)} \int_P (\xi-\alpha)^{(-w-n-1)} \left\{\int \dots \int \alpha^u f(\alpha) d\alpha \dots d\alpha\right\} d\alpha \quad (36)$$

$$= \frac{e^{-\pi u i}\Gamma_2(w+1)u(u-1)\dots(u-n+1)(-1)^n}{4\pi\sin(\pi u)} \times \int_P \alpha^{u-n} \left\{\int \dots \int (\xi-\alpha)^{-w-1} f(\alpha) d\alpha \dots d\alpha\right\} d\alpha \quad (37)$$

Since the Pochhammer contour doesn't intersect the singularities at $\alpha = 0$ and $\alpha = \xi$ in (35) and for any u and w, $\int_P \alpha^u f(\alpha)(\xi-\alpha)^{-w-1} d\alpha$ is bicomplex holomorphic $\forall\ \xi \in X \setminus \mathcal{O}_2,\ \xi \neq 0$.

Now, we are at the stage from where we can state our Propositions.

Proposition 4.1. *Let $f(\xi)$ be bicomplex holomorphic function on the simply connected open set $X \subset \mathbb{C}_2$ which contains the point $\xi = 0$. Also, let $f(0) \neq 0$.*

(i) If $\xi \in X \setminus \mathcal{O}_2,\ \xi \neq 0,\ \varrho_1 \neq -\frac{1}{2}(m+n)$, and $\varrho_2 \neq -\frac{i}{2}(m-n)$; $m, n \in \mathbb{N} \cup \{0\}$ such that $u \neq 0$, then $D_\xi^w \xi^u f(\xi)$ is an entire function of w (where ξ and u are fixed).

(ii) If $\xi \in X \setminus \mathcal{O}_2,\ \xi \neq 0,\ z_1 = \frac{1}{2}(m+n)$, and $z_2 = \frac{i}{2}(m-n)$; $m, n \in \mathbb{N} \cup \{0\}$, then $D_\xi^w \xi^u f(\xi)$ is an entire function of u (where ξ and w are fixed). When $\xi \in X \setminus \mathcal{O}_2$, $\xi \neq 0,\ z_1 \neq \frac{1}{2}(m+n)$, and $z_2 \neq \frac{i}{2}(m-n)$; $m, n \in \mathbb{N} \cup \{0\}$, then $D_\xi^w \xi^u f(\xi)$ is a meromorphic function of u whose only singularities are simple poles at non-zero u such that $\varrho_1 = -\frac{1}{2}(m+n)$ and $\varrho_2 = -\frac{i}{2}(m-n)$; $m, n \in \mathbb{N} \cup \{0\}$ or a subset thereof.

(iii) If $\varrho_1 \neq -\frac{1}{2}(m+n)$ and $\varrho_2 \neq -\frac{i}{2}(m-n)$; $m, n \in \mathbb{N} \cup \{0\}$ such that $u \neq 0$, then $D_\xi^w \xi^u f(\xi) = \xi^{u-w} g(w, u; \xi)$, where $g(w, u; \xi)$ is a bicomplex holomorphic function of ξ on X.

Proof. From (35), we see that the only singularities of $D_\xi^w \xi^u f(\xi)$ can be identified from the coefficient of the integral. Thus for $\xi \in X \setminus \mathcal{O}_2,\ \xi \neq 0$, the only possible singularities are at non-zero w such that $z_1 = -\frac{1}{2}(m+n)$ and $z_2 = -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$ and u such that $\varrho_1 = \pm\frac{1}{2}(m+n)$ and $\varrho_2 = \pm\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$. Out of these set of singularities, some of them can be removed. From (36), it is clear that the singularities at non-zero w such that $z_1 = -\frac{1}{2}(m+n)$ and $z_2 = -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$ can be removed and because n is arbitrary, we infer that w has no analyticity constraints. From (37), we can exclude the singularities at the point u such that $\varrho_1 = \frac{1}{2}(m+n)$ and $\varrho_2 = \frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$. From (35), we can state that for fixed w and fixed $\xi \in X \setminus \mathcal{O}_2$, $\xi \neq 0$, $D^w_\xi \xi^u f(\xi)$ has simple poles at non-zero u such that $\varrho_1 = -\frac{1}{2}(m+n)$ and $\varrho_2 = -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$. By expanding $\xi^u f(\xi)$ in a power series in ξ and then operating term-wise with D^w_ξ, we obtain that $D^w_\xi \xi^u f(\xi) = \xi^{u-w} g(w,u;\xi)$, where $g(w,u;\xi)$ is a bicomplex holomorphic function of ξ on X. □

Case- II: $\delta = 1$.

Proposition 4.2. *Let $f(\xi)$ be a bicomplex holomorphic function of ξ on the simply connected open set X, containing the point $\xi = 0$. Also, let $f(0) \neq 0$.*

(i) If $\xi \in X \setminus \mathcal{O}_2$, $\xi \neq 0$, $\varrho_1 \neq -\frac{1}{2}(m+n)$, and $\varrho_2 \neq -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$ such that $u \neq 0$, then $D^w_\xi \xi^u \log(\xi) f(\xi)$ is an entire function of w (where ξ and u are fixed).

(ii) If $\xi \in X \setminus \mathcal{O}_2$, $\xi \neq 0$, $z_1 = \frac{1}{2}(m+n)$, and $z_2 = \frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$, then $D^w_\xi \xi^u \log(\xi) f(\xi)$ is an entire function of u (where ξ and w are fixed). When $z_1 \neq \frac{1}{2}(m+n)$, $z_2 \neq \frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$, $\xi \in X \setminus \mathcal{O}_2$, and $\xi \neq 0$, then $D^w_\xi \xi^u \log(\xi) f(\xi)$ is a meromorphic function of u, whose only singularities are simple poles at non-zero u such that $\varrho_1 = -\frac{1}{2}(m+n)$ and $\varrho_2 = -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$.

(iii) If $\varrho_1 \neq -\frac{1}{2}(m+n)$ and $\varrho_2 \neq -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$ such that $u \neq 0$, then $D^w_\xi \xi^u \log(\xi) f(\xi) = \xi^{u-w}\{\log(\xi)A(w,u;\xi) + B(w,u;\xi)\}$, where $A(w,u;\xi)$ and $B(w,u;\xi)$ are bicomplex holomorphic functions of ξ on X.

Proof. This Proposition can be proved similar to Proposition 4.1. □

Remark 4.3. *Similar arguments to Propositions 4.1 and 4.2 can be made for integration also.*

4.2 The Law of Exponent

In below Theorems, we state the law of exponent for the differential operators applied on functions of the type $\xi^u f(\xi)$ and $\xi^u \log(\xi) f(\xi)$.

Theorem 4.4. *Let $f(\xi)$ be bicomplex holomorphic on the simply connected open set X having the point $\xi = 0$. Also, we assume $f(0) \notin \mathcal{O}_2$, $\varrho_1 \neq -\frac{1}{2}(m+n)$, and $\varrho_2 \neq -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$ such that $u \neq 0$.*

(i) If $w \neq u$, $\varrho_1 - z_1 \neq -\frac{1}{2}(m+n)$, and $\varrho_2 - z_2 \neq -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$, then for $\xi \in X \setminus \mathcal{O}_2$ and $\xi \neq 0$, we have

$$D^{w'}_\xi D^w_\xi \xi^u f(\xi) = D^{w'+w}_\xi \xi^u f(\xi),$$

where $w' = z'_1 + jz'_2$.

(ii) If $\varrho_1 - z_1 = -\frac{1}{2}(m+n)$ and $\varrho_2 - z_2 = -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$, then for $\xi \in X \setminus \mathcal{O}_2$ and $\xi \neq 0$, we have

$$\begin{aligned} &D^{w'}_\xi D^w_\xi \xi^u f(\xi) \\ &= D^{w'+w}_\xi \xi^u f(\xi) \\ &\quad - \sum_{n=0}^{w-u-1} \frac{f^{(n)}(0)\Gamma_2(u+n+1)\xi^{u-w-w'+n}}{n!\Gamma_2(u-w-w'+n+1)}. \end{aligned}$$

Proof. Expanding $F(\xi) = \xi^u f(\xi)$ in Maclaurin series and operating term-wise D^w_ξ and then after $D^{w'}_\xi$, we can reach the proof. □

Theorem 4.5. *Let $f(\xi)$ be bicomplex holomorphic on the simply connected open set X which contains the point $\xi = 0$. Also, assume $f(0) \notin \mathcal{O}_2$, $\varrho_1 \neq -\frac{1}{2}(m+n)$, and $\varrho_2 \neq -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$ such that $u \neq 0$.*

(i) If $w \neq u$, $\varrho_1 - z_1 \neq -\frac{1}{2}(m+n)$, and $\varrho_2 - z_2 \neq -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$, then

$$D^{w'}_\xi D^w_\xi \xi^u \log(\xi) f(\xi) = D^{w'+w}_\xi \xi^u \log(\xi) f(\xi). \quad (38)$$

(ii) However, if $\varrho_1 - z_1 = -\frac{1}{2}(m+n)$ and $\varrho_2 - z_2 = -\frac{i}{2}(m-n)$; $m,n \in \mathbb{N}\cup\{0\}$, then $D^{w'}_\xi D^w_\xi \xi^u \log(\xi) f(\xi)$ is undefined unless $z'_1 = \frac{1}{2}(m+n)$ and $z'_2 = \frac{i}{2}(m-n)$, in which case (38) remains true.

Proof. Expanding $F(\xi) = \xi^u(\log\xi) f(\xi)$ in Maclaurin series and operating term-wise D^w_ξ and then after $D^{w'}_\xi$, we can reach to the proof. □

4.3 Generalizations of the Leibniz Rule

The complex version of generalized Leibniz rule can be seen in [18, p. 263]. The series representation obtained in [37, eq. 84] has a difficulty under which we can not be able to interchange the both functions appearing at right hand side but interchange is obvious on left side. To overcome this difficulty we generalize the Leibniz formula. The necessary generalization is

$$\begin{aligned} &D^w_\xi f(\xi) g(\xi) \\ &= \sum_{n=-\infty}^{\infty} \binom{w}{n+\kappa} D^{w-n-\kappa}_\xi f(\xi) D^{n+\kappa}_\xi g(\xi) \end{aligned} \quad (39)$$

In this series κ is an arbitrary bicomplex number. This generalization can be obtained by using simple idempotent method. Our interest centers on finding the region of convergence of series (39) whose complex counter part is given in [16, 17]. Let ${}_1\xi,\ {}_2\xi,\ {}_3\xi, \cdots$ be singular points of the functions $f(\xi)$ and

$g(\xi)$ in the bicomplex space (where the set of singular points can be uncountably infinite). For each singularity ${}_i\xi$, it take its idempotent components ${}_i\xi_1$ and ${}_i\xi_2$. Now, from each ${}_i\xi_1$ and ${}_i\xi_2$ draw the line segments to the origin in adjacent complex spaces and call these ${}_i\mathrm{L}_1$ and ${}_i\mathrm{L}_2$, respectively. Next, consider the half-planes ${}_i\mathrm{HP}_1$ and ${}_i\mathrm{HP}_2$, whose boundary is the orthogonal bisector of ${}_i\mathrm{L}_1$ and ${}_i\mathrm{L}_2$, respectively and which contains the origin. The region of convergence of (39) is formed by the intersection of all such open half-planes shown in Fig. 4. We

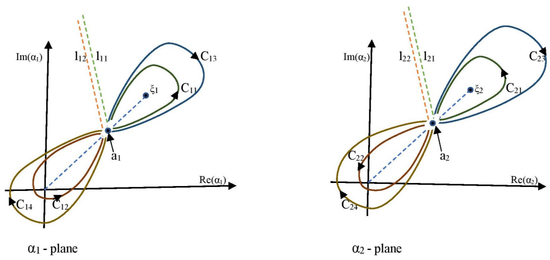

Figure 4. The region of convergence for Leibniz rule

may generalise (39) further by inserting the parameter a, where $0 < a \leq 1$, as

$$D_\xi^w f(\xi)g(\xi) = \sum_{n=-\infty}^{\infty} \binom{w}{an+\kappa} D_\xi^{w-an-\kappa} f(\xi) D_\xi^{an+\kappa} g(\xi) a. \quad (40)$$

Let $an = \mu$. Then letting a approaching zero will transform (40) into a integral formula and we get the integral version of the Leibniz rule as follows:

$$D_\xi^w f(\xi)g(\xi) = \int_{-\infty}^{\infty} \binom{w}{\mu+\kappa} D_\xi^{w-\mu-\kappa} f(\xi) D_\xi^{\mu+\kappa} g(\xi) d\mu, \quad \mu \in \mathbb{R}.$$

Remark 4.6. *In Fig.4 position of points in both planes should not characterize as an ideal representation.*

4.4 Generalization of the Chain Rule

Here, we discuss the generalized chain rule in bicomplex space. For the complex variable functions the generalized chain rule [18, p. 265] is defined as

$$D_{\xi_1}^{w_1} f_1(g_1(\xi_1)) = \sum_{n=0}^{\infty} \frac{{}_1U_n(\xi_1) D_{g_1(\xi_1)}^n f_1(g_1(\xi_1))}{n!}, \quad (41)$$

where

$${}_1U_n(\xi_1) = \sum_{r=0}^{n} \binom{n}{r} \{-g_1(\xi_1)\}^r D_{\xi_1}^{w_1}[g_1(\xi_1)^{n-r}].$$

In (41,) we require that $g_1^{-1}(0) = 0$. Similarly, changing subscript by 2, we get

$$D_{\xi_2}^{w_2} f_2(g_2(\xi_2)) = \sum_{n=0}^{\infty} \frac{{}_2U_n(\xi_2) D_{g_2(\xi_2)}^n f_2(g_2(\xi_2))}{n!}, \quad (42)$$

where

$${}_2U_n(\xi_2) = \sum_{r=0}^{n} \binom{n}{r} \{-g_2(\xi_2)\}^r D_{\xi_2}^{w_2}[g_2(\xi_2)^{n-r}]$$

and $g_2^{-1}(0) = 0$. Since

$$\begin{aligned} &\left\{D_{\xi_1}^{w_1} f_1(g_1(\xi_1))\right\} e_1 + \left\{D_{\xi_2}^{w_2} f_2(g_2(\xi_2))\right\} e_2 \\ &= \left\{e_1 D_{\xi_1}^{w_1} + e_2 D_{\xi_2}^{w_2}\right\} \{f_1(g_1(\xi_1))e_1 + f_2(g_2(\xi_2))e_2\} \\ &= D_\xi^w f(g(\xi)), \end{aligned}$$

hence, taking (41) and (42) as idempotent components, we have

$$\begin{aligned} &D_\xi^w f(g(\xi)) \\ &= \left\{D_{\xi_1}^{w_1} f_1(g_1(\xi_1))\right\} e_1 + \left\{D_{\xi_2}^{w_2} f_2(g_2(\xi_2))\right\} e_2 \\ &= \left\{\sum_{n=0}^{\infty} \frac{{}_1U_n(\xi_1) D_{g_1(\xi_1)}^n f_1(g_1(\xi_1))}{n!}\right\} e_1 \\ &\quad + \left\{\sum_{n=0}^{\infty} \frac{{}_2U_n(\xi_2) D_{g_2(\xi_2)}^n f_2(g_2(\xi_2))}{n!}\right\} e_2 \\ &= \sum_{n=0}^{\infty} \frac{U_n(\xi) D_{g(\xi)}^n f(g(\xi))}{n!}, \end{aligned}$$

where

$$U_n(\xi) = \sum_{r=0}^{n} \binom{n}{r} \{-g(\xi)\}^r D_\xi^w [g(\xi)^{n-r}]$$

and $g^{-1}(0) = 0$.

Theorem 4.7. *Let $f(\xi)$ and $g(\xi)$ be bicomplex holomorphic functions on $X \subset \mathbb{C}_2$ and $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with $Re(z_1) < |Im(z_2)|$. Then*

$$D_\xi^w (f(\xi) + g(\xi)) = D_\xi^w f(\xi) + D_\xi^w g(\xi).$$

Proof. We proceed

$$\begin{aligned} &D_\xi^w (f(\xi) + g(\xi)) \\ &= \left\{e_1 D_{\xi_1}^{w_1} + e_2 D_{\xi_2}^{w_2}\right\} \\ &\quad \{(f_1(\xi_1) + g_1(\xi_1))e_1 + (f_2(\xi_2) + g_2(\xi_2))e_2\} \\ &= \{D_{\xi_1}^{w_1}(f_1(\xi_1) + g_1(\xi_1))\}e_1 \\ &\quad + \{D_{\xi_2}^{w_2}(f_2(\xi_2) + g_2(\xi_2))\}e_2 \\ &= D_{\xi_1}^{w_1} f_1(\xi_1)e_1 + D_{\xi_1}^{w_1} g_1(\xi_1)e_1 \\ &\quad + D_{\xi_2}^{w_2} f_2(\xi_2)e_2 + D_{\xi_2}^{w_2} g_2(\xi_2)e_2 \\ &= \left\{D_{\xi_1}^{w_1} f_1(\xi_1)e_1 + D_{\xi_2}^{w_2} f_2(\xi_2)e_2\right\} \\ &\quad + \left\{D_{\xi_1}^{w_1} g_1(\xi_1)e_1 + D_{\xi_2}^{w_2} g_2(\xi_2)e_2\right\} \\ &= \left\{e_1 D_{\xi_1}^{w_1} + e_2 D_{\xi_2}^{w_2}\right\} \{f_1(\xi_1)e_1 + f_2(\xi_2)e_2\} \\ &\quad + \left\{e_1 D_{\xi_1}^{w_1} + e_2 D_{\xi_2}^{w_2}\right\} \{g_1(\xi_1)e_1 + g_2(\xi_2)e_2\} \\ &= D_\xi^w f(\xi) + D_\xi^w g(\xi). \end{aligned}$$

□

A similar argument can be proved for integration also.

5 Applications

The Maxwell's equations describe the behavior of electric field **E** and magnetic field **H** [43]. According to various mediums Maxwell's equations attain various forms. By the time, researchers defined many generalizations of these equations by using different fractional operators. Some of them can be found in [19, 24].

(a) If we restrict $w \in \mathbb{R}^+$, then fractional Maxwell's equations can be identified as (in vacuum)

$$\triangledown^w \times \mathbf{E} = -\mu_0 \frac{\partial^w \mathbf{H}}{\partial t^w} \tag{43}$$

$$\triangledown^w \times \mathbf{H} = \epsilon_0 \frac{\partial^w \mathbf{E}}{\partial t^w} \tag{44}$$

$$\triangledown^w \cdot \mathbf{E} = 0 \tag{45}$$

$$\triangledown^w \cdot \mathbf{H} = 0, \tag{46}$$

where μ_0 is permeability and ϵ_0 is permittivity of medium. Let us consider a new bicomplex vector field [1]

$$\mathbf{F} = \sqrt{\epsilon_0}\mathbf{E} + j\sqrt{\mu_0}\mathbf{H}.$$

We have

$$\begin{aligned}
\triangledown^w \times \mathbf{F} &= \triangledown^w \times \{\sqrt{\epsilon_0}\mathbf{E} + j\sqrt{\mu_0}\mathbf{H}\} \\
&= \sqrt{\epsilon_0}\, \triangledown^w \times \mathbf{E} + j\sqrt{\mu_0}\, \triangledown^w \times \mathbf{H} \\
&= -\mu_0\sqrt{\epsilon_0}\frac{\partial^w \mathbf{H}}{\partial t^w} + j\epsilon_0\sqrt{\mu_0}\frac{\partial^w \mathbf{E}}{\partial t^w} \\
&= j\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon_0}\frac{\partial^w}{\partial t^w}\{\sqrt{\epsilon_0}\mathbf{E} + j\sqrt{\mu_0}\mathbf{H}\} \\
&= j\sqrt{\mu_0\epsilon_0}\frac{\partial^w \mathbf{F}}{\partial t^w} \\
&= j\frac{1}{c}\frac{\partial^w \mathbf{F}}{\partial t^w}, \quad \text{where } c = \frac{1}{\sqrt{\mu_0\epsilon_0}}.
\end{aligned}$$

Proceeding in the similar manner, we reach

$$\triangledown^w \cdot \mathbf{F} = 0.$$

Hence, four fractional Maxwell's equations can be written in two equations as

$$\triangledown^w \times \mathbf{F} = j\frac{1}{c}\frac{\partial^w \mathbf{F}}{\partial t^w} \tag{47}$$

$$\triangledown^w \cdot \mathbf{F} = 0. \tag{48}$$

Hence, we have seen that, with the assistance of a bicomplex vector field which combines both the electric and magnetic fields, the number of unknown quantities is practically reduced by half or in other words the advantage of using the tool of bicomplexification of vector field is that we are able to represent four Maxwell's equations (43)-(46) into two equations (47) and (48) only.

(b) Now, we apply the mathematical concept of bicomplex-fractionalization to Maxwell's equations in a source-free domain. For this, initially we consider Maxwell's equations in a form involving the wavenumber k and rather than the medium permittivity and permeability, the medium intrinsic impedance eta, i.e.

$$\triangledown^w \times \mathbf{E} = -ik\eta\mathbf{H}$$

$$\triangledown^w \times \mathbf{H} = i\frac{k}{\eta}\mathbf{E}$$

for a $e^{i\mu t}$ time convention.

Let us consider the bicomplex vector field as

$$\mathbf{F} = \frac{1}{\sqrt{\eta}}\mathbf{E} + j\sqrt{\eta}\mathbf{H} \tag{49}$$

where each directional component of **F** is a scalar bicomplex function that combines the relevant field directional components. On applying fractional curl of order w, (49) becomes

$$\begin{aligned}
\triangledown^w \times \mathbf{F} &= \triangledown^w \times \left\{\frac{1}{\sqrt{\eta}}\mathbf{E} + j\sqrt{\eta}\mathbf{H}\right\} \\
&= \frac{1}{\sqrt{\eta}}\{\triangledown^w \times \mathbf{E}\} + j\sqrt{\eta}\{\triangledown^w \times \mathbf{H}\} \\
&= \frac{1}{\sqrt{\eta}}\{-ik\eta\mathbf{H}\} + j\sqrt{\eta}\left\{i\frac{k}{\eta}\mathbf{E}\right\} \\
&= ijk\left\{j\sqrt{\eta}\mathbf{H} + \frac{1}{\sqrt{\eta}}\mathbf{E}\right\} \\
&= ijk\left\{\frac{1}{\sqrt{\eta}}\mathbf{E} + j\sqrt{\eta}\mathbf{H}\right\} \\
&= ijk\mathbf{F}.
\end{aligned}$$

Hence, again we are able to reduce the number of equations by half.

A lot of work on Maxwell's equations have been done so far in many Clifford algebras [44]. In our work, in source-free medium, Maxwell's equations are reduced to a single equation in the same way that other forms of Clifford algebras are, which makes the study easier to extract electric and magnetic fields from a bi-vector filed. We discussed here Maxwell's equations in fractional sense in which we used Riemann-Liouville fractional differential operator in place of ordinary operators. Advantages of this technique can be made as follows: (i) The fractional differential operators can be used to build a fractional non-local electrodynamics with power law non-locality. [19]. (ii) Non-local properties in classical dynamics can be described by such fractional representations. (iii) Power-like tails are an essential characteristic of fractional equation solutions for non-integer derivatives with regard to coordinates which will be discussed in our next work and it will be shown that the Helmholtz equation is no longer necessary in the development of final solution of these Maxwell's equations. The such bicomplexification of fractional Maxwell's equations enables us to extraction of several closed form solutions which are not easily derivable via standred analytical techniques.

6 Conclusions

In this article, a generalization of Riemann-Liouville fractional operators is obtained which can be applied on the functions of bicomplex variable and discussed some properties with region of convergence. Some examples are calculated to illustrate these operators. As applications, we constructed the system of Maxwell's equations in various mediums with the help of these operators and reduced the number of Maxwell's equations by half. This reduction of equations through bicomplexification of vector field gives an easy method to solve these Maxwell's equations.

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REFERENCES

[1] Agarwal R., Goswami M.P., Agarwal R.P., Venkataratnam K., Baleanu D., "Solution of Maxwell's wave equation in bicomplex space", Romanian Journal of Physics, vol. 62, no. 115, pp. 1-11, 2017. https://rjp.nipne.ro/2017_62_5-6/RomJPhys.62.115.pdf

[2] Anastassiu H.T., Atlamazoglou P.E., Kaklamani D.I., "Application of bicomplex (quaternion) algebra of fundamental electromagnetics: a lower order alternative to the Helmholtz equation", Transaction on Antennas and Propagation, vol. 51, no. 8, pp. 2130-2136, 2003. DOI: 10.1109/TAP.2003.810231

[3] Catarino P., "Bicomplex k-Pell quarternions", Springer-Verlag GmbH, vol. 19, pp. 65-76, 2019. https://doi.org/10.1007/s40315-018-0251-5

[4] Dixit D., "Bicomplex version of Cayley Hamilton Theorem", International Journal of Research and Scientific Innovation , vol. 3 no. 3, pp. 66-70, 2016. https://www.rsisinternational.org/IJRSI/Issue25 /66-70.pdf

[5] Ghosh C., "Bicomplex Mobius transformation", arXiv:1706.07699v1 [math.CV] 22 Jun 2017, pp. 1-9, 2017. https://arxiv.org/pdf/1706.07699

[6] Goyal S., Mathur T., Agarwal R., "Bicomplex gamma and beta functions", J. Rajasthan Acad. Phy. Sci., vol. 5, no. 1, pp. 131-142, 2006. https://www.researchgate.net/publication/26704 8967_Bicomplex_gamma_and_beta_functions

[7] Kumar J., "Bicomplex Dirichlet series", Journal of Emerging Technologies and Innovative Research, vol. 5, pp. 1-23, 2018. https://www.academia.edu/37687470/Bicomplex _Dirichlet_Series

[8] Banerjee A., "On the quantum mechanics of bicomplex Hamiltonian system", Annals of Physics, vol. 377, pp. 493-505, 2017. https://doi.org/10.1016/j.aop.2017.01.006

[9] Aydin T., "Bicomplex Fibonacci quaternions", Chaos, Solitons & Fractals", vol. 106, pp. 147-153, 2018. https://doi.org/10.1016/j.chaos.2017.11.026

[10] Agarwal R., Goswami M.P., Agarwal R.P., "Mellin transform in bicomplex space and its applications", Stud. Univ. Babes-Bolyai Math., vol. 62, no. 2, pp. 217-232, 2017. DOI: 10.24193/subbmath.2017.2.08

[11] Goswami M.P., Jha N., "Triple Laplace transform in bicomplex space with application", Mathematics and Statistics, vol. 8, no. 4, pp. 443-450, 2020. DOI: 10.13189/ms.2020.080411

[12] Cerejeiras P., Fu Y., Gomes N., "Bicomplex signals with sparsity constraints", Mathematical Methods in the Applied Sciences, vol. 2018, pp. 1-19, 2018. DOI: 10.1002/mma.5059. https://doi.org/10.1002/mma.5059

[13] Hashimoto K., Kato S., "Bicomplex extensions of zero mean curvature surfaces in $\mathbb{R}^{2,1}$ and $\mathbb{R}^{2,1}$," Journal of Geometry and Physics, vol. 138, pp. 223-240, 2019. https://doi.org/10.1016/j.geomphys.2018.12.017

[14] Elizarrarás M.E.L., Regalado C.O.P., Shapiro M., "Singularities of bicomplex holomorphic functions", Mathematical Methods in the Applied Sciences, vol. 2021, pp. 1-16, 2021. https://doi.org/10.1002/mma.7522.

[15] Kobayashi M., "Bicomplex-valued twin-hyperbolic Hopfield neural networks", Neurocomputing, vol. 434, pp. 203-210, 2021. https://doi.org/10.1016/j.neucom.2020.12.109

[16] Osler T.J., "A correction to Leibniz rule for fractional derivatives", SIAM J. Math. Anal., vol. 4, no. 3, pp. 456-459, 1973. https://doi.org/10.1137/0504040

[17] Osler T.J., "Leibniz rule for fractional derivatives-generalized and an application to infinite series", SIAM J. Math. Anal., vol. 18, no. 3, pp. 658-674, 1970. https://www.researchgate.net/publication/26872 4526_Generalized_Leibniz_rule_for_an_extended_fr actional_derivative_operator_with_applications_to _special_functions

[18] Lavoie J.L., Osler T., Tremblay R., "Fractional derivatives and special functions", SIAM J. Math. Anal., vol. 18, no. 2, pp. 240-268, 1976. https://www.researchgate.net/publication/24238 6925_Fractional_Derivatives_and_Special_Function s

[19] Tarasov V.E., "Fractional vector calculus and fractional Maxwell's equations", Annals of Physics, vol. 323, no. 11, pp. 2756-2778, 2008. https://doi.org/10.1016/j.aop.2008.04.005

[20] Jumarie G., "Cauchy's integral formula via the modified Riemann-Liouville derivative for analytic functions of fractional order", Applied Mathematics Letters, vol. 23, no. 12, pp. 1444-1450, 2010. https://doi.org/10.1016/j.aml.2010.08.001

[21] Fernandez A., "A complex analysis approach to Atangana-Baleanu fractional calculus", Mathematical Methods in the Applied Sciences, vol. 2019, pp. 1-18, 2019. DOI:10.1002/mma.5754.

[22] Rezapour M., Sijuwade A., Asaki T., "A new sigmoidal fractional derivative for regularization", AIMS Mathematics, vol. 5, no. 4, pp. 3284-3297, 2020. doi: 10.3934/math.2020211.

[23] Bhangale N., Kachhia K.B., "Fractional electromagnetic waves in plasma and dielectric media with Caputo generalized fractional derivative", Revista Mexicana de Fisica, vol. 66, no. 6, pp. 848-855, 2020. https://www.scielo.org.mx/pdf/rmf/v66n6/0035-001X-rmf-66-06-848.pdf

[24] Salam F., "Fractional approach of Maxwell's equations in the curved space", Journal of Taibah University for Science, vol. 7, no. 3, pp. 173-179, 2013. https://doi.org/10.1016/j.jtusci.2013.04.005

[25] Sejdic E., Djurovic I., Stankovic L., "Fractional Fourier transform as a signal processing tool: An overview of recent developments", Signal Processing, vol. 91, no. 6, pp. 1351-1369, 2011. https://doi.org/10.1016/j.sigpro.2010.10.008

[26] Lan K., "Linear first order Riemann-Liouville fractional differential and perturbed Abel's integral equations", Journal of Differential Equations, vol. 306, pp. 28-59, 2022. https://doi.org/10.1016/j.jde.2021.10.025

[27] Torres D.F.M., "Cauchy's formula on nonempty closed sets and a new notion of Riemann–Liouville fractional integral on time scales", Applied Mathematics Letters, vol. 121, pp. 1-6, 2021. https://doi.org/10.1016/j.aml.2021.107407

[28] Gu Z., "Spectral collocation method for nonlinear Riemann-Liouville fractional terminal value problems", Journal of Computational and Applied Mathematics, vol. 398, pp. 1-12, 2021. https://doi.org/10.1016/j.cam.2021.113640

[29] Fernandez A., Ustaoğlu C., "On some analytic properties of tempered fractional calculus", Journal of Computational and Applied Mathematics, vol. 366, pp. 1-14, 2020. https://doi.org/10.1016/j.cam.2019.112400

[30] Garrappaa A., Giusticd A., Mainardie F., "Variable-order fractional calculus: A change of perspective", Communications in Nonlinear Science and Numerical Simulation, vol. 102, pp. 1-16, 2021. https://doi.org/10.1016/j.cnsns.2021.105904

[31] Atangana A., Baleanu D., "New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model", Therm. Sci., vol. 20, no. 2, pp. 763-769, 2016. DOI: 10.2298/TSCI160111018A

[32] Haubold H.J., Mathai A.M., Saxena R.K., Mittag-Leffler functions and their applications, Journal of Applied Mathematics vol. 2011, pp. 1-51, 2011. https://downloads.hindawi.com/journals/jam/2011/298628.pdf

[33] Haubold H.J., Mathai A.M., Saxena R. "Mittag–Leffler Functions and Fractional Calculus in Special Functions for Applied Scientists", Springer, 2018. DOI: 10.1155/2011/298628

[34] Luna M., Shapiro M., Struppa D., Vajiac A., "Bicomplex Holomorphic Functions- The Algebra, Geometry and Analysis of Bicomplex Numbers", Frontiers in Mathematics, 2015. DOI 10.1007/978-3-319-4.

[35] Luna M., Shapiro M., Struppa D., Vajiac A., "Bicomplex Numbers and their elementary functions", CUBO-A Mathematical Journal, vol. 14, pp. 61-80, 2012. https://www.researchgate.net/publication/257428323_Bicomplex_Numbers_and_Their_Elementary_Functions

[36] Price G.B., "An Introduction to Multicomplex-Spaces and Functions (Pure and Applied Mathematics)", Taylor and Francis, 1991. https://doi.org/10.1201/9781315137278

[37] Goswami M.P., Kumar R., Riemann-Liouville fractional operators of bicomplex order and its properties", Math. Meth. App. Sci., vol. 45, no. 10, pp. 5699-5720, 2022. https://doi.org/10.1002/mma.8135 doi:10.1002/mma.8135

[38] Coloma A., Teodoro A.D., Tocachi D.O., Ponce F., "Fractional elementary bicomplex functions in the Riemann-Liouville sense", Advances in Applied Clifford Algebras, vol. 31, no. 63, pp. 1-29, 2021. https://link.springer.com/content/pdf/10.1007/s00006-021-01165-0.pdf?pdf=button

[39] Miller K.S., Ross B., "An Introduction to the Fractional Calculus and Fractional Differential Equations", John Wiley and Sons, 1993. http://oldpm.umd.edu/an_introduction_to_the_fractional_calculus_and_fractional_differential_equations.pdf

[40] Samko S.G., Kilbas A.A., Marichev O.I., "Fractional Integrals and Derivatives-Theory and Applications", Gordon and Breach Science Publishers, 1993. http://www.gbv.de/dms/hebis-darmstadt/toc/32759916.pdf

[41] Podlubny I., "Geometric and physical interpretation of fractional integration and fractional differentiation', Fractional Calculus and Applied Sciences, vol. 5, no. 4, pp. 367-386, 2002. https://arxiv.org/abs/math/0110241

[42] Lavoie J.L., Osler T.J., Tremblay R., "Fundamental properties of fractional derivatives via Puchhammer integral", Lecture Nots in Mathematics on Fractional Calculus and Its Applications, No. 457, Springer-Verlag, Verlin-Heidelberg-New-York, pp. 323-356, 1975. https://doi.org/10.1007/BFb0067118

[43] Lonngren K.E., Savov S.V., "Fundamentals of Electromagnetics with Matlab", SciTech Publishing, 2005. https://library.uoh.edu.iq/admin/ebooks/60654-fundamentals-of-electromagnetics-with-matlab.pdf

[44] Mohazzabi P., Wielenberg1 N.J., Alexander G.C., "A New Formulation of Maxwell's Equations in Clifford Algebra", Journal of Applied Mathematics and Physics, vol. 5, no. 8, pp. 1575-1588, 2017. DOI: 10.4236/jamp.2017.58130

Appendix

The Incomplete Gamma Function

Many of the special functions encountered in the study of fractional calculus are inextricably linked to the classical incomplete gamma function. The incomplete gamma function for complex arguments is defined by

$$\gamma^*(z,t) = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(z+k+1)}. \tag{50}$$

It is an entire function of both z and t. Let us consider (50) as follows:

$$\gamma^*(w_1,\xi_1) = e^{-\xi_1}\sum_{k=0}^{\infty}\frac{\xi_1^k}{\Gamma(w_1+k+1)} \tag{51}$$

and

$$\gamma^*(w_2, \xi_2) = e^{-\xi_2} \sum_{k=0}^{\infty} \frac{\xi_2^k}{\Gamma(w_2 + k + 1)}. \tag{52}$$

Now, combining (51) and (52) with coefficients e_1 and e_2, respectively, we have

$$\begin{aligned} &\{\gamma^*(w_1, \xi_1)\} e_1 + \{\gamma^*(w_2, \xi_2)\} e_2 \\ &= \left\{ e^{-\xi_1} \sum_{k=0}^{\infty} \frac{\xi_1^k}{\Gamma(w_1 + k + 1)} \right\} e_1 \\ &\quad + \left\{ e^{-\xi_2} \sum_{k=0}^{\infty} \frac{\xi_2^k}{\Gamma(w_2 + k + 1)} \right\} e_2 \\ &= e^{-(\xi_1 e_1 + \xi_2 e_2)} \sum_{k=0}^{\infty} \frac{(\xi_1 e_1 + \xi_2 e_2)^k}{\Gamma(w_1 e_1 + w_2 e_2 + k + 1)} \\ &= e^{-\xi} \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(w + k + 1)} \\ &= \Upsilon^*(w, \xi). \end{aligned} \tag{53}$$

We call $\Upsilon^*(w, \xi)$ bicomplex incomplete gamma function. Hence, we can conclude that

$$\Upsilon^*(w, \xi) = \{\gamma^*(w_1, \xi_1)\} e_1 + \{\gamma^*(w_2, \xi_2)\} e_2. \tag{54}$$

Also, for $a \in \mathbb{C}_2$

$$\begin{aligned} \Upsilon^*(w, a\xi) &= \{\gamma^*(w_1, a\xi_1)\} e_1 + \{\gamma^*(w_2, a\xi_2)\} e_2 \\ &= e^{-a\xi} \sum_{k=0}^{\infty} \frac{(a\xi)^k}{\Gamma(w + k + 1)} \end{aligned} \tag{55}$$

In a similar way, we can obtain E_ξ, C_ξ, and S_ξ for bicomplex arguments as follows:

$$E_\xi(w, a) = \xi^w \sum_{k=0}^{\infty} \frac{(a\xi^k)}{\Gamma_2(w + k + 1)}, \tag{56}$$

$$S_\xi(w, a) = t^w \sum_{k_{odd}}^{\infty} \frac{(-1)^{(k-1)/2} (a\xi)^k}{\Gamma_2(w + k + 1)}, \tag{57}$$

$$C_\xi(w, a) = \xi^w \sum_{k_{even}}^{\infty} \frac{(-1)^{k/2} (a\xi)^k}{\Gamma_2(w + k + 1)}. \tag{58}$$