Mathematics and Statistics 11(4): 669-675, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110408

# Existence, Uniqueness, and Stability Results for Fractional Differential Equations with Lacunary Interpolation by the Spline Method

**Ridha G. Karem[1], Karwan H. F. Jwamer[2,*], Faraidun K. Hamasalh[3]**

[1]Department of Mathematics, College of Basic Education, University of Sulaimani, Iraq
[2]Department of Mathematics, College of Science, University of Sulaimani, Iraq
[3]Department of Mathematics, College of Education, University of Sulaimani, Iraq

*Received April 16, 2023; Revised June 30, 2023; Accepted July 20, 2023*

***Cite This Paper in the Following Citation Styles***

***(a):*** *[1] Ridha G. Karem, Karwan H. F. Jwamer, Faraidun K. Hamasalh , "Existence, Uniqueness, and Stability Results for Fractional Differential Equations with Lacunary Interpolation by the Spline Method," Mathematics and Statistics, Vol. 11, No. 4, pp. 669 - 675, 2023. DOI: 10.13189/ms.2023.110408.*

***(b):*** *Ridha G. Karem, Karwan H. F. Jwamer, Faraidun K. Hamasalh (2023). Existence, Uniqueness, and Stability Results for Fractional Differential Equations with Lacunary Interpolation by the Spline Method. Mathematics and Statistics, 11(4), 669 - 675. DOI: 10.13189/ms.2023.110408.*



**Abstracts** Although there are theoretical conclusions about the existence, uniqueness, and properties of solutions to ordinary and partial differential equations, only the simplest and most straightforward particular problems can usually be solved explicitly, especially when nonlinear terms are involved, and we typically develop approximation. In order to resolve the form problem of fractional order beginning value (1) by lacunary interpolation with a fractional degree spline function, the main goal of this paper is to investigate and improve some approximate solutions as well as new approximate solution techniques that have been proposed for the first time. From a practical standpoint, the numerical solution of these differential equations is crucial because only a tiny portion of equations can be resolved analytically. For fractional differential equations that are sensitive to the beginning conditions, we provide a fractional spline approach. The polynomial coefficient-based spline interpolation must be constructed using the Caputo fractional integral and derivative. For the given spline function, a stability analysis is completed after investigating error boundaries. The numerical rationale for the suggested technique is thought to use three cases. The outcomes demonstrate how effective the spline fractional technique is in interpolating the coefficient with fractional polynomials. Finally, to demonstrate the effectiveness and correctness of the suggested strategy, general procedure programs are created in MATLAB and used to a number of instructive cases.

**Keywords** Stability Analysis, Fractional Polynomial, Fractional Derivative

## 1. Introduction

Spline capacities can be connected to numerical approximations of Standard fractional differential conditions and indispensably conditions as well as lacunary insertion, addition, and information fitting. From [1], starting esteem issues happen in numerous areas of the sciences and designing, so a numerical arrangement is required instead of an expository one. (See [2], and [3]) different spline capacities have been utilized already by creators as an arrangement for introductory esteem issues. This paper assists to create these thoughts in order to solve fragmentary starting esteem issues.

A spline may be a specific work characterized piecewise utilizing polynomials in science. Spline introduction is for the most part favored to polynomial addition in inserting issues since it provides comparative results when utilizing moo polynomials of first-order while avoiding Rung's influence of more advanced degrees. A fragmentary subordinate with the Caputo and Reimann-Liouville equation could be a subordinate of any non-integer arrange, genuine or complex, and is profitable for modeling energetic wonders within the science and innovation fields. However, for some difficulties the fractional differential equations cannot provide results. Therefore, instead of fractional order methods, analytical or numerical methods are used to obtain the best approximation. To get the leading approximations, explanatory or numerical techniques are used (see [5, 6]). The created spline polynomial in the first section, the error boundaries in the second section, and the solidity examination in the third section may all be considered of as being repeated in the current article. It has been looked at how the third component, which presents our numerical data compiled, results from proper arrangements. This section looks to be the outcomes and conclusions of the thought, (See [1], [7], [9], and [13]).

The construction and application of a novel class $C^{\frac{9}{2}}$-approximation approach are used with the lacunary spline to solve a fractional beginning value problem numerically.

$$y^{\left(\frac{3}{2}\right)}(x) = f(x,y), \qquad y(x_0) = y_0, \quad y'(x_0) = y'_0, \;\; x \in [0,1] \tag{1}$$

## 2. Preliminaries

Alternative definitions for Taylor's theorem and fractional derivatives, both of which we used in our study, are provided in this section. The most common methods for defining fractional derivatives are the Riemann-Liouville and Caputo derivatives (see [4, 5, and 6]).

**Definition 2.1:** Assume that $\alpha > 0, x > \beta, \alpha, x, \beta \in \mathbb{R}$. defines the Caputo fractional derivative of order $\alpha > 0$ by [4,5]

$$D_\beta^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_\beta^x \dfrac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}}\, ds \;, m-1 < \alpha < m, m \in \mathbb{N} \\ \dfrac{d^m}{dx^m} f(x)\,, \alpha = m, m \in \mathbb{N}. \end{cases}$$

**Definition 2.2:** Assume that $\alpha > 0, x > \beta, \alpha, x, \beta \in \mathbb{R}$. defines the Riemann-Liouville fractional derivative of order $\alpha > 0$ by [6,7]

$$D_\beta^\alpha g(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dx^m} \displaystyle\int_\beta^x \dfrac{g(t)}{(x-t)^{\alpha+1-m}}\, ds\,, m-1 < \alpha < m, m \in \mathbb{N} \\ \dfrac{d^m}{dx^m} g(x)\,, \alpha = m, m \in \mathbb{N}. \end{cases}$$

**Definition 2.3:** Assume that $D_\beta^{z\,\alpha}\, G(x) \in \mathbb{C}[a,b]$ for $z = 0,1,\dots,m+1$, and $0 < \alpha \le 1$. Then we have the Taylor series expansion about $x = \tau$

$$g(x) = \sum_{i=0}^{m} \frac{(x-\tau)^{i\,\alpha}}{\Gamma(i\,\alpha+1)} D_\beta^{i\,\alpha} g(\tau) + \frac{\left(D_\beta^{(m+1)\alpha} g\right)(\varphi)}{\Gamma((m+1)\alpha+1)} (x-\tau)^{(m+1)\alpha}, \text{ with } \beta \le \varphi \le x, \text{ for all } x \in [a,b]$$

where $D_\beta^{z\,\alpha} = D_\beta^{\alpha} . D_\beta^{\alpha} \dots D_\beta^{\alpha}$ (z times) [5].

## 3. Theories relating to the Spline Technique

Create an approximation of the fractional differential equation solution in this part, taking stability and error boundaries into account in relation to the subsequent theorems. This is accomplished by combining a spline function with the fractional polynomial of the coefficient.

**Theorem 3.1:** Given the actual figures $D^{\left(\frac{3}{2}\right)} S_j, j = 1,,\dots, M-1, S_0, D^{\left(\frac{1}{2}\right)} S_0$ and $D' S_0$, then there exists a unique spline $S(x) \in S_{\left(m,\frac{9}{2}\right)}$, like that:

$$S(x_0) = f(x_0),$$

$$D^{\left(\frac{3}{2}\right)} S_i(x) = D^{\left(\frac{3}{2}\right)} f_i(x)\,, \;\; i = 0,1,2,\dots,n \;\; \text{and} \tag{2}$$

$D^{\left(\frac{1}{2}\right)} S(x) = D^{\left(\frac{1}{2}\right)} f(x)$, where $x = x_0, x_n$.

Proof:

We created the construction spline function from [1,10,11] utilizing the spline function with the [0,1] range and the fractional polynomial $S(x) \in S_{(m,\frac{9}{2})}$:

$$P(x) = P(0)\, A_0(x) + P^{(\frac{1}{2})}(0)A_1(x) + P^{(\frac{5}{2})}(0)A_2(x) + P^{(\frac{3}{2})}(0)A_3(x) + P^{(\frac{3}{2})}(\lambda)A_4(x) + P^{(\frac{7}{2})}(0)A_5(x).$$

Where

$$A_0(x) = 1,$$
$$A_1(x) = \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}},$$
$$A_2(x) = \frac{8}{15\sqrt{\pi}} x^{\frac{5}{2}} - \frac{64}{315\sqrt{\pi}\lambda^2} x^{\frac{9}{2}},$$
$$A_3(x) = \frac{4}{3\sqrt{\pi}} x^{\frac{3}{2}} - \frac{64}{315\lambda^3\sqrt{\pi}} x^{\frac{9}{2}},$$
$$A_4(x) = \frac{64}{315\lambda^3\sqrt{\pi}} x^{\frac{9}{2}}$$
$$A_5(x) = \frac{16}{105\sqrt{\pi}} x^{\frac{7}{2}} - \frac{32}{315\lambda\sqrt{\pi}} x^{\frac{9}{2}} \quad (3)$$

Consider the step size $= x_j + t\lambda h\ 0 \le t \le 1$, use the same wording for $S(x)$ in $[x_{j-1}, x_j]$. Since $S(x) \in C^{\frac{9}{2}}$ and $S(x_j^+) = S(x_j^-)$ to $D^{(\frac{7}{2})}S(x_j^-) = D^{(\frac{7}{2})}S(x_j^+)$, respectively, for $j = 1,, \dots, N-1$, leads to the following linear system of equations:

$$S_i = S_{i-1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} S_{i-1}^{(\frac{1}{2})} + \frac{104}{315\sqrt{\pi}} h^{\frac{5}{2}} S_{i-1}^{(\frac{5}{2})} + \frac{356}{315\sqrt{\pi}} h^{\frac{3}{2}} S_{i-1}^{(\frac{3}{2})} + h^{\frac{3}{2}} \frac{64}{315\sqrt{\pi}} S_{i-1+\lambda}^{(\frac{3}{2})} + \frac{16}{315\sqrt{\pi}} h^{\frac{7}{2}} S_{i-1}^{(\frac{7}{2})} \quad (4)$$

$$h^{\frac{1}{2}} S_i^{(\frac{1}{2})} = h^{\frac{1}{2}} S_{i-1}^{(\frac{1}{2})} + h^{\frac{5}{2}} \left(\frac{2\lambda^2-1}{4\lambda^2}\right) S_{i-1}^{(\frac{5}{2})} + h^{\frac{3}{2}} \left(\frac{4\lambda^3-1}{4\lambda^3}\right) S_{i-1}^{(\frac{3}{2})} + h^{\frac{3}{2}} \left(\frac{1}{4\lambda^3}\right) S_{i-1+\lambda}^{(\frac{3}{2})} + h^{\frac{7}{2}} (\frac{8\lambda-6}{48\lambda}) S_{i-1}^{(\frac{7}{2})} \quad (5)$$

$$h^{\frac{3}{2}} S_i^{(\frac{3}{2})} = \frac{\lambda^2-1}{\lambda^2} h^{\frac{5}{2}} S_{i-1}^{(\frac{5}{2})} + \frac{\lambda^3-1}{\lambda^3} h^{\frac{3}{2}} S_{i-1}^{(\frac{3}{2})} + \frac{1}{\lambda^3} h^{\frac{3}{2}} S_{i-1+\lambda}^{(\frac{3}{2})} + \frac{\lambda-1}{2\lambda} h^{\frac{7}{2}} S_{i-1}^{(\frac{7}{2})} \quad (6)$$

$$h^{\frac{5}{2}} S_i^{(\frac{5}{2})} = \frac{\lambda^2-3}{\lambda^2} h^{\frac{5}{2}} S_{i-1}^{(\frac{5}{2})} - \frac{3}{\lambda^3} h^{\frac{3}{2}} S_{i-1}^{(\frac{3}{2})} + \frac{3}{\lambda^3} h^{\frac{3}{2}} S_{i-1+\lambda}^{(\frac{3}{2})} + \frac{2\lambda-3}{2\lambda} h^{\frac{7}{2}} S_{i-1}^{(\frac{7}{2})}, \quad (7)$$

$$h^{\frac{7}{2}} S_i^{(\frac{7}{2})} = -\frac{6}{\lambda^2} h^{\frac{5}{2}} S_{i-1}^{(\frac{5}{2})} - \frac{6}{\lambda^3} h^{\frac{3}{2}} S_{i-1}^{(\frac{3}{2})} + \frac{6}{\lambda^3} h^{\frac{3}{2}} S_{i-1+\lambda}^{(\frac{3}{2})} + \frac{\lambda-3}{\lambda} h^{\frac{7}{2}} S_{i-1}^{(\frac{7}{2})} \quad (8)$$

The proof of the theorem completed.

**Corollary:** Let $\lambda = 1$ in **Theorem 3.1**, then the following system is stable.

**Proof:** Directly from the equation (4), (5), (6), (7) and (8) we obtain the following system:

$$S_i = S_{i-1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} S_{i-1}^{(\frac{1}{2})} + \frac{104}{315\sqrt{\pi}} h^{\frac{5}{2}} S_{i-1}^{(\frac{5}{2})} + \frac{420}{315\sqrt{\pi}} h^{\frac{3}{2}} S_{i-1}^{(\frac{3}{2})} + \frac{16}{315\sqrt{\pi}} h^{\frac{7}{2}} S_{i-1}^{(\frac{7}{2})} \quad (9)$$

$$h^{\frac{1}{2}} S_i^{(\frac{1}{2})} = h^{\frac{1}{2}} S_{i-1}^{(\frac{1}{2})} + \frac{1}{4} h^{\frac{5}{2}} S_{i-1}^{(\frac{5}{2})} + h^{\frac{3}{2}} S_{i-1}^{(\frac{3}{2})} + \frac{1}{24} h^{\frac{7}{2}} S_{i-1}^{(\frac{7}{2})} \quad (10)$$

$$h^{\frac{5}{2}} S_i^{(\frac{5}{2})} = -2\, h^{\frac{5}{2}} S_{i-1}^{(\frac{5}{2})} - \frac{1}{2} h^{\frac{7}{2}} S_{i-1}^{(\frac{7}{2})} \quad (11)$$

$$h^{\frac{7}{2}} S_i^{(\frac{7}{2})} = -6\, h^{\frac{5}{2}} S_{i-1}^{(\frac{5}{2})} - 2\, h^{\frac{7}{2}} S_{i-1}^{(\frac{7}{2})}, \quad (12)$$

For $i = 1,, \dots, N-1$

And in the section of stability analysis, we will show that the above system will be stable.

**Hint:** We may demonstrate the existence and originality of the spline method model using the aforementioned theorem. Additional evidence that the spline technique construction's convergence analysis is accurate comes from the following theorems:

**Theorem 3.2:** Assume that the spline fractional described in Theorem 3.1, S(x), is correct. $D^{(\frac{1}{2})}f, D^{(\frac{3}{2})}f \in C^{\frac{7}{2}}[0,1]$ and that $D^{(j)}f(0) = 0,\ j = 1,2$ then for any $x \in [0,1]$ we have

$$\left| S_i^{(\frac{7}{2})} - y_i^{(\frac{7}{2})} \right| \le \left| \frac{\lambda-3}{\lambda} - 1 \right| y_{i-1}^{(\frac{7}{2})} + \left| \frac{6}{\lambda^2} h^{-1} \right| y_{i-1}^{(\frac{5}{2})} \quad (13)$$

$$\left| S_i^{(\frac{5}{2})} - y_i^{(\frac{5}{2})} \right| \le \left| \frac{\lambda^2-3}{\lambda^2} - 1 \right| y_{i-1}^{(\frac{5}{2})} + \left| \frac{2\lambda-3}{2\lambda} - 1 \right| h y_{i-1}^{(\frac{7}{2})} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{i-1}^{(3)} \quad (14)$$

$$\left|S_i^{\left(\frac{3}{2}\right)} - y_i^{\left(\frac{3}{2}\right)}\right| \le \left|\frac{\lambda^2-1}{\lambda^2} - 1\right| h y_{i-1}^{\left(\frac{5}{2}\right)} + \left|\frac{\lambda-1}{2\lambda} - \frac{1}{2}\right| h^2 y_{i-1}^{\left(\frac{7}{2}\right)} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{i-1}^{(2)} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{i-1}^{(3)}. \tag{15}$$

**Proof**: Since

$$h^{\frac{3}{2}} S_i^{\left(\frac{3}{2}\right)} = \frac{\lambda^2-1}{\lambda^2} h^{\frac{5}{2}} S_{i-1}^{\left(\frac{5}{2}\right)} + \frac{\lambda^3-1}{\lambda^3} h^{\frac{3}{2}} S_{i-1}^{\left(\frac{3}{2}\right)} + \frac{1}{\lambda^3} h^{\frac{3}{2}} S_{i-1+\lambda}^{\left(\frac{3}{2}\right)} + \frac{\lambda-1}{2\lambda} h^{\frac{7}{2}} S_{i-1}^{\left(\frac{7}{2}\right)}, \tag{16}$$

$$h^{\frac{5}{2}} S_i^{\left(\frac{5}{2}\right)} = \frac{\lambda^2-3}{\lambda^2} h^{\frac{5}{2}} S_{i-1}^{\left(\frac{5}{2}\right)} - \frac{3}{\lambda^3} h^{\frac{3}{2}} S_{i-1}^{\left(\frac{3}{2}\right)} + \frac{3}{\lambda^3} h^{\frac{3}{2}} S_{i-1+\lambda}^{\left(\frac{3}{2}\right)} + \frac{2\lambda-3}{2\lambda} h^{\frac{7}{2}} S_{i-1}^{\left(\frac{7}{2}\right)}, \tag{17}$$

$$h^{\frac{7}{2}} S_i^{\left(\frac{7}{2}\right)} = \frac{\lambda-3}{\lambda} h^{\frac{7}{2}} S_{i-1}^{\left(\frac{7}{2}\right)} - \frac{6}{\lambda^2} h^{\frac{5}{2}} S_{i-1}^{\left(\frac{5}{2}\right)} - \frac{6}{\lambda^3} h^{\frac{3}{2}} S_{i-1}^{\left(\frac{3}{2}\right)} + \frac{6}{\lambda^3} h^{\frac{3}{2}} S_{i-1+\lambda}^{\left(\frac{3}{2}\right)}. \tag{18}$$

Then, using the Taylor series for the analytic function y(x), and after some simplifications, we obtain:

$$\left|S_i^{\left(\frac{7}{2}\right)} - y_i^{\left(\frac{7}{2}\right)}\right| \le \left|\frac{\lambda-3}{\lambda} - 1\right| y_{i-1}^{\left(\frac{7}{2}\right)} + \left|\frac{6}{\lambda^2} h^{-1}\right| y_{i-1}^{\left(\frac{5}{2}\right)}, \tag{19}$$

$$\left|S_i^{\left(\frac{5}{2}\right)} - y_i^{\left(\frac{5}{2}\right)}\right| \le \left|\frac{\lambda^2-3}{\lambda^2} - 1\right| y_{i-1}^{\left(\frac{5}{2}\right)} + \left|\frac{2\lambda-3}{2\lambda} - 1\right| h y_{i-1}^{\left(\frac{7}{2}\right)} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{i-1}^{(3)}, \tag{20}$$

$$\left|S_i^{\left(\frac{3}{2}\right)} - y_i^{\left(\frac{3}{2}\right)}\right| \le \left|\frac{\lambda^2-1}{\lambda^2} - 1\right| h y_{i-1}^{\left(\frac{5}{2}\right)} + \left|\frac{\lambda-1}{2\lambda} - \frac{1}{2}\right| h^2 y_{i-1}^{\left(\frac{7}{2}\right)} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{i-1}^{(2)} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} y_{i-1}^{(3)}. \tag{21}$$

# 4. Stability Analysis

Study is being done on the provided techniques (9), (10), (11) and (12) in order to assess its stability study and a means of putting the equation to the test.

$$\text{If } S_i^{\left(\frac{3}{2}\right)} = \beta^{\frac{3}{2}} S_i,\ \beta \in \mathbb{R},\ y(x_0) = y_0, y'(x_0) = {y'}_0, \tag{22}$$

$$S_i = S_{i-1} + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} S_{i-1}^{\left(\frac{1}{2}\right)} + \frac{104}{315\sqrt{\pi}} h^{\frac{5}{2}} S_{i-1}^{\left(\frac{5}{2}\right)} + \frac{420}{315\sqrt{\pi}} h^{\frac{3}{2}} \beta^{\frac{3}{2}} S_{i-1} + \frac{16}{315\sqrt{\pi}} h^{\frac{7}{2}} S_{i-1}^{\left(\frac{7}{2}\right)}$$

$$h^{\frac{1}{2}} S_i^{\left(\frac{1}{2}\right)} = h^{\frac{1}{2}} S_{i-1}^{\left(\frac{1}{2}\right)} + \frac{1}{4} h^{\frac{5}{2}} S_{i-1}^{\left(\frac{5}{2}\right)} + h^{\frac{3}{2}} \beta^{\frac{3}{2}} S_{i-1} + \frac{1}{24} h^{\frac{7}{2}} S_{i-1}^{\left(\frac{7}{2}\right)},$$

$$h^{\frac{5}{2}} S_i^{\left(\frac{5}{2}\right)} = -2\, h^{\frac{5}{2}} S_{i-1}^{\left(\frac{5}{2}\right)} - \frac{1}{2} h^{\frac{7}{2}} S_{i-1}^{\left(\frac{7}{2}\right)},$$

$$h^{\frac{7}{2}} S_i^{\left(\frac{7}{2}\right)} = -6\, h^{\frac{5}{2}} S_{i-1}^{\left(\frac{5}{2}\right)} - 2\, h^{\frac{7}{2}} S_{i-1}^{\left(\frac{7}{2}\right)},$$

$$S_i = K S_{i-1},\ I = 1, \ldots, N-1 \tag{23}$$

$$\text{where } S_i = \begin{bmatrix} S_i \\ S_i^{\left(\frac{1}{2}\right)} \\ S_i^{\left(\frac{5}{2}\right)} \\ S_i^{\left(\frac{7}{2}\right)} \end{bmatrix} \text{ and } S_{i-1} = \begin{bmatrix} S_{i-1} \\ S_{i-1}^{\left(\frac{1}{2}\right)} \\ S_{i-1}^{\left(\frac{5}{2}\right)} \\ S_{i-1}^{\left(\frac{7}{2}\right)} \end{bmatrix}$$

$$K = \begin{pmatrix} (1 + \frac{420}{315\sqrt{\pi}} h^{\frac{3}{2}} \beta^{\frac{3}{2}}) & \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} & \frac{104}{315\sqrt{\pi}} h^{\frac{5}{2}} & \frac{16}{315\sqrt{\pi}} h^{\frac{7}{2}} \\ h^{\frac{3}{2}} \beta^{\frac{3}{2}} & h^{\frac{1}{2}} & \frac{1}{4} h^{\frac{5}{2}} & \frac{1}{24} h^{\frac{7}{2}} \\ 0 & 0 & -2\, h^{\frac{5}{2}} & -\frac{1}{2} h^{\frac{7}{2}} \\ 0 & 0 & -6\, h^{\frac{5}{2}} & -2\, h^{\frac{7}{2}} \end{pmatrix}$$

**Theorem 4.1:** If $K$ has n independent columns, $K^{-1}$ exists, and $Kx = y$ has a unique solution u, then $K$ is non-singular.

**Proof:** Since we have a matrix K from the linear system of equation (22), if $|K| \neq 0$, then $K^{-1}$ is exists, and the system is a unique solution [12].

**Theorem 4.2:** The system of equations (23), using the fractional spline method, is stable.

**Proof:**

Assume that $|\lambda_i| \leq 1$ is true and that T is a complex conjugate matrix. If all complex eigenvalues have negative real components, as the characteristic polynomial, continued stability of the characteristic equation is expected [12].

## 5. Numerical Results

Using the methodology demonstrated in the two numerical examples in this section, all calculations are completed. To specify the spline for fractional interpolation class $C^{\frac{7}{2}}$ and confirm the mathematical viability of the suggested approach, three fractional initial value problems are taken into consideration. The usage of the results in these two contexts highlights the value of the suggested method. The terms $e, e^{(\frac{1}{2})}$ and $e^{(1)}$ indicate the biggest mistakes in magnitude.

$$|e(x)| = |S(x) - f(x)| \, , \left|D^{\left(\frac{1}{2}\right)}e(x)\right| = \left|D^{\left(\frac{1}{2}\right)}S(x) - D^{\left(\frac{1}{2}\right)}y(x)\right| \, ,$$

and

$$|D'e(x)| = |D'S(x) - D'y(x)| \text{ , respectively.}$$

**Example 1:** Think about the differential equation for fractions as [14-15]

$$D^2y(x) - x^2D^{(3/2)}y(x) - \sqrt{x}D^{(1/2)}y(x) - x^{1/3}y(x) = 6\sqrt{\pi}x - \frac{16}{5}x^3 - x^{10/3}\sqrt{\pi}$$

$$y(0) = y'(0) = 0 \, , 0 \leq x \leq 1$$

The exact solution is given by $(x) = \sqrt{\pi}\, x^3$.

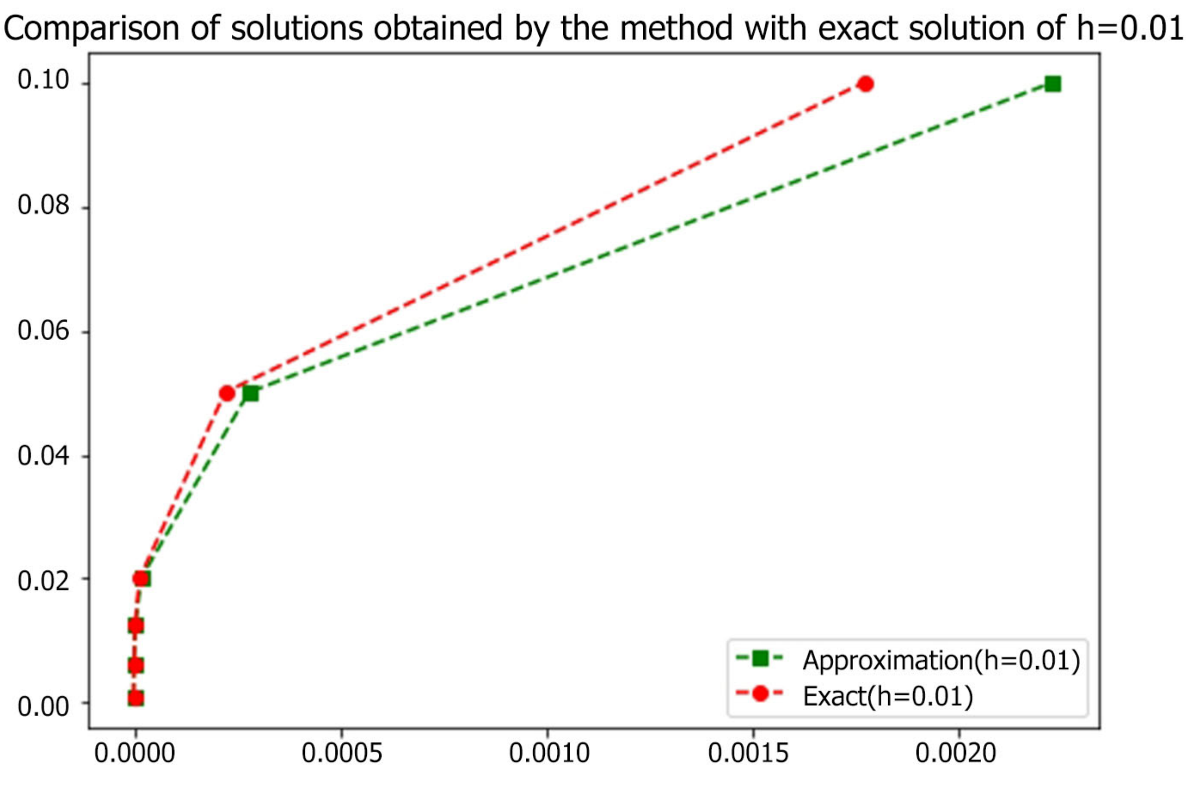


**Figure 1.** Numerical Result for Example 1

**Table 1.** Absolute error of $S(x)$, and its derivative of example 1.

| X | $\|S(x)-f(x)\|$ | $\left\|S^{(\frac{1}{2})}(x)-f^{(\frac{1}{2})}(x)\right\|$ | $\|S'(x)-f'(x)\|$ | Exact Solution | Approximate Solution |
|---|---|---|---|---|---|
| 0.001 | 4.58515966 $10^{-10}$ | 8.030086432 $10^{-8}$ | 7.509641214 $10^{-6}$ | 1.77245385 $10^{-9}$ | 2.230969817 $10^{-9}$ |
| 0.00625 | 1.11942071 $10^{-7}$ | 7.841896410 $10^{-6}$ | 2.933458773 $10^{-4}$ | 4.32727991 $10^{-7}$ | 5.446700614 $10^{-7}$ |
| 0.0125 | 8.95525121 $10^{-7}$ | 4.436086768 $10^{-5}$ | 1.173393246 $10^{-3}$ | 3.46182393 $10^{-6}$ | 4.357349048 $10^{-6}$ |
| 0.02 | 3.66794346 $10^{-6}$ | 1.436516788 $10^{-4}$ | 3.003954483 $10^{-3}$ | 1.41796308 $10^{-5}$ | 1.784757426 $10^{-5}$ |
| 0.05 | 5.72860289 $10^{-5}$ | 1.420032967 $10^{-3}$ | 1.878015871 $10^{-2}$ | 2.21556732 $10^{-4}$ | 2.788427602 $10^{-4}$ |
| 0.1 | 4.57227644 $10^{-4}$ | 8.046113568 $10^{-3}$ | 7.523344280 $10^{-2}$ | 1.77245385 $10^{-3}$ | 2.229681494 $10^{-3}$ |

**Example2:** Consider the following nonlinear FIVP

$$D^{(3/2)}y(t)+D^{(2)}y(t)=\frac{\Gamma(6)}{\Gamma(4.5)}\,t^{7/2}-\frac{3\Gamma(5)}{\Gamma(3.5)}t^{\frac{5}{2}}+\frac{\Gamma(4)}{\Gamma(2.5)}t^{\frac{3}{2}}+[t^5-3t^4+2t^3]^2$$

The precise answer is $(t)=t^5-3t^4+2t^3$, with the initial condition being $y(0)=0=y'(0)$

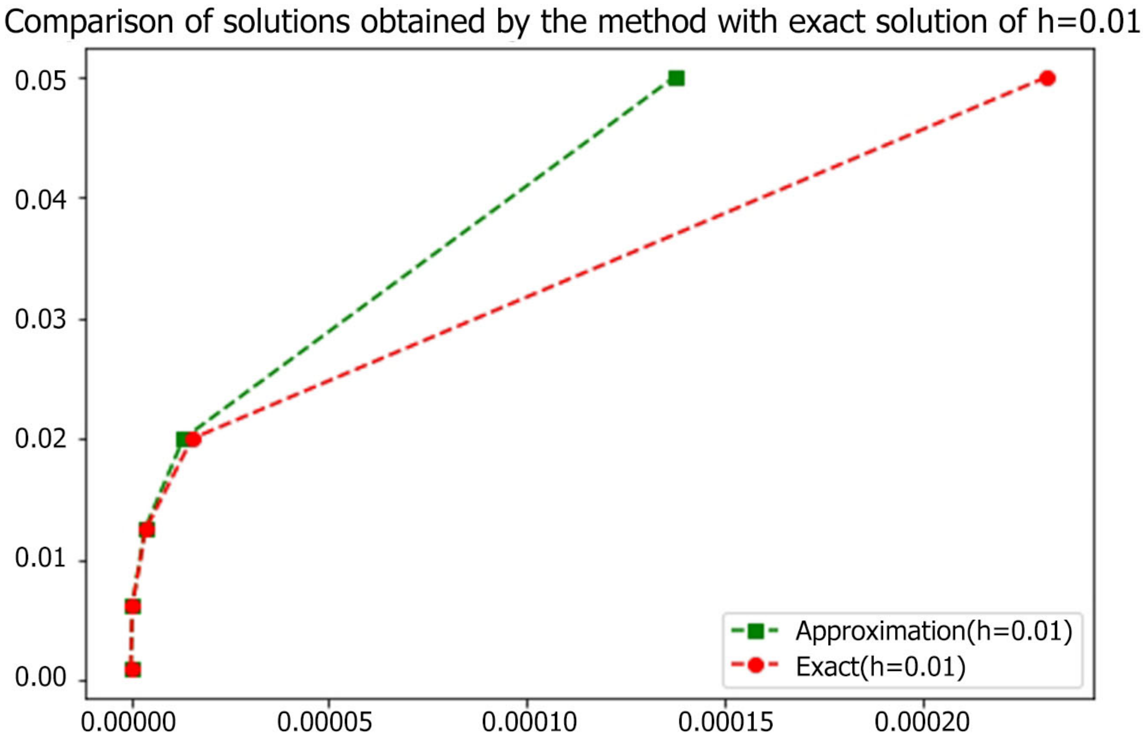


**Figure 2.** Numerical Result for Example 2

**Table 2.** Absolute error of $S(x)$, and its derivative of example 2.

| X | $\|S(x)-f(x)\|$ | $\left\|S^{(\frac{1}{2})}(x)-f^{(\frac{1}{2})}(x)\right\|$ | $\|S'(x)-f'(x)\|$ | Exact Solution | Approximate Solution |
|---|---|---|---|---|---|
| 0.001 | 1.864171816 $10^{-10}$ | 6.301350199 $10^{-7}$ | 1.957907717 $10^{-5}$ | 1.99700100 $10^{-9}$ | 2.183418181 $10^{-9}$ |
| 0.00625 | 3.653050043 $10^{-9}$ | 6.011771117 $10^{-5}$ | 7.420715087 $10^{-4}$ | 4.83713150 $10^{-7}$ | 4.873662000 $10^{-7}$ |
| 0.0125 | 2.503828939 $10^{-7}$ | 3.310442872 $10^{-4}$ | 2.864594028 $10^{-3}$ | 3.83331298 $10^{-6}$ | 3.582930094 $10^{-6}$ |
| 0.02 | 2.194090785 $10^{-6}$ | 1.037813935 $10^{-3}$ | 7.022271067 $10^{-3}$ | 1.55232000 $10^{-5}$ | 1.332910921 $10^{-5}$ |
| 0.05 | 9.406902677 $10^{-5}$ | 8.964207383 $10^{-3}$ | 3.645376718 $10^{-2}$ | 2.31562500 $10^{-4}$ | 1.374934732 $10^{-4}$ |
| 0.1 | 1.383344970 $10^{-3}$ | 3.942630170 $10^{-2}$ | 1.002948476 $10^{-1}$ | 1.71000000 $10^{-3}$ | 3.266550298 $10^{-4}$ |

# 6. Conclusions

Rarely has the spline function been investigated. As a result, it is advised to use the fractional spline interpolation function as a method for identifying fractional beginning value issues. The last two examples and figures and tables demonstrated that the proposed algorithm can deliver positive results and also produces a decent approximation to the given answer, demonstrating that a small step size h must be chosen.

This recently developed algorithm targets the approximation of the higher order derivatives as well as the approximation of the fractional initial value problems. Additionally, a spline technique based on the parameter $\lambda \in [0,1]$, is constructed, and $\lambda = 1$ is taken and stability work is done on it.

# REFERENCES


[1] Faraidun K. Hamasalh, Amina H. Ali, "Stability Analysis of Some Fractional Differential Equations by Special type of Spline Function", Journal Zanko Sulaimani-Part A., Vol. 19, no. 1, pp. 191-196, 2017, doi.org/10.17656/jzs.10596.

[2] Srivastava, R., "On Lacunary Interpolation through g-Splines", International Journal of Innovative Research in Science, Engineering and Technology, Vol. 4, No. 6, pp. 4667-4670, 2015, DOI:10.15680/IJIRSET.2015.0406118.

[3] Svante, W., "Spline function in data analysis", Technimetrics, American Statistical Association, Vol. 16, No. 1, pp. 1-11, 1974, doi.org/10.2307/1267485.

[4] Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J. "Fractional Calculus: Models and Numerical Methods", World Scientific: Singapore, Volume 3, 2012, http://dx.doi.org/10.1142/10044.

[5] Daftardar-Gejji, V. "Fractional Calculus and Fractional Differential Equations"; Springer, Gateway East, 2019.

[6] Athassawat Kammanee, "Numerical Solution of Fractional Differential Equations with variable Coefficients by Taylor Basis Functions", Kyungpook Mathematical Journal, KYUNGPOOK Math. J., Vol. 61, 2021, pp. 383-393, doi.org/10.5666/KMJ.2021.61.2.383.

[7] Deo, S. G., Lakshmlkantham, V. and Raghavendra, V. "Textbook of Ordinary Differential Equations", 2nd Edition, Megraw Hill Education, 1997.

[8] E. L Tarazi, M. N. and Karaballi, A. A., "On Even-Degree Splines with Application to quadratures", Journal of approximation theory, Vol. 60, No. 2, pp. 157-167, 1990, doi.org/10.1016/0021-9045(90)90080-A.

[9] Ghosh, U. Sarkar, S. and Das, S. "Solution of System of Linear Fractional Differential Equations with Modified Derivative of Jumarie Type", American Journal of Mathematical Analysis, Vol. 3, No. 3, pp. 72-84, 2015, DOI: 10.12691/ajma-3-3-3.

[10] Birkhoff, G. and Priver, A., "Hermite interpolation errors for derivatives", J Math Phys. Vol. 46, pp. 440-447, 1967, doi.org/10.1002/sapm1967461440.

[11] Varma, A. and Howell, G., "Best error bounds for derivatives in two point Birkhoff interpolation problems", J. Approx. Theory. Vol. 38, pp. 258-268, 1983, doi.org/10.1016/0021-9045(83)90132-6.

[12] Lipschutz, S., "Theory and problem of linear algebra", Schaum Publishing Co (McGraw-Hill) 1st.ed. (1968).

[13] Maleknejad, Khosrow and Rashidinia, Jalil and Jalilian, Hamed, "Quintic Spline functions and Fredholm integral equation", Computational Methods for Differential Equations, Vol. 9, no. 1, 2021, pp. 211-224, doi. 10.22034/CMDE.2019.31983.1492.

[14] Faraidun K. Hamasalh, "Fractional Polynomial Spline for solving Differential Equations of Fractional Order", Math. Sci. Lett. 4, No. 3, pp. 291-296, 2015, http://dx.doi.org/10.12785/msl/040312.

[15] Faraidun K. Hamasalh , Seaman S. Hamasalh, "Spline Fractional Polynomial for Computing Fractional Differential Equations", Journal of University of Babylon for Pure and Applied Sciences, Vol. 30, No. 2, pp. 68-79, 2022, https://doi.org/10.29196/jubpas.v30i2.4185