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Jacobson Graph of Matrix Rings

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Abstract Some researchers have studied some properties of the Jacobson graph of commutative rings. In this study, we expand these results by examining the Jacobson graph of a non-commutative ring with identity, where we focus on the case of matrix rings. Initially, we update the definition of the Jacobson graph of non-commutative rings as a directed graph. Then we find that the Jacobson graph of the matrix rings case is undirected. We can classify matrices based on rank by viewing the matrix as a linear transformation. The main result is that the order of the matrix rank values will be proportional to the order of the matrix degrees as vertices of the graph. So that one can identify the maximum and minimum degrees in this graph. Sequentially, we describe the graph properties starting from the Jacobson graph of matrices over fields, then expanding to the Jacobson graph of matrices over local commutative rings and the Jacobson graph of matrices over non-local rings. In the end, we also give different results on the Jacobson graph of triangular matrices. The main contribution of this paper is to review the relationship between the aspects of linear algebra in the form of matrix rings and combinatorics in the form of diameter and vertex degree on this graph.

Keywords Jacobson Graph, Matrix Ring, Diameter, Degree

1 Introduction

The Jacobson graph was first defined for commutative rings in 2013 by Azimi et al. [1]. Several subsequent papers developed various properties of the Jacobson graph of commutative rings. In [2], Azimi studied the Jacobson graph's properties of cycles and paths. In [3], Akbari et al. traced the clique number of Jacobson graphs. The classification of toroidal and projective Jacobson graphs has been widely discussed (see [4],[5], [6],[7]). Various other properties are provided in [8].

Recently, several researchers have tried to generalize the concept of the Jacobson graph. In 2018, Ghayour formed a generalization of the Jacobson graph called the n-array Jacobson graph (see [9]). Subsequently, Humaira et al. found another generalization in 2020, namely the Jacobson matrix graph (see [10]).

The purpose of the present article is to expand the results of the Jacobson graph by examining it in the case of finite non-commutative rings. We assume a non-commutative ring that contains an identity. This article will focus on the case of the matrix ring as a representation of a finite non-commutative ring.

In this paper, we discuss the diameter, the degree of vertices, and the components in the Jacobson graph of matrix rings. We divide this paper into 6 sections. In the first section, we introduce the literature review and background of the research. In the second section, there are several definitions and notations used for this paper. In the third section, we will discuss the Jacobson graph of the matrix over a field. In the fourth section, we will discuss the properties of the Jacobson graph of a matrix over a finite commutative ring. In the fifth section, we will explore the special case of a Jacobson subgraph of a triangular matrix. In the last section, we close with the conclusion.

2 Definitions and notations

Most of the graph notation used in this paper refers to the book of Bondy and Murty [11]. Let G be a graph. We define the *diameter* of G, denoted by $diam(G)$, as the maximum distance between any two vertices on G. Let v be any vertices in

G. We define the *degree* of v, denoted by $d(v)$, as the number of vertices adjacent to v in G. Denote $\delta(G)$ as the *minimum degree* and $\Delta(G)$ as the *maximum degree* of G. We call G an empty graph if there is no edge in G. Graph G is a *complete graph* if every vertex in G is adjacent. Graph G is a *complete bipartite graph* if there are two partitions V_1, V_2 such that every vertex in V_1 is adjacent to every vertex in V_2. Let v be a vertex in G. We call v an *isolated vertex* if v is not adjacent to other vertices in G.

Let R be a commutative ring and $J(R)$ be the Jacobson radical of R, i.e., the intersection of all maximal ideals of R. *The Jacobson graph* of R, denoted by $\mathfrak{J}_R$, is a graph with vertex set $R\backslash J(R)$, where two distinct vertices x, y are adjacent if and only if $1 - xy$ is not a unit element in R.

As we work on the case of a non-commutative ring, the definition of the Jacobson graph needs to be updated because the non-commutative properties will give the graph's direction. Let R be a non-commutative ring with identity 1. $J(R)$ is the Jacobson radical of R. The Jacobson graph of R, denoted by $\mathfrak{J}_R$, is a directed graph with the vertex set $R\backslash J(R)$ and two distinct vertices are said to be adjacent from A to B if and only if $1 - AB$ is not a unit element.

3 Jacobson graph of the matrix over finite fields

Let us first recall the basic properties of matrices over fields. A field in this article refers to a finite field. Let F be a field. The $n \times n$ matrix over F, $M_n(F)$, is an endomorphism ring of the vector space F^n over F.

Notate I_n as the identity matrix in $M_n(F)$. Suppose $A \in M_n(F)$. The subspace $\ker(A) = \{x \in F^n | Ax = 0\}$ is called the kernel of A and the subspace $im(A) = \{Ax | x \in F^n\}$ is called the image of A. The dimension of $\ker(A)$ is called the nullity of A and is denoted by $null(A)$. The dimension of $im(A)$ is called the rank of A and is denoted by $rk(A)$. The relation between the dimension kernel and the image is written as $rk(A) + null(A) = n$.

Denote $U(M_n(F))$ as the unit set of $M_n(F)$. Consider A as a linear transformation, then $A \in U(M_n(F))$ means that $rk(A) = n$ and $null(A) = 0$. The opposite statement that $A \notin U(M_n(F))$ says that $rk(A) < n$ and $null(A) \geq 1$. In this case, we can write that there exists a nonzero $y \in F^n$ such that $Ay = 0$. We define a nonzero $x \in F^n$ to be a fixed point of A if it satisfies $Ax = x$. The following lemma describes the adjacency of $\mathfrak{J}_{M_n(F)}$ in general form.

Lemma 3.1. *Let F be a finite field. In the Jacobson graph of $M_n(F)$, two vertices A, B are adjacent from A to B if and only if AB has a fixed point.*

Proof. Suppose A, B are two distinct vertices in $\mathfrak{J}_{M_n(F)}$. Suppose A is adjacent to B, i.e., $I_n - AB \notin U(M_n(F))$. This implies $null(I_n - AB) \geq 1$. There exists a nonzero $x \in F^n$ such that $(I_n - AB)x = 0$. In this case, we obtain $ABx = x$. We conclude that AB has a fixed point. Otherwise, let nonzero y be a fixed point of AB such that $ABy = y$. We have $(I - AB)y = 0$, then $I - AB \notin U(M_n(F))$. Thus there is an arc from A to B. □

Lemma 3.2. *Let F be a finite field. Let A, B be two distinct vertices in $\mathfrak{J}_{M_n(F)}$. If there is an arc from A to B, then there is also an arc from B to A.*

Proof. If there is an arc from A to B, then by Lemma 3.1 there is a nonzero x as a fixed point of AB such that $ABx = x$. It is easy to show that Bx is nonzero and is a fixed point for BA. Thus, there is an array from B to A. □

The statement in Lemma 3.2 shows that for the field F, the graph $\mathfrak{J}_{M_n(F)}$ always has an alternating arrow. This gives rise to the following remarks.

Remark 3.3. *If F is a field, then $\mathfrak{J}_{M_n(F)}$ is an undirected graph.*

The following theorem explains the diameter properties of the Jacobson graph.

Theorem 3.4. *If F is a finite field, then $diam(\mathfrak{J}_{M_n(F)}) \leq 3$.*

Proof. Let $A, B \in V(\mathfrak{J}_{M_n(F)})$ be two distinct and nonadjacent vertices. This means that for every nonzero $x \in F^n$, we have $ABx \neq x$. We divide this into two cases, $rk(A) \geq 2$ or $rk(B) \geq 2$, and $rk(A) = rk(B) = 1$.

1. Assume that $rk(A) \geq 2$. Let $y \in F^n$. Since $B \neq 0$, $By \neq 0$. Since $rk(A) \geq 2$, there is an $x \in F^n$ such that $\{Ax, By\}$ is linearly independent in F^n. Obviously, in this case, $x \neq 0$ and $y \neq 0$. The set $\{Ax, By\}$ can be expanded to a basis, say S as a basis of F^n, such that $\{Ax, By\} \subset S$. The linear operator on F^n can be constructed by specifying the image of a basis in F^n. Thus, by using the basis S above, we make a linear transformation $C : F^n \to F^n$, which means $C(Ax) = x$ and $C(By) = y$. We obtain $x \neq 0$ and $y \neq 0$, which are fixed points of CA and CB, respectively.

2. If $rk(A) = 1$ and $rk(B) = 1$. This case is divided into two subcases. In the case of $im(A) \cap im(B) = \{0\}$, there is an $x, y \in F^n$ such that Ax, By is linearly independent. By case 1, we can construct $C \in \mathfrak{J}_{M_n(F)}$. Hence, $A-C-B$ is a path in $\mathfrak{J}_{M_n(F)}$. Suppose $im(A) = im(B)$. Let $x \in F^n$, where $im(A) = \langle Ax \rangle$. In this case, $x \neq 0$ and $Ax \neq 0$. We may extend $\{Ax\}$ and $\{x\}$ into the basis of F^n, i.e., $\{Ax, b_2, \ldots, b_n\}$, and $\{x, z_2, \ldots z_n\}$, respectively. Define $C : F^n \to F^n$ as a linear operator with $C(A(x)) = x$ and $C(b_i) = z_i$ for $i = 1, \ldots, n$. We obtain that AC is an edge and $rk(C) \geq 2$. If CB is not an edge, then, using case 1, there is a $D \in M_n(F)$ such that $C - D - B$ is a path in graph $\mathfrak{J}_{M_n(F)}$.

□

Corollary 3.5. *Let F be a field. The graph $\mathfrak{J}_{M_n(F)}$ has a diameter exactly 2 if and only if $F \cong \mathbb{Z}_2$.*

Let $A, B \in M_n(F)$. Define the equivalent relation $\sim$ with $A \sim B$ if there exists a nonsingular matrix $P, Q \in M_n(F)$ such that $A = PBQ^{-1}$. Suppose that $A \in V(\mathfrak{J}_{M_n(F)})$. Define the set

$$N(A) := \{C \in V(\mathfrak{J}_{M_n(F)}) | I - AC \notin U(M_n(F))\}$$

where $\eta(A)$ denotes the cardinality of $N(A)$.

Theorem 3.6. *Let $A, B \in V(\mathfrak{J}_{M_n(F)})$. Then $A \sim B$ if and only if $\eta(A) = \eta(B)$.*

Proof. If $A \sim B$, then there exists a matrix $P, Q \in M_n(F)$ such that $A = PBQ^{-1}$. Construct a mapping $\lambda : N(A) \to N(B)$ with the rule that for any $C \in N(A)$, $\lambda(C) = Q^{-1}CP$ holds. Since $I - AC = I - PBQ^{-1}C \notin U(M_n(F))$, $I - \lambda(C)B = I - Q^{-1}CPB \notin U(M_n(F))$. Consequently, $\lambda(C) \in N(B)$. Thus, $\eta(A) = \eta(B)$. We leave the proof of the opposite statement after Theorem 3.9. □

Lemma 3.7. *Let F be a finite field and A be a vertex in $\mathfrak{J}_{M_n(F)}$.*

- *If $I - A^2 \notin U(M_n(F))$, then $d(A) = \eta(A) - 1$*
- *If $I - A^2 \in U(M_n(F))$, then $d(A) = \eta(A)$.*

Proof. If $I - A^2 \notin U(M_n(F))$, then A is adjacent to itself. Since the Jacobson graph is a simple undirected graph, that edge must be removed. Therefore, $d(A) = \eta(A) - 1$. For $I - A^2 \in U(M_n(F))$ we can easily see that the set of vertices adjacent to A is $N(A)$. Therefore, $d(A) = \eta(A)$. □

Remark 3.8. *Let $A, B \in M_n(F)$. Then $A \sim B$ if and only if $rk(A) = rk(B)$.*

By Remark 3.8, we have n equivalent classes with respect to $\sim$ in $M_n(F)$, which for every matrix A with $rk(A) = k$ is equivalent to matrix block $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, where I_k is the identity matrix in $M_k(F)$. The following theorem will explain the involvement of rank in determining the degree of a vertex.

Theorem 3.9. *Let $A, B \in V(\mathfrak{J}_{M_n(F)})$. If $rk(A) < rk(B)$ then $d(A) < d(B)$.*

Proof. Suppose $|F| = q$. First, we need to calculate $\eta(I_n)$ on $M_n(F)$. For every A adjacent to I_n, A has a nonzero fixed point $x \in F^n$ such that $Ax = x$. The number of fixed points on F^n is $\frac{q^n-1}{q-1}$. For each fixed point x, there are $q^{n(n-1)} - 1$ possible choices of A such that $Ax = x$. In addition, it is also known that if $I_n = A$, then $I_n x = x$. Therefore

$$\eta(I_n) = \frac{q^n - 1}{q - 1}\left(q^{n(n-1)} - 1\right) + 1. \tag{3.1}$$

For any vertices where rk equals k in $M_n(F)$, $k < n$, we obtain

$$\eta\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} = \left(\frac{q^k - 1}{q - 1}\left(q^{k(k-1)} - 1\right) + 1\right) q^{n^2 - k^2}.$$

Using mathematical induction, for $n = 2$ we obtain

$$\eta\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = q^3 < q^3 + q^2 - q = \eta(I_2).$$

Assume that for $n = k$, the statement is true. We will prove that for $n = k + 1$

$$\begin{aligned} \eta\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} &= \eta(I_k) q^{(k+1)^2 - k^2} \\ &= \left(\tfrac{q^k-1}{q-1}\left(q^{k(k-1)} - 1\right) + 1\right) q^{2k-1} \\ &< \tfrac{q^{k+1}-1}{q-1}\left(q^{k+1(k)} - 1\right) + 1 \\ &= \eta(I_{k+1}). \end{aligned}$$

Therefore, for every $A, B \in V(\mathfrak{J}_{M_n(F)})$ where $rank(A) = k$ and $rank(B) = k + 1$, $d(A) = \eta(I_k) + \epsilon < \eta(I_{k+1}) + \epsilon = d(B)$ for $\epsilon = 0$ or 1. □

The opposite statement in Theorem 3.6 can be proved by contradiction. Suppose A is not equivalent to B. Since $\eta(A) = \eta(B)$ and by Theorem 3.9, $rk(A) \geq rk(B)$. Since A and B are not equivalent, $rk(B) < rk(A)$ so $\dot{(B)} < \dot{(A)}$. Contradiction.

Theorem 3.9 directly shows that the minimum degree of the Jacobson graph of $M_n(F)$ occurs at a matrix where its rank dimension is equal to 1, and the maximum degree is at a vertex where its rank dimension is equal to n.

$$\delta(\mathfrak{J}_{M_n(F)}) = \eta\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = q^{n^2-1} - 1$$

$$\Delta(\mathfrak{J}_{M_n(F)}) = \eta(I_n) = \frac{q^n - 1}{q - 1}\left(q^{n(n-1)} - 1\right) + 1.$$

In the following lemma, we will count the number of vertices in each rank dimension in $\mathfrak{J}_{M_n(F)}$.

Lemma 3.10. *Let $R_k^n = \{A \in M_n(F) | rk(A) = k\}$ and $|F| = q$. Then*

1. *$|R_0^n| = 1$.*
2. *$|R_1^1| = q - 1$ and $|R_1^n| = (q^n - 1) + (q^{n-1} - 1)q + |R_1^{n-1}|q^2$.*
3. *$|R_n^n| = \prod_{i=0}^{n-1}(q^n - q^i)$.*
4. *for $2 \leq k \leq n - 1$,
$|R_k^n| = |R_k^{n-1}|q^{2k} + |R_{k-1}^{n-1}|(q^{k-1}(q^n - q^{k-1}) + (q^{n-1} - q^{k-1})q^k) + |R_{k-2}^{n-1}|(q^{n-1} - q^{k-2})(q^n - q^{k-1})$.*

Proof. 1. It is obvious that $R_0^n = \{0\}$. Therefore

$$|R_0^n| = 1.$$

2. We have $R_1^1 = F \backslash \{0\}$. Then $|R_1^1| = q - 1$. Let $R_1^n = \{A \in M_n(F) | rk(A) = 1\}$. For $A \in R_1^n$, we may write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

with A_{11} is a submatrix size $(n-1) \times (n-1)$. We divide into two cases :

(a) If $rk(A_{11}) = 0$ then $rk\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = 0$ or $rk\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = 1$. If $rk\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = 0$ then $\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ is not zero vector. If $rk\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = 1$ then $\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ is linear combination of column vector of $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$. The number of choices for A in this case is $(q^n - 1) + (q^{n-1} - 1)q$.

(b) If $rk(A_{11}) = 1$ then A_{21} can be chosen arbitrarily and $\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ is linear combination of column vector of $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$. By this case the number of choices for A is $|R_1^{n-1}|q^2$.

Based on (a) and (b), we conclude that

$$|R_1^n| = (q^n - 1) + (q^{n-1} - 1)q + |R_1^{n-1}|q^2. \quad (3.2)$$

3. The set R_n^n is invertible matrix set. By the Theorem in [12] we have

$$|R_n^n| = \prod_{i=0}^{n-1}(q^n - q^i).$$

4. For any n and $2 \le k \le n-1$, $R_k^n = \{A \in M_n(F) | rk(A) = k\}$. Let $A \in R_k^n$, we may write $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. There are 3 cases, $A_{11} \in R_k^{n-1}$, or $A_{11} \in R_{k-1}^{n-1}$, or $A_{11} \in R_{k-2}^{n-1}$.

(a) If $A_{11} \in R_k^{n-1}$. Then A_{21} is linear combination of row vectors of A_{11}. We may choose $\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ as linear combination of column vectors of $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$. The number of choice A in this case is $|R_k^{n-1}|(q^k)^2$.

(b) If $A_{11} \in R_{k-1}^{n-1}$, then A_{21} is linear combination of row vectors of A_{11} or A_{21} is not.

- If A_{21} is linear combination of row vectors of A_{11}, then $rk\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = k-1$. Vector $\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ must be linear independent to column vectors of $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$. The number of choice A in this case is $|R_{k-1}^{n-1}|(q^{k-1}(q^n - q^{k-1}))$.
- If A_{21} is not linear combination of row vectors of A_{11}, then $rk\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = k$. Thus $\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ must be linear combination of column vectors of $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$. We have $|R_{k-1}^{n-1}|((q^{n-1} - q^{k-1})q^k)$ choices for this case.

(c) If $A_{11} \in R_{k-1}^{n-1}$, then $rk\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = k-1$ thus A_{21} must be linear independent with row vectors of A_11. For $\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}$ can be choosen from independent vectors of column vector of $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$. Therefore, the choice of A in this case is $|R_{k-2}^{n-1}|(q^{n-1} - q^{k-2})(q^n - q^{k-1})$

Based on the three cases above, we may conclude
$|R_k^n| = |R_k^{n-1}|q^{2k} + |R_{k-1}^{n-1}|(q^{k-1}(q^n - q^{k-1}) + (q^{n-1} - q^{k-1})q^k) + |R_{k-2}^{n-1}|(q^{n-1} - q^{k-2})(q^n - q^{k-1})$.

□

Example 3.11. *The Jacobson graph of the matrix over $\mathbb{Z}_2$, $\mathfrak{J}_{M_n(\mathbb{Z}_2)}$, has $M_2(\mathbb{Z}_2)\backslash\{0\}$ as the vertex set, which consists of 9 vertices with rank 1 and 6 vertices with rank 2. At any vertex A with $rk(A) = 1$, we obtain $\eta(A) = 8$. Therefore, $d(A) = 7$ or 8. Meanwhile, for any vertex B with $rk(B) = 2$, we obtain $\eta(B) = 10$. Therefore, $d(B) = 9$ or 10. The following is the Jacobson graph of $M_2(\mathbb{Z}_2)$ in Figure 1.*

4 Jacobson graph of the matrix over rings

We begin by recalling some of the structures of commutative rings with nonzero identities. A ring R is called local if it has only one maximal ideal, i.e., $\mathfrak{m}$, where the quotient ring $R/J(R) = R/\mathfrak{m}$ is a field. The set of matrices over R, $M_n(R)$ can be viewed as an endomorphism of the free module R^n over R. Suppose R is a local ring, then the set of matrices on the quotient field can be written as $M_n(R/J(R)) = \{A + M_n(J(R)) | A \in M_n(R)\}$.

Remark 4.1. *Let R be a finite local ring. Then $A \in U(M_n(R))$ if and only if $A + M_n(J(R)) \in U(M_n(R/J(R)))$.*

Based on Remark 4.1, for finite local ring R it is easier to redefine the Jacobson graph of $M_n(R)$, i.e., the graph with vertex set $M_n(R) \backslash M_n(J(R))$ and two distinct vertices A, B are adjacent if and only if $(I_n - AB) + M_n(J(R)) \notin U(M_n(R/J(R)))$. With this simplification, we obtained a direct relationship between graphs $\mathfrak{J}_{M_n(R)}$ and $\mathfrak{J}_{M_n(R/J(R))}$ as follows. Recall that an empty graph is a graph that has no edges.

Lemma 4.2. *Let R be a finite local ring and let $R/J(R)$ be the quotient field.*

1. *$A + M_n(J(R))$ is adjacent to $B + M_n(J(R))$ in $\mathfrak{J}_{M_n(R/J(R))}$ if and only if every vertex in $A + M_n(J(R))$ is adjacent to $B + M_n(J(R))$ in $\mathfrak{J}_{M_n(R)}$.*
2. *$I - A^2 \notin U(M_n(R))$ if and only if each vertex in $A + M_n(J(R))$ forms a complete subgraph in $\mathfrak{J}_{M_n(R)}$.*

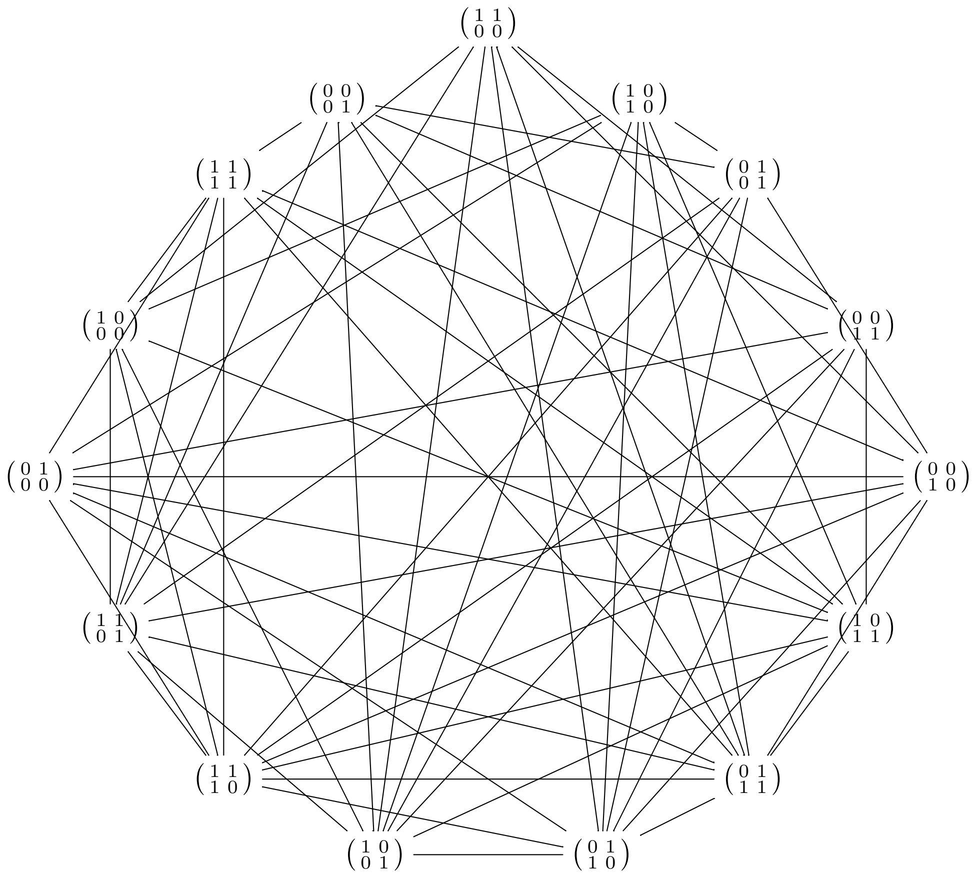

Figure 1. graf $\mathfrak{J}_{M_2(\mathbb{Z}_2)}$

3. *$I - A^2 \in U(M_n(R))$ if and only if each vertex in $A + M_n(J(R))$ forms an empty subgraph in $\mathfrak{J}_{M_n(R)}$.*

Proof. In the first statement it is sufficient to show that for any vertex $C \in A + M_n(J(R)), D \in B + M_n(J(R)))$, such as $C = A + M_1, D = B + M_2, M_1, M_2 \in M_n(J(R))$, C is adjacent to D. Note that $I - CD \in I - AB + M_n(J(R)) \notin U(M_n(R/J(R)))$. For the second statement, if $I - A^2 \notin U(M_n(R))$, then for all vertices $C, D \in A + M_n(J(R))$, $I - CD \in I - A^2 + M_n(J(R)) \notin U(M_n(R/J(R)))$. The set of vertices in $A + M_n(J(R))$ forms a complete subgraph. The reverse direction is obvious. For the third statement, take any two vertices $E, F \in A + M_n(J(R))$, then $I-EF \in I-A^2+M_n(J(R)) \in U(M_n(R/J(R))$. Therefore, E is never adjacent to F. □

Theorem 4.3. *Let R be a finite local ring. The graph $\mathfrak{J}_{M_n(R)}$ is connected and $diam(\mathfrak{J}_{M_n(R)}) \leq 3$.*

Proof. Take any vertices A, B that are not adjacent to each other. We divide this into two cases. The first case is $A + M_n(J(R)) = B + M_n(J(R))$. Since A is not adjacent to B and because $\mathfrak{J}_{M_n(R/J(R))}$ is a connected graph (by Theorem 3.4), it is always possible to choose a neighbor of $A + M_n(J(R))$, for example $C + M_n(J(R))$, such that A and B can be associated with every vertex in $C + M_n(J(R))$. The second case is $A + M_n(J(R)) \neq B + M_n(J(R))$, which is not adjacent in $\mathfrak{J}_{M_n(R/J(R))}$. By using the proof of Theorem 3.4, we obtain that the longest path connecting $A + M_n(J(R))$ and $B + M_n(J(R))$ has length 3. Let us say that the vertex is $\bar{X} = X + M_n(J(R))$ and $\bar{Y} = Y + M_n(J(R))$ such that $A + M_n(J(R) - \bar{X} - \bar{Y} - B + M_n(J(R))$. Then $A - X - Y - B$. Thus, we obtain $diam(\mathfrak{J}_{M_n(R)}) \leq 3$. □

Corollary 4.4. *Let R be a finite local ring. Then $diam(\mathfrak{J}_{M_n(R)}) = 2$ if and only if $R/J(R) \cong \mathbb{Z}_2$ and $diam(\mathfrak{J}_{M_n(R)}) = 3$ if and only if $|R/J(R)| \geq 3$.*

The special form that distinguishes the Jacobson graph of the matrix over a field and the matrix over a finite local ring is in the degree of each vertex. Above we have explained that the Jacobson graph on the matrix over a field has $2n$ sets of vertices with different degrees, which are grouped based on their rank. Likewise, in a finite local ring there are $2n$ sets of vertices with different degrees, but there is an expansion at each vertex so that each vertex has more degrees than the graph in the matrix over a field.

Proposition 4.5. *Let R be a local finite with $|R/J(R)| = q$. Take any vertices $A \in \mathfrak{J}_{M_n(R)}$ and let $\bar{R}^n_k = \{A + M_n(J(R)) \in M_n(R/J(R)) | rk(A + M_n(J(R))) = k\}$. Then*

1. *If $I - A^2 \notin U(M_n(R))$ then*

$$d(A) = |\bar{R}^n_k||J(R)|^{n^2} + |J(R)|^{n^2} - 1$$

2. *If $I - A^2 \in U(M_n(R))$ then*

$$d(A) = \bar{R}_k^n |J(R)|^{n^2}$$

Proof. Let $rk(A + M_n(J(R))) = k$, we have

$$d(A + M_n(J(R)) = |\bar{R}_k^n|$$

in $\mathfrak{J}_{M_n(R/J(R))}$. The number of vertices adjacent to A, which is not in $A + M_n(J(R))$ is $d(A + M_n(J(R)).|J(R)|^{n^2}$.

1. If $I - A^2 \notin U(M_n(R))$, then vertex set $A + M_n(J(R))$ form complite induced subgraph in $\mathfrak{J}_{M_n(R)}$. The number of vertices adjacent to A in $A + M_n(J(R))$ is $|J(R)|^{n^2} - 1$. Therefore,
$d(A) = d(A + M_n(J(R)).|J(R)|^{n^2} + |J(R)|^{n^2} - 1 = |\bar{R}_k^n|.|J(R)|^{n^2} + |J(R)|^{n^2} - 1$.

2. If $I - A^2 \in U(M_n(R))$, then A is not adjacent to all vertices in $A + M_n(J(R))$. We obtain

$$d(A) = d(A + M_n(J(R))).|J(R)|^{n^2} = |\bar{R}_k^n|.|J(R)|^{n^2}.$$

□

A finite non-local ring with a nonzero identity can be decomposed into the direct sum of finite local rings $R_1, \ldots, R_k$, such that

$$R = R_1 \oplus \ldots \oplus R_k.$$

The Jacobson radical and unit of the ring R can be written as $J(R) = J(R_1) \oplus ... \oplus J(R_k)$ and $U(R) = U(R_1) \oplus ... \oplus U(R_k)$. The quotient ring of a finite non-local ring is $R/J(R) = R_1/J(R_1) \oplus \ldots R_k/J(R_k)$, where this quotient ring is a semi-simple ring. The set of matrices of size $n \times n$ over R can be written as

$$M_n(R) = M_n(R_1) \oplus ... \oplus M_n(R_k).$$

Any $A \in M_n(R)$ can be written as

$$A = A_1 \times A_2 \times ... \times A_k = \times_{i=1}^k A_i$$

with $A_i \in M_n(R_i)$. If $A \notin U(M_n(R))$, then there is $i \in \{1, \ldots, k\}$ so $A_i \notin U(M_n(R_i))$. In the Jacobson graph of the non-local ring R, $\mathfrak{J}_{M_n(R)}$, the vertex set is $M_n(R) \backslash M_n(J(R))$ and two distinct vertices A, B are adjacent if there exists an $i \in \{1, \ldots, k\}$ such that $I_n - A_i B_i + M_n(J(R_i)) \notin U(M_n(R_i/J(R_i)))$.

Theorem 4.6. *Let R be a finite non-local ring. Then $diam(\mathfrak{J}_{M_n(R)}) \leq 3$.*

Proof. Suppose $A = A_1 \times \ldots \times A_k, B = B_1 \times \ldots B_k$ two distinct vertices that are not adjacent in $\mathfrak{J}_{M_n(R)}$. Then there exists a $i, j \in \{1, \ldots, k\}$ such that $A_i \notin M_n(J(R_i))$ and $B_j \notin M_n(J(R_j))$. We divide this into two cases. The first case is $i \neq j$. We can choose a vertex C with a certain C_i and C_j such that $I - A_i C_i \notin U(M_n(R_i))$ and $I - C_j B_j \notin U(M_n(R_j))$. Thus, A and B are adjacent to C. The second case is $i = j$. Based on Theorem 3.4 and Theorem 4.3, the maximum length of the maximum path connecting A_i and B_i is 3 in $\mathfrak{J}_{M_n(R_i)}$. We can choose the vertices $D = D_1 \times \ldots \times D_k$ and $E = E_1 \times \ldots \times E_k$ with A_i, D_i, E_i, B_i forming a path of size 3 in $\mathfrak{J}_{M_n(R_i)}$. Therefore, $A - D - E - B$ and $diam(\mathfrak{J}_{M_n(R)}) \leq 3$. □

Corollary 4.7. *Let R be a finite non-local ring. Then*

- $Diam(\mathfrak{J}_{M_n(R)}) = 2$ *if and only if* $R \cong R_1 \oplus ... \oplus R_k$, *where there exists* $i \in \{1, \ldots, k\}$ *such that* $R_i/J(R_i) \cong \mathbb{Z}_2$.
- $Diam(\mathfrak{J}_{M_n(R)}) = 3$ *if and only if* $|R_i/J(R_i)| \geq 3$ *for every* $i \in \{1, \ldots, k\}$.

The following proposition describes the degree of each vertex of the Jacobson graph of the matrix over a finite non-local ring. Of course, there is a close relationship with the previous local ring case, but the graph obtained will be denser than in the local ring case and the field case.

Proposition 4.8. *Suppose $R = R_1 \oplus \ldots \oplus R_k$ is a finite non-local ring. Suppose $|R_i| = q_i$. For any vertex $A \in \mathfrak{J}_{M_n(R)}$, $A = A_1 \times \ldots \times A_k$, with $A_i \in M_n(R_i)$. We write $d(A_i)$ as degree of A_i in $\mathfrak{J}_{M_n(R_i)}$. Then*

$$d(A) = \prod_{i=1}^{k} q_i^{n^2} - \prod_{i=1}^{k}(q_i^{n^2} - d(A_i)).$$

Proof. Suppose $R = R_1 \oplus \ldots \oplus R_k$. For any vertex A in $\mathfrak{J}_{M_n(R)}$, with $A = A_1 \times \ldots \times A_k$. A is adjacent to B in $\mathfrak{J}_{M_n(R)}$, where $B = B_1 \times \ldots \times B_k$, if there is an i such that A_i is adjacent to B_i in $\mathfrak{J}_{M_n(R_i)}$. To simplify the calculation, it is better to count the number of vertices B that are not adjacent to A, because if A is not adjacent to B then for every i, it satisfies A_i is not adjacent to B_i in $\mathfrak{J}_{M_n(R_i)}$. Based on Proposition 4.5, we obtain that the number of vertices B_i that are not adjacent to A_i in $\mathfrak{J}_{M_n(R_i)}$ is $q_i^{n^2} - d(A_i)$. Therefore, the number of vertices B that are not adjacent to A is $\prod_{i=1}^{k}(q_i^{n^2} - d(A_i))$. The degree of A is

$$d(A) = \prod_{i=1}^{k} q_i^{n^2} - \prod_{i=1}^{k}(q_i^{n^2} - d(A_i)).$$

□

5 Jacobson graph of the triangular matrix over rings

In this section, we will discuss the special case of a Jacobson graph of an upper triangular matrix. Of course, the same applies to a lower triangular matrix, because upper triangular matrices are isomorphic to lower triangular matrices. The properties obtained for this subgraph are quite different from the previous result above. Suppose R is a finite commutative ring. Denote the upper triangular matrix over R by $T_n(R)$.

$$T_n(R) = \{A = (a_{ij}) \in M_n(R) | \forall i > j, a_{ij} = 0\}.$$

Define the subset of $T_n(R)$,

$$S_n(R) := \{A = (a_{ij}) \in T_n(R) | a_{ii} \in J(R)\}.$$

Proposition 5.1. *Let R be a finite commutative ring. The Jacobson graph of $T_n(R)$ is not a connected graph. Vertex set $S_n(R)$ forms an empty subgraph.*

Proof. We will prove that the set of vertices in $S_n(R)$ forms an empty subgraph in $\mathfrak{J}_{T_n(R)}$. Take any $A \in S_n(R)$. For every $B \in S_n(R)$, $AB \in S_n(R)$. Then $I - AB \in U(T_n(R))$. Therefore, all vertices in $S_n(R)$ are not adjacent to each other. For each vertex $C \in T_n(R) \backslash S_n(R)$, then $AC \in S_n(R)$. Therefore, A is not adjacent to C. Therefore the vertex in $S_n(R)$ is not adjacent to the vertices $T_n(R) \backslash S_n(R)$. □

As an initial illustration, the following will show an example of the graph $\mathfrak{J}_{T_2(\mathbb{Z}_2)}$.

Example 5.2. *Let $T_2(\mathbb{Z}_2)$ be the upper triangular matrix of $\mathbb{Z}_2$. The Jacobson graph of $T_2(\mathbb{Z}_2)$ forms two subgraphs. Clearly, $S_2(\mathbb{Z}_2)$ has only 1 member, i.e., $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which is an isolated vertex. The second component consists of 6 vertices that form a connected subgraph. The shape of this graph can be seen in Figure 2.*

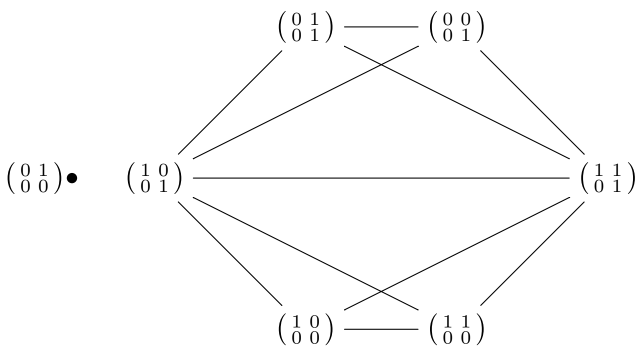

Figure 2. Graf $\mathfrak{J}_{T_2(\mathbb{Z}_2)}$

Denote the connected subgraph of $T_n(R)$ by $\mathfrak{J}^*_{T_n(R)}$, where the vertex set is $T_n(R) \backslash S_n(R)$. We will investigate the diameter of $\mathfrak{J}^*_{T_n(R)}$ in the following theorem.

Theorem 5.3. *Let R be a finite commutative ring. Then $diam(\mathfrak{J}^*_{T_n(R)}) \leq 3$.*

Proof. Suppose $A, B \in V(\mathfrak{J}^*_{T_n(R)})$. Assume A is not adjacent to B. Suppose $A = (a_{ij})$ and $B = (b_{ij})$. Since $A, B \notin S_n(R)$, then there is an i, j such that $a_{ii}, b_{jj} \notin J(R)$. If $i \neq j$, then we can choose the matrix $C = (c_{ij}) \in T_n(R)$ with $1 - a_{ii}c_{ii} \notin U(R)$ and $1 - b_{jj}c_{jj} \notin U(R)$. If $i = j$, then we can choose the matrix $D = (d_{ij}) \in T_n(R)$ and $E = (e_{ij}) \in T_n(R)$ such that $1 - a_{ii}d_{ii}, 1 - d_{kk}e_{kk}, 1 - e_{ii}b_{ii} \notin U(R)$ for some k. Thus, $diam(\mathfrak{J}^*_{T_n(R)}) \leq 3$. □

6 Conclusions

The Jacobson graph of any matrix ring is undirected. We obtain the results regarding diameter, degrees, and the number of components of the Jacobson graph. The main result is to determine the degree of a vertex in Jacobson graph of matrix ring, we only need to look at the rank of the matrix. In the future, one can continue with other graph properties of the Jacobson graph of a matrix ring like the independent set, dominating number, and chromatic number. Also, another form of non-commutative ring with an identity other than the ring matrix can be found and can complement the properties of the Jacobson graph over the non-commutative ring.

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