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The Generalized Inverse of Picture Fuzzy Matrices

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Abstract The generalized inverse is crucial in matrix theory. In many applications, such as control systems, robotics, and signal processing, the generalized inverse of matrices is critical. The generalized inverse of a picture fuzzy matrix is critical to solving a variety of real-world problems. Because of their ability to handle uncertain and imprecise medical data, applications of the generalized inverse of picture fuzzy matrix have gained significant attention in the medical field. Numerous researchers have investigated generalized inverses in fuzzy matrices and intuitionistic matrices. The fuzzy picture is an effective mathematical model for dealing with uncertain real-world issues. The picture fuzzy matrix is a generalization of the classical fuzzy matrix and the intuitionistic fuzzy matrix. In this research, a method for determining the generalized inverse (g-inverse) of a picture fuzzy matrix is implemented. In addition, the concept of a standard basis for picture fuzzy vectors is established. A few results related to the g-inverse of a fuzzy picture matrix are premeditated with relevant examples. An algorithm for evaluating the generalized inverse of a fuzzy picture matrix is provided. This study concludes with an application of the g-inverse of a picture fuzzy matrix.

Keywords Picture Fuzzy (PictF), Picture Fuzzy Set (PictFS), Fuzzy Matrices (FM), Picture Fuzzy Matrices (PictFMs), Picture Fuzzy Permutation Matrix (PictFPM), Picture fuzzy vectors (PictFVs), Generalized Inverse (GI), Relational Equation (RI), Semi Inverse (SI)

1 Introduction

Traditional logic finds it difficult to determine if something is true or false, to solve uncertain real-life problems. So as a precaution, Zadeh [1] developed the concept of Fuzzy set theory which plays an authoritative part in decisiveness making under uncertain situations. The matrix is crucial in many fields of research and engineering. Unfortunately one can not succesful with normal matrices due to various types of uncertainties confronted in the real life problems, such difficulties are solved by FM [2]. The concept of FM was initiated by Hashimoto in 1983. After that lot of works have been done on

FM[2, 3, 4]. To handle much uncertainty the above theory was generalized by Atanassov [5] as Intuitionistic Fuzzy Set (IFS) which correspond relevant and irrelevent degrees singly such the sum of the two degrees should not exceed one. By and by IFS has been appealed by numerous researchers in a variety of fields [6, 7, 8, 9, 10, 11, 12] and it is observed that prime abstraction of neutrality degree is missing in IFS. The thought of the above degree can be observed in a variety of contexts, such as when human opinions involve more yes, abstain, no, or rejection responses.

On the other deal, neutrality in medical diagnosis can be taken into retainer. For example, the symptoms temperature, headache on the diseases stomach and chest problems may not effect. Also, the symptoms of pain in stomach as well as chest have neutral burden on the illness like viral fever, malaria, typhoid etc. In this regards, in [13, 14], Coung founded the new belief with the neutrality and named it as a PictFS. It triply consists of neutral membership along with the membership degree present in IFS. PictFS enthralled authors as it is directly applied to daily life problems.

In 2020 PictFM and its diligence was studied by Shovan Dogra and Madhumangal Pal [15]. Regular matrices and GI play a significant role in in many fields of sciences. Some results on GI of IFMs were designed by M.Pal and Pradhan[16]. Khan S.K and A.Pal talked over the GI of IFMs [17] and Rajkumar Pradhan and Madhumangal Pal defined the GI of Atanassov's IFMs[18]. Murugadas.P contrived some implication operations on PictFMs (2021) [19].

In this article, we boil down our care to GI of PictFM with some properties and a uncomplicated method for determining the GI of PictFM is established. Some outcomes related to the GI of PictFM are studied with right exemplications. At the end of this study, an application of GI of PictFM is demonstrated.

2 Preliminary Definitions

For preliminary definitions regarding this topic see [13, 15].

Definition 2.1. Let $P = (\langle p_{\mu}, , p_{\eta}, p_{\nu}\rangle)$ be a PictFM, then multiplication by PictF element(Scalar multiplication), $c = \langle c^{'}, c^{''}, c^{'''}\rangle$is defined as
$c.P = \langle c^{'} \wedge p_{\mu}, c^{''} \wedge p_{\eta}, c^{'''} \vee p_{\nu}\rangle$

3 Operations on PictFSs and PictFMs

Throughout the article $\mathscr{P}_{kl}$ denotes PictFMs of order $k \times l$ and $\mathscr{P}_k$ denotes PictFMs of order $k \times k$. In arithmetic operations, the values of all the three memberships are needed. So all elements of PictFM are the members of $< P >$, where $< P >= \{\langle a_{\mu}, a_{\eta}, a_{\nu}\rangle | a_{\mu}, a_{\eta}, a_{\nu} \in [0, 1]$ and $0 \leq a_{\mu} + a_{\eta} + a_{\nu} \leq 1\}$
To study respective properties, the algebraic operations such as componentwise addition ($\vee$) and multiplications ($\wedge$) of PictFSs are given as:

Definition 3.1. For $a = \langle a', a'', a'''\rangle, b = \langle b', b'', b'''\rangle \in PicFS$,
we define Joint ($\vee$) and meet ($\wedge$) operations as,
1) $\langle a', a'', a'''\rangle \vee \langle b', b'', b'''\rangle = \langle \max(a', b'),\ \max(a'', b''),$ $\min(a''', b''') \rangle = \langle (c', c'', c''')\rangle$ if $c' + c'' + c''' \leq 1$,
otherwise find $\max\{c', c'', c'''\}$ and replace $\max\{c', c'', c'''\}$ by
1- (sum of the rest of the components)
2) $\langle a', a'', a'''\rangle \wedge \langle b', b'', b'''\rangle = \langle \min(a', b'),\ \min(a'', b''),$ $\max(a''', b''')\rangle$
3) $a^c = \langle a''', a'', a'\rangle$

Definition 3.2. For PictFMs,
$M1 = \left(\langle m1'_{ij}, m1''_{ij}, m1'''_{ij}\rangle\right)_{m\times n}$,
$M2 = \left(\langle m2'_{ij}, m2''_{ij}, m2'''_{ij}\rangle\right)_{m\times n}$.
Define
1) $M1 \vee M2 = \left(\langle m1'_{ij} \vee m2'_{ij}, m1''_{ij} \vee m2''_{ij}, m1'''_{ij} \wedge m2'''_{ij}\rangle\right)$
2) $M1 \wedge M2 = \left(\langle m1'_{ij} \wedge m2'_{ij}, m1''_{ij} \wedge m2''_{ij}, m1'''_{ij} \vee m2'''_{ij}\rangle\right)$
3) $M1 \times M2 = (\langle \vee(m1'_{ij} \wedge m2'_{ij}), \vee(m1''_{ij} \wedge m2''_{ij}),$ $\wedge(m1'''_{ij} \vee m2'''_{ij})\rangle)$
4) $M1^T = \left(\langle m1'_{ji}, m1''_{ji}, m1'''_{ji}\rangle\right)$ ($M1^T$ is transpose of $M1$)
5) $M1 \leq M2$ iff $m1'_{ij} \leq m2'_{ij}, m1''_{ij} \leq m2''_{ij}, m1'''_{ij} \geq m2'''_{ij}$.
6) $M1^c = \left(\langle m1'''_{ij}, m1''_{ij}, m1'_{ij}\rangle\right)$
Here after $M1M2$ means $M1 \times M2$.

4 Picture fuzzy vector space

Let V_n be the set of all n-tuples

$$(\langle x_{1\mu}, x_{1\eta}, x_{1\nu}\rangle, \langle x_{2\mu}, x_{2\eta}, x_{2\nu}\rangle, ..., \langle x_{n\mu}, x_{n\eta}, x_{n\nu}\rangle)$$

over $\langle P\rangle$. An element of V_n is called PictFV of dimension n, where $x_{i\mu}, x_{i\eta}$ and $x_{i\nu}$ are the three membership values of the component x_i and also they form a PictFVs with respect to usual componentwise addition and multiplications.

Definition 4.1. [Linear combination of PictFVs]

Let $S = \{b_1, b_2, ..., b_p\}$ be a PictFVs set each having elements. The finite sum $\sum_{i=1}^{p} \alpha_i b_i$ where $b_i \in S$ and $\alpha \in [0,1]$ is the linear combinations of elements of the set S that is the set of all linear combinations of the elements of S and it is called the span of S and it is marked by $\langle S \rangle$.

The terminologies of Independent, Dependent and Basis of PictFVs are similar to classical vectors.

Definition 4.2. [Standard Basis of PictFVs]

A basis B_1 of a PictFVs W_1 is a standard basis (SB) if and only if $\alpha_{ij} b_i = b_i$. whenever $b_i = \sum_{j=1}^{n} \alpha_{ij} b_j$ for $b_i, b_j \in B_1$ and $\alpha_{ij} \in [0,1]$.

Example 4.1. Let S=$\{b_1, b_2, b_3\}$ be a subset of V_3 given by
$b_1 = (\langle .5, .4, .1\rangle, \langle .4, .3, .2\rangle, \langle .4, .2, .3\rangle)$,
$b_2 = (\langle .4, .5, .1\rangle, \langle .3, .4, .1\rangle, \langle .3, .3, .2\rangle)$ and
$b_3 = (\langle .4, .3, .2\rangle, \langle .5, .4, .1\rangle, \langle .4, .3, .3\rangle)$.
Here S is independent set, since $b_1 \neq \alpha_1 b_2 + \alpha_2 b_3$, $b_2 \neq \alpha_3 b_1 + \alpha_4 b_3$ and $b_3 \neq \alpha_5 b_1 + \alpha_6 b_2$.
So $\{b_1, b_2, b_3\}$ is a basis for $\langle S \rangle$, is the SB.
Since $b_1 = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3$ holds good.
For $\alpha_1 = \langle .5, .4, .1\rangle$,$\alpha_2 = \langle .4, .2, .3\rangle$,$\alpha_3 = \langle .4, .2, .2\rangle$
Similarly for b_2 and b_3.

Definition 4.3. [Row space(RS) and Column space(CS)]

Let $B = (\langle b_{ij\mu}, b_{ij\eta}, b_{ij\nu}\rangle) \in \mathscr{P}_{m\times k}$ be a PictFM. Then the element $\langle b_{ij\mu}, b_{ij\eta}, b_{ij\nu}\rangle$ is the ij^{th} entry of B and $B_{i*}(B_{*j})$ denotes the i^{th} row (j^{th} column) of B.

The RS $R(B)$ of B is the subspace of V_k generated by the rows $\{B_{i*}\}$ of B. The CS $C(B)$ of B is the subspace of V_m generated by the columns $\{B_{*j}\}$ of B.

5 Generalized Inverse of Picture fuzzy Matrix

The GI of a PictFM is researched here.

Definition 5.1. [Generalized inverse]

A PictFM $B \in \mathscr{P}_{k\times m}$ is said to be regular if for another PictFM, $X \in \mathscr{P}_{m\times k} \ni BXB = B$. All such X is called a GI's of B and the collection is denoted by B^-.

The GI of a PictFM is not unique (i.e) a PictFM has many GIs. All such GIs of B are denoted by $B\{1\}$.

Theorem 5.1. The non-zero rows of a PictFM B form a SB. If B satisfies the equation $BPB = B$ for some PictFPM P then, B is regular.

Proof: Let $PB = X$, here B is rearranged to get X. Clearly X is an idempotent PictFM, (i.e) $X^2 = X$, and the RS of X and B are same. Since SBs are unique $B = PX$, for some PictFPM P. Then,
$BP^TB = PXP^TPX = PXX = PX = B$
(i.e) $BPB = B$.
Therefore, B is regular.

Definition 5.2. For a PictFM $B \in \mathscr{P}_{k\times m}$ and $H \in \mathscr{P}_{m\times k}$ is said to be $B\{2\}$ or outer inverse (OI) of B if $HBH = H$.

If $BHB = B$ and $HBH = H$, then H is said to be $\{1,2\}$ inverse or semi-inverse(SI) of B.

If $BHB = B$ and $(BH)^T = BH$, H is said to be $\{1,3\}$ inverse or a least square(LS) GI of B and it is named as $B\{1,3\}$.

Again, H is said to be $\{1,4\}$ inverse or a minimum norm (MN) GI of B if $BHB = B$ and $(HB)^T = HB$ and it is named as $B\{1,4\}$.

No algorithm is getable to find GI of PictFM. Forthcomming is a simple algorithm to evaluate GI of a PictFM.

Algorithm (To find the GI of a PictFM B)

Step 1: Is the non-zero rows of a PictFM B form a SB or not for the RS of B.?
Step 2: If so find a PictFPM $P \ni BPB = B$.
Step 3: Select a PictFM $Q \ni QB = B$.
Step 4: Then PQ is the GI of B.
Step 5: $PQBPQ$ is the SI of B.
The matrix PQ is a GI of B.
[since $B(PQ)B = BP(QB) = BPB = B$]
The example illustrates the above algorithm to find GI of B.

Example 5.1. Consider a PictFM,
$$B = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.5, 0.1\rangle & \langle 0.3, 0.3, 0.2\rangle \end{bmatrix}$$
The rows of B form a SB.

Now, consider the PictFPM,

$$P = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix},$$

$BPB = B$ holds good.

Now, for the PictFM,

$$Q = \begin{bmatrix} \langle 0.5, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.3, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \end{bmatrix},$$

$QB = B$ holds good.

So the GI of B is,

$$PQ = \begin{bmatrix} \langle 0.3, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \\ \langle 0.5, 0.4, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix} = X$$

which fulfills the equation $BXB = B$.

Also, $PQBPQ = \begin{bmatrix} \langle 0.3, 0.3, 0.2\rangle & \langle 0.4, 0.5, 0.1\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.4, 0.1\rangle \end{bmatrix} = H$ be the SI of B. [since $BHB = B$ and $HBH = H$]

If each rows of a PictFM D can be written as a linear combination of the rows of PictFM B, then we write $R(D) \subseteq R(B)$. If $R(D) \subseteq R(B)$ and vice versa, then $R(B) = R(D)$.

Theorem 5.2. For a regular PictFM $B \in \mathscr{P}_{k\times m}$, if H is the GI of B. Then

(a) $H^T \in B^T\{1\}$.

(b) For PictFPMs P and Q, $Q^T HP^T \in PBQ\{1\}$.

(c) BH and HB are idempotent.

Proof: (a) For a GI H be B.

$\Rightarrow BHB = B$ holds good.

Taking transpose on both sides, we get

$B^T H^T B^T = B^T \Rightarrow H^T \in B^T\{1\}$.

(b) P and Q are invertible and $P^{-1} = P^T$, $Q^{-1} = Q^T$.

Now, $PBQ(Q^T HP^T)PBQ = PB(QQ^T)H(P^T P)BQ$

$= PBHBQ$ (since $QQ^T = I$,$PP^T = I$)

$= PBQ$. (since $BHB = B$)

$\Rightarrow Q^T HP^T \in PBQ\{1\}$.

(c) Again, $(BH)(BH) = (BHB)H$

$= BH$ (since $BHB = B$)

Also $(HB)(HB) = (HBH)B$

$= HB$ (since $HBH = H$)

Thus BH and HB are idempotent.

Example 5.2. Let $B = \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.3, 0.4, 0.1\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.5, 0.3, 0.2\rangle \end{bmatrix}$ be a PictFM and one of its GI is

$$H = \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.3, 0.3, 0.3\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.5, 0.3, 0.2\rangle \end{bmatrix}$$

Now, $B^T H^T B^T = \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.3, 0.4, 0.1\rangle & \langle 0.5, 0.3, 0.2\rangle \end{bmatrix}$

$= B^T$.

$\Rightarrow H^T \in B^T\{1\}$.

Let $P = \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \\ \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \end{bmatrix}$ and let

$$Q = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}$$

be two PictFPMs.

Now, $Q^T HP^T = \begin{bmatrix} \langle 0.4, 0.3, 0.2\rangle & \langle 0.5, 0.3, 0.2\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.3, 0.3, 0.3\rangle \end{bmatrix}$ and

$$PBQ = \begin{bmatrix} \langle 0.3, 0.4, 0.1\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$$

Also,

$$PBQ(Q^T HP^T)PBQ = \begin{bmatrix} \langle 0.3, 0.4, 0.1\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$$

$$\begin{bmatrix} \langle 0.4, 0.3, 0.2\rangle & \langle 0.5, 0.3, 0.2\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.3, 0.3, 0.3\rangle \end{bmatrix}$$

$$\begin{bmatrix} \langle 0.3, 0.4, 0.1\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.3, 0.4, 0.1\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix} = PBQ.$$

$\Rightarrow Q^T HP^T \in PBQ\{1\}$.

$$(BH)^2 = (BH)(BH) = \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.3, 0.4, 0.1\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.5, 0.3, 0.2\rangle \end{bmatrix}$$

$$\begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.3, 0.4, 0.1\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.5, 0.3, 0.2\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.3, 0.4, 0.1\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.5, 0.3, 0.2\rangle \end{bmatrix} = BH.$$

$\Rightarrow (BH)^2 = BH$. Similarly we prove $(BH)^2 = BH$

Hence HB and BH are idempotent PictFMs.

Theorem 5.3. Let B be a PictFM, $Y, Z \in B\{1\}$ and $X = YBZ$, then $X \in B\{1, 2\}$ (i.e) X is a SI of B.

Proof: Since $Y, Z \in B\{1\}$

$\Rightarrow BYB = B$ and $BZB = B$.

As, $X = YBZ$. So, $BXB = B(YBZ)B$

$= (BYB)ZB$

$= BZB$

$= B.$

Also

$XBX = (YBZ)B(YBZ)$

$= Y(BZB)(YBZ)$

$= Y(BYB)Z$

$= YBZ$

$= X.$

Hence X is a SI of the PictFM B.

Example 5.3. $B = \begin{bmatrix} \langle .4,.4,.0\rangle & \langle .4,.4,.0\rangle \\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle \end{bmatrix}$

be a PictFM.

Let $Y = \begin{bmatrix} \langle 0.5,0.4,0.0\rangle & \langle 0.5,0.4,0.1\rangle \\ \langle 0.0,0.0,0.5\rangle & \langle 0.5,0.3,0.1\rangle \end{bmatrix}$ and

$Z = \begin{bmatrix} \langle 0.6,0.4,0.0\rangle & \langle 0.5,0.4,0.1\rangle \\ \langle 0.0,0.0,0.5\rangle & \langle 0.6,0.3,0.1\rangle \end{bmatrix}$

be two of its GI of B of type $B\{1\}$.

Then , $X = YBZ = \begin{bmatrix} \langle 0.4,0.4,0.0\rangle & \langle 0.4,0.4,0.1\rangle \\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle \end{bmatrix}$

Then $BXB = B$ and $XBX = X$ are true.

Thus X is a SI of the PictFM B.

Theorem 5.4. For PictFM B and $X \in B\{1\}$, $X \in B\{2\}$ if and only if $R(BX) = R(X)$.

Proof: $X \in B\{2\} \Rightarrow XBX = B$ (i.e) $B \in X\{1\}$

So, $R(X) = R(BX)$ (since BX is idempotent).

On the other hand, let $R(BX) = R(B)$, then for B and X, if the BX is defined so, $R(BX) \subseteq R(X)$.

(i.e) $X = YBX$, for a few $Y \in \mathscr{P}_k$.

Thus $X(BX) = (YBX)BX$

or $XBX = Y(BXB)X = YBX = X$.

Hence $X \in B\{2\}$.

Theorem 5.5. Let $B \in \mathscr{P}_m$ be symmetric and idempotent (S and I) PictFM then B itself a LS GI.

Proof: Since B is symmetric, $B^T = B$ and B is idempotent , $B^2 = B$.

Then $BP = B$ if $P = I_n$.

then $BPB = BB = B^2 = B \quad \Rightarrow B \in B\{1\}$.

Now $(BX)^T = X^T B^T = X^T B = B^T B$

(since taking $X = B$,as B itself a GI)

$= BB = BX.$

$\Rightarrow B \in B\{1,3\}.$

Theorem 5.6. Let $B \in \mathscr{P}_m$ be a S and I PictFM then B itself a MN GI.

Proof: Here $B^T = B$ and $B^2 = B$.

For $P = I_n, PB = B$.

then $BPB = BB = B^2 = B$.

$\Rightarrow B \in B\{1\}$.

Now $(XB)^T = B^T X^T = BX^T = BB^T$

(since taking $X = B$, as B itself a GI)

$= BB = XB.$

$\Rightarrow B \in B\{1,4\}.$

Example 5.4. Let $B = \begin{bmatrix} \langle 0.4,0.4,0.1\rangle & \langle 0.3,0.4,0.2\rangle \\ \langle 0.3,0.4,0.2\rangle & \langle 0.4,0.4,0.2\rangle \end{bmatrix}$

be a symmetric PictFM,

Now

$B^2 = BB = \begin{bmatrix} \langle 0.4,0.4,0.1\rangle & \langle 0.3,0.4,0.2\rangle \\ \langle 0.3,0.4,0.2\rangle & \langle 0.4,0.4,0.2\rangle \end{bmatrix}$

$\begin{bmatrix} \langle 0.4,0.4,0.1\rangle & \langle 0.3,0.4,0.2\rangle \\ \langle 0.3,0.4,0.2\rangle & \langle 0.4,0.4,0.2\rangle \end{bmatrix}$

$= \begin{bmatrix} \langle 0.4,0.4,0.1\rangle & \langle 0.3,0.4,0.2\rangle \\ \langle 0.3,0.4,0.2\rangle & \langle 0.4,0.4,0.2\rangle \end{bmatrix}$

$= B.$

This shows that B is S and I and B satisfy the relation $BXB = B$ for $X = B$, itself.

Again $(BB)^T = (B)^2 = B = BB$.

So $B \in B\{1,3\}$ and $B\{1,4\}$.

Theorem 5.7. Let $B \in \mathscr{P}_m$ be a S and I PictFM then $\{B+G :$ for all PictFM, $G \in \mathscr{P}_m$ such that $BH \geq BG\}$ is the set of all SI of B, dominating B.

Proof: Here B is S and I PictFM, B itself is a SI.

Let $\gamma = \{B + G : \forall$ PictFM $G \in \mathscr{P}_m \ni BH \geq BG\}$.

If $H \in B\{1,3\}$, then $H \geq B$.

Let $H - B = G$ since $B\{1,3\} \subseteq B\{1\}$

$H \geq B + G \geq B.$

$\Rightarrow BH \geq B(B+G) \geq BB \geq B$ ......(1)

Now H is a SI and B itself SI so $B\{1,3\}$ consists of all solutions for X of $BX = B$ (since B is idempotent)

Thus $BH = B$.

$\Rightarrow B(B+G) = B$

(i.e) $B \geq BG$.

Now by (1) $BH \geq BG$.

$(B+G) \in \gamma$.

Thus each SI H we have a unique element in γ.

On the other side, for each $H \in \gamma, H = B+G \geq B$ with $B \geq BG$.

Hence, $BH = B + BG = B$.

$\Rightarrow H \in B\{1,3\}$.

Example 5.5. Consider the PictFM,
$B = \begin{bmatrix} \langle 0.4,0.4,0.1\rangle & \langle 0.3,0.4,0.2\rangle \\ \langle 0.3,0.4,0.2\rangle & \langle 0.4,0.4,0.2\rangle \end{bmatrix}$

Here $B^2 = B$ and $B^T = B$.

For the PictFM, $H = \begin{bmatrix} \langle 0.5,0.4,0.1\rangle & \langle 0.3,0.4,0.2\rangle \\ \langle 0.3,0.4,0.2\rangle & \langle 0.5,0.3,0.2\rangle \end{bmatrix}$.

$BHB = B$ and $(BH)^T = BH \Rightarrow H \in B\{1,3\}$.

For the PictFM , $G = \begin{bmatrix} \langle 0.3,0.4,0.2\rangle & \langle 0.2,0.3,0.4\rangle \\ \langle 0.3,0.4,0.3\rangle & \langle 0.3,0.4,0.3\rangle \end{bmatrix}$.

$BH = \begin{bmatrix} \langle 0.4,0.4,0.1\rangle & \langle 0.3,0.4,0.2\rangle \\ \langle 0.3,0.4,0.2\rangle & \langle 0.4,0.4,0.2\rangle \end{bmatrix}$ and

$BG = \begin{bmatrix} \langle 0.3,0.4,0.2\rangle & \langle 0.3,0.4,0.3\rangle \\ \langle 0.3,0.4,0.2\rangle & \langle 0.3,0.4,0.3\rangle \end{bmatrix}$

Note that, $BH \geq BG$.

Then $B + G = \begin{bmatrix} \langle 0.4,0.4,0.1\rangle & \langle 0.3,0.4,0.2\rangle \\ \langle 0.3,0.4,0.2\rangle & \langle 0.4,0.4,0.2\rangle \end{bmatrix}$

$= B \in B\{1,3\}$.

Theorem 5.8. Let $B \in \mathscr{P}_m$ be a S and I PictFM then $\{B+K :$ for all PictFM, $K \in \mathscr{P}_m$ such that $BH \geq KB\}$ is the collection of all MN inverse of B, exceeding B.

Proof: Here B is S and I PictFM, B itself a MN inverse.

Let $\gamma = \{B + K$: for all PictFM $K \in \mathscr{P}_m$ such that $BH \geq KB\}$.

If $H \in B\{1,3\}$, then $H \geq B$.

Let $H - B = K$ since $B\{1,4\} \subseteq B\{1\}$

$H \geq B + K \geq B$.

$\Rightarrow BH \geq B(B+K) \geq BB \geq B$ ......(2)

Now $H \in B\{1,4\}$ and B MN inverse so, as the set $B\{1,4\}$ contains of all solutions for X of $BX = B$ (as B is idempotent)

Thus $BH = B$.

$\Rightarrow B(B+K) = B$

(i.e)$B \geq KB$.

Now by (2) $BH \geq KB$.

(B+K) $\in \gamma$.

Hence each $H \in B\{1,4\}$ $\exists$ a unique element in γ.

In reverse, for each $H \in \gamma, H = B + K \geq B$ with $B \geq KB$.

Hence, $BH = B + KB = B$.

$\Rightarrow H \in B\{1,4\}$.

Example 5.6. Consider the PictFM,
$B = \begin{bmatrix} \langle 0.5,0.4,0.1\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.3,0.2\rangle & \langle 0.5,0.3,0.2\rangle \end{bmatrix}$

Here B^2=B and B^T=B.

For the PictFM, $H = \begin{bmatrix} \langle 0.5,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.3,0.2\rangle & \langle 0.5,0.4,0.1\rangle \end{bmatrix}$.

$BHB = B$ and $(BH)^T = BH \Rightarrow H \in B\{1,4\}$.

For the PictFM, $K = \begin{bmatrix} \langle 0.4,0.3,0.3\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.2,0.4\rangle & \langle 0.4,0.2,0.4\rangle \end{bmatrix}$.

$HB = \begin{bmatrix} \langle 0.5,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.3,0.2\rangle & \langle 0.5,0.3,0.2\rangle \end{bmatrix}$ and

$KB = \begin{bmatrix} \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.2,0.4\rangle & \langle 0.4,0.2,0.4\rangle \end{bmatrix}$

Note that, $HB \geq KB$.

Then $B + K = \begin{bmatrix} \langle 0.5,0.4,0.1\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.3,0.2\rangle & \langle 0.5,0.3,0.2\rangle \end{bmatrix}$

$= B \in B\{1,4\}$.

6 An Application of Generalized Inverse of PictFM

Here the GI of PictFMs is used to find the solution of PictF REs .

Let us consider the system of PictF REs $BX = C$

Where,

$$B = \begin{bmatrix} \langle 0.5,0.4,0.1\rangle & \langle 0.4,0.3,0.2\rangle & \langle 0.5,0.2,0.2\rangle \\ \langle 0.5,0.3,0.1\rangle & \langle 0.4,0.3,0.1\rangle & \langle 0.5,0.2,0.1\rangle \end{bmatrix},$$

$$X = \begin{bmatrix} \langle x_{1\mu}, x_{1\eta}, x_{1\nu} \rangle \\ \langle x_{2\mu}, x_{2\eta}, x_{2\nu} \rangle \\ \langle x_{3\mu}, x_{3\eta}, x_{3\nu} \rangle \end{bmatrix}$$

and $C = \begin{bmatrix} \langle 0.6, 0.2, 0.2 \rangle \\ \langle 0.5, 0.3, 0.2 \rangle \end{bmatrix}$

Each matrix X that fullfils the equation $BX = C$ is called its solution and the set $\Omega(BC) = B^{-}\mathrm{C}$ denotes the set of all solutions.

A PictFM may have many multiple GIs.

Consider one of the GIs of the PictFM B, which is

$B^{-} = \begin{bmatrix} \langle 0.6, 0.4, 0.0 \rangle & \langle 0.5, 0.3, 0.2 \rangle \\ \langle 0.5, 0.3, 0.1 \rangle & \langle 0.4, 0.2, 0.2 \rangle \\ \langle 0.5, 0.2, 0.2 \rangle & \langle 0.5, 0.2, 0.0 \rangle \end{bmatrix}$ then

$$X \in \Omega(BC) = B^{-}\mathrm{C} = \begin{bmatrix} \langle 0.6, 0.3, 0.2 \rangle \\ \langle 0.5, 0.2, 0.2 \rangle \\ \langle 0.5, 0.2, 0.2 \rangle \end{bmatrix}$$

is one of the solutions of the above system of equations.

7 Conclusions

In medical research, the generalized inverse of the picture fuzzy matrix has emerged as a valuable tool for addressing the issues posed by uncertain and inaccurate medical data. Medical personnel, on the other hand, can improve their decision-making processes, resulting in enhanced efficiency and personalized patient care. Here, we introduced the concept of SB for PictFVs. A PictFM's GI is defined, and an algorithm for determining a PictFM's GI is provided. The GI feature is not available on all PictFMs. The conditions for the existence of the GI are discussed in this article. Some PictFM GI results are investigated using relevant examples. More research is needed to determine whether GI is present in all PictFMs. Finally, there is a GI exercise at the end of the paper.

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