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NE-Nil Clean Rings and Their Generalization

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Abstract This article presents the concept of a NE-nil clean ring, which is a generalization of the strongly nil clean ring. A ring R is considered NE-nil clean if, for every a in R, there exists a_1 in R such that $aa_1 = \delta$ with $a - a_1 = q$ and $a_1 q = qa_1$, where q is nilpotent and δ is idempotent. This article's aim is to introduce a new type of ring, the NE-nil clean ring, and provide the fundamental properties of this ring. We also establish the relationship between NE-nil clean rings and 2-Boolean rings. Additionally, we demonstrate that the Jacobson radical $\mathcal{J}(\mathcal{R})$ and the right singular ideal $\gamma(\mathcal{R})$ over NE-nil clean ring are nil ideals. Among other results, we prove that every strongly nil clean ring and every weak * nil clean ring are NE-nil clean. We establish that a strongly 2-nil clean ring and NE-nil clean ring are equivalent. Furthermore, we introduce and investigate NT-nil clean ring, that is a ring with every a in R, there exists a_1 in R such that $aa_1 = t$ with $a - a_1 = q$ and $a_1 q = qa_1$, where t is a tripotent and q is nilpotent, by showing that these rings are a generalization of NE-nil clean rings. We provide the basic properties of these rings and explore their relationship with NE-nil clean and Zhou rings.

Keywords Idempotents, Tripotents, Strongly Nil Clean, Strongly 2-nil Clean

1 Introduction

We assume that all rings are associative with an identity. To indicate the set of units in R, we use the symbol $\mathcal{UN}(\mathcal{R})$. Additionally, $\mathcal{ID}(\mathcal{R})$ represents the set of idempotents, while $\mathcal{NL}(\mathcal{R})$ denotes the set of nilpotents. We also write Z_n to refer to the ring of integers modulo n, $\mathcal{J}(\mathcal{R})$ for the Jacobson radical of $\mathcal{R}$, and $Tr(\mathcal{R})$ for the set of tripotent elements in $\mathcal{R}$.

In 1977, W.K. Nicholson [1] stated the notion of a clean ring, that is a ring where every element of $\mathcal{R}$ is a sum of an idempotent and a unit. Later in 1999, Nicholson [2] defined the strongly clean ring if the idempotent and the unit commute. For several years many authors worked in this field (c.f. for example) [3], [4] and [5].

In 2013, Diesl [6] gave the idea of nil clean ring, which is a ring with each element is a sum of an idempotent and a nilpotent, and he proved that every nil clean is a clean ring, later in 2016. A strongly nil clean ring was proposed by T. Kosan [7] as a ring $\mathcal{R}$ where each member may be written as the sum of an idempotent and a nilpotent that commute. For several years, nil, strongly nil clean ring and related topics have drawn the attention of many authors (c.f. for example), [8], [9] and [10].

A strongly 2-nil clean ring was introduced by Chen and Sheibani [11] in 2017, as a ring with every element of $\mathcal{R}$ is a sum of two idempotents and nilpotent that commute. Indeed, this ring generalizes a strongly nil clean ring. This class of rings are studied by many authors (c.f. for example). [12] and [13].

2 Preliminaries

We begin with some well-known definitions and results, which we need it in the present work.

Definition 2.1. *[1]. An element a in a ring $\mathcal{R}$ is assumed to be clean, if it can be expressed as a sum of an idempotent element δ and a unit u, such that $a = \delta + u$. A ring $\mathcal{R}$ is a clean ring if all elements of $\mathcal{R}$ are clean. If further $u\delta = \delta u$, $\mathcal{R}$ is assumed to be strongly clean.*

Definition 2.2. *[6]. An element a in a ring $\mathcal{R}$ is called a nil clean element if $a = \delta + q$, $\delta \in \mathcal{ID}(\mathcal{R})$ and $q \in \mathcal{NL}(\mathcal{R})$. If further $\delta q = q\delta$, $\mathcal{R}$ is considered to be a strongly nil clean ($\mathcal{SNC}$ for short).*

Definition 2.3. *[11]. An element a in a ring $\mathcal{R}$ is assumed to be a strongly 2-nil clean if $a = \delta + \delta_1 + q$, $\delta, \delta_1 \in \mathcal{ID}(\mathcal{R})$ and $q \in \mathcal{NL}(\mathcal{R})$ that commute with one another. $\mathcal{R}$ will say to be a strongly 2-nil clean (strongly 2-$\mathcal{NC}$ for short) if all it is elements are strongly 2-$\mathcal{NC}$.*

Definition 2.4. *[14]. An element a in a ring $\mathcal{R}$ is called a weak nil clean element of the ring $\mathcal{R}$, if $a = q - \delta$ or $a = q + \delta$, for some $q \in \mathcal{NL}(\mathcal{R}), \delta \in \mathcal{ID}(\mathcal{R})$, and if each element is weak nil clean, then $\mathcal{R}$ is assumed to be a weak nil clean ring. Further, if $a = q - \delta$ or $a = q + \delta$ with $q\delta = \delta q$, Then $\mathcal{R}$ is called weak* nil clean.*

Lemma 2.5. *[11]. The following two relations are equivalent, for any ring $\mathcal{R}$.*

1. *$\mathcal{R}$ is strongly 2-$\mathcal{NC}$.*
2. *For each $a \in \mathcal{R}$, $a = \delta - \delta_1 + q$, where δ, δ_1 are idempotents and q is a nilpotent that commute one another.*

Definition 2.6. *[15]. If $t = t^3$, the element $t \in \mathcal{R}$ is called tripotent.*

Definition 2.7. *[16]. A ring $\mathcal{R}$ is considered to be strongly trinil clean, if for any $a \in \mathcal{R}$, it can be written as the sum of two elements t and q, where $t \in Tr(\mathcal{R}), q \in \mathcal{NL}(\mathcal{R})$ and $tq = qt$.*

Theorem 2.8. *[11]. For any a in $\mathcal{R}$, the following three issues are equivalent.*

1. *$\mathcal{R}$ is strongly 2-$\mathcal{NC}$.*
2. *$a - a^3 \in \mathcal{NL}(\mathcal{R})$.*
3. *a^2 is $\mathcal{SNC}$ element.*

The next lemma duo to Lam [17], plays a crucial role in several of our proofs.

Lemma 2.9. *[17]. For any ring $\mathcal{R}$.*

1. *The element $1 + q$ is a unit for every $q \in \mathcal{NL}(\mathcal{R})$.*
2. *If $u \in \mathcal{UN}(\mathcal{R})$ and $uq = qu$, then $u + q \in \mathcal{UN}(\mathcal{R})$.*

Definition 2.10. *[18]. The right singular ideal of $\mathcal{R}$, denoted by $\gamma(\mathcal{R}) = \{a \in \mathcal{R} : r(a)$ is essential right ideal$\}$, where $r(a)$ is the right annihilator of a. If every right ideal of $\mathcal{R}$ is essential, $\mathcal{R}$ will say to be a right uniform ring.*

Definition 2.11. *[19]. If every $a \in \mathcal{R}$ may be expressed as the sum of two tripotents and a nilpotent that commute then the ring $\mathcal{R}$ is Zhou nil-clean.*

3 NE-nil clean rings

Our main focus in this section is to introduce the notion of NE-nil clean rings and to explore its fundamental properties.

Definition 3.1. *Let a in $\mathcal{R}$, then a is called NE-nil clean, if there is $a_1 \in \mathcal{R}$, such that $aa_1 = \delta$ with $a - a_1 = q$ and $qa_1 = a_1q$, where $q \in \mathcal{NL}(\mathcal{R})$, $\delta \in \mathcal{ID}(\mathcal{R})$. $\mathcal{R}$ is called NE-nil clean ($\mathcal{NE} - \mathcal{NC}$ for short) if all elements in $\mathcal{R}$ are $\mathcal{NE} - \mathcal{NC}$.*

Example 3.2. 1. *Consider the ring*

$$M_2(Z_3) = \left\{ \begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & \alpha_3 \end{bmatrix}, \alpha_1, \alpha_2, \alpha_3 \in Z_3 \right\}.$$

Then $\mathcal{NL}(M_2(Z_3)) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \right\}$ *and*

$$\mathcal{ID}(M_2(Z_3)) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

By direct calculation, one easily check that $M_2(Z_3)$ is a $\mathcal{NE} - \mathcal{NC}$ ring.

2. *Consider the ring Z_{10}. Then $\mathcal{NL}(Z_{10}) = \{\bar{0}\}$ and $\mathcal{ID}(Z_{10}) = \{\bar{0}, \bar{1}, \bar{5}, \bar{6}\}$. Clearly, Z_{10} is not $\mathcal{NE} - \mathcal{NC}$, since the element $\bar{2}$ is not $\mathcal{NE} - \mathcal{NC}$.*

We shall begin this section with the following lemma, which contains several easy statements, which will frequently use throughout this work.

Lemma 3.3. *Suppose $\mathcal{R}$ is a $\mathcal{NE} - \mathcal{NC}$ ring, then*

1. *$aq = qa$.*
2. *$aa_1 = a_1a = \delta$.*
3. *$\delta q = q\delta$.*
4. *a^2 is an $\mathcal{SNC}$ element.*

Proof. 1. Take $aa_1 = \delta$ with $a - a_1 = q$ and $qa_1 = a_1q$. Then $aq = a_1q + q^2$ and $qa = qa_1 + q^2$, this implies $aq = qa$.

2. Let $a = a_1 + q$, then $aa_1 = a_1^2 + qa_1$ and $a_1a = a_1^2 + a_1q$, this gives $aa_1 = a_1a = \delta$.

3. Let $aa_1 = \delta$, then $q\delta = q(aa_1) = (qa)a_1$, since $qa = aq$, by part(1) we have $q\delta = a(qa_1) = a(a_1q) = (aa_1)q = \delta q$.

4. Let $aa_1 = \delta$, with $a - a_1 = q$ and $qa_1 = a_1q$, then $a^2 - aa_1 = aq$, gives $a^2 = \delta + aq$. since $aq = qa$ by part(1), then $aq \in \mathcal{NL}(\mathcal{R})$. On the other hand, $a^3 = a\delta + a^2q = \delta a + aqa$ using part(1) we get $a\delta = \delta a$. Thus, $\delta(aq) = (aq)\delta$. Therefore, a^2 is an $\mathcal{SNC}$-element. □

Here, we list the $\mathcal{NE} - \mathcal{NC}$ rings fundamental properties.

Proposition 3.4. *Suppose $\mathcal{R}$ is a $\mathcal{NE} - \mathcal{NC}$ ring. Then for all $a \in \mathcal{R}$ and for some $a_1 \in \mathcal{R}$, then.*

1. *$a^2 - a_1^2 \in \mathcal{NL}(\mathcal{R})$.*
2. *$a^2 - a^4 \in \mathcal{NL}(\mathcal{R})$.*

Proof. 1. Suppose $a \in \mathcal{R}$, then $aa_1 = \delta$ for some $a_1 \in \mathcal{R}$ with $a - a_1 = q$ and $qa_1 = a_1q$, $\delta \in \mathcal{ID}(\mathcal{R})$ and $q \in \mathcal{NL}(\mathcal{R})$. Using Lemma 3.3, $a^2 = \delta + aq$, but $a_1^2 = \delta + a_1q$, so $a^2 - a_1^2 = \delta + aq - (\delta - a_1q) = aq + a_1q = (a + a_1)q$, since $q(a + a_1) = (a + a_1)q$, then $q(a + a_1) \in \mathcal{NL}(\mathcal{R})$. Hence, $a^2 - a_1^2 \in \mathcal{NL}(\mathcal{R})$.

2. Let $a \in \mathcal{R}$, then by Lemma 3.3, $a^2 = \delta + aq$, this gives $a^2 - aq = \delta$. Hence, $(a^2 - aq)^2 = a^2 - aq$. This implies $a^2(a^2 - 2aq + q^2) = a^2 - aq$, so $a^4 - 2a^3q + a^2q^2 = a^2 - aq$. Thus, $a^2 - a^4 = -2a^3q + a^2q^2 - aq = (-2a^3 + a^2q + a)q \in \mathcal{NL}(\mathcal{R})$. □

Proposition 3.5. *Let $\mathcal{R}$ be $\mathcal{NE} - \mathcal{NC}$ ring, then $\mathcal{J}(\mathcal{R}) \subseteq \mathcal{NL}(\mathcal{R})$.*

Proof. Suppose $a \in \mathcal{J}(\mathcal{R})$. Then there is $a_1 \in \mathcal{R}$, such that $aa_1 = \delta$ with $a - a_1 = q$ and $qa_1 = a_1q$, where $\delta = \delta^2$ and q is nilpotent. But $aa_1 \in \mathcal{J}(\mathcal{R})$ shows that $\delta \in \mathcal{J}(\mathcal{R})$, this gives $\delta = 0$. By Lemma 3.3, $a^2 = \delta + aq$, so $a^2 = aq \in \mathcal{NL}(\mathcal{R})$, thus, $a \in \mathcal{NL}(\mathcal{R})$. □

Proposition 3.6. *Let $\mathcal{R}$ be $\mathcal{NE} - \mathcal{NC}$ ring. Then $\gamma(\mathcal{R}) \subseteq \mathcal{NL}(\mathcal{R})$.*

Proof. Given $0 \neq a$ in $\gamma(\mathcal{R})$, there is a_1 in $\mathcal{R}$ such that $aa_1 = \delta$ with $a - a_1 = q$ and $qa_1 = a_1q$, $\delta \in \mathcal{ID}(\mathcal{R})$, $q \in \mathcal{NL}(\mathcal{R})$, so $a^2 = \delta + aq$. Let $c \in r(a^2) \cap \delta\mathcal{R}$, then $c = \delta\mathcal{R}$, for some $r \in \mathcal{R}$, and $a^2c = 0$. So $(\delta + aq)\delta r = 0$, as $\delta = \delta^2$, then $(1 + aq)\delta r = 0$. Since $aq \in \mathcal{NL}(\mathcal{R})$, then by Lemma 2.9, $1 + aq = u \in \mathcal{UN}(\mathcal{R})$. This gives $u.\delta r = 0$, thus, $\delta r = c = 0$. But $r(a^2)$ is a non-trivial and essential, then $\delta\mathcal{R} = 0$, so $\delta = 0$. Therefore, $aq = a^2 \in \mathcal{NL}(\mathcal{R})$, so $a \in \mathcal{NL}(\mathcal{R})$. □

Proposition 3.7. *If $\mathcal{R}$ is $\mathcal{NE} - \mathcal{NC}$ ring with idempotents 0 and 1, then $\mathcal{R}$ is a local ring.*

Proof. Choose $a \in \mathcal{R}$, then there is $a_1 \in \mathcal{R}$, such that $aa_1 = \delta$ with $a - a_1 = q$ and $qa_1 = a_1q$, $\delta \in \mathcal{ID}(\mathcal{R})$, $q \in \mathcal{NL}(\mathcal{R})$. If $\delta = 1$, then $aa_1 = 1$, this implies $a^2 = 1 + aq$, by Lemma 2.9, $1 + aq \in \mathcal{UN}(\mathcal{R})$, so $a \in \mathcal{UN}(\mathcal{R})$. On the other hand, if $\delta = 0$, then $aa_1 = 0$, this gives $a^2 = aq \in \mathcal{NL}(\mathcal{R})$, so a is nilpotent, and then $1 + a \in \mathcal{UN}(\mathcal{R})$. Thus, $\mathcal{R}$ is local. □

Proposition 3.8. *If $\mathcal{R}$ is a $\mathcal{NE} - \mathcal{NC}$ ring. Then for any idempotent δ, there exists a unit u and an idempotent δ_1 such that $u\delta_1 = \delta$.*

Proof. Let $\delta = \delta^2 \in \mathcal{R}$, then $\delta a_1 = \delta_1$ for some $a_1 \in \mathcal{R}$, with $\delta - a_1 = q$ and $qa_1 = a_1q$, where $\delta_1 \in \mathcal{ID}(\mathcal{R})$ and $q \in \mathcal{NL}(\mathcal{R})$. Then $\delta - \delta_1 = \delta q$, this implies $\delta(1 - q) = \delta_1$, but $1 - q = u \in \mathcal{UN}(\mathcal{R})$. Then we have $\delta u = \delta_1$. Hence, $\delta = u^{-1}\delta_1$. □

Definition 3.9. *A 2-Boolean ring is a ring with every a in $\mathcal{R}$, $a^2 \in \mathcal{ID}(\mathcal{R})$.*

Proposition 3.10. *Suppose the ring $\mathcal{R}$ is $\mathcal{NE} - \mathcal{NC}$ with every nilpotent element is a difference of two idempotents that commute. Then $\mathcal{R}$ is a 2-Boolean.*

Proof. Take $a \in \mathcal{R}$, then by Lemma 3.3, $a^2 = \delta + aq$, where $\delta \in \mathcal{ID}(\mathcal{R})$ and $aq \in \mathcal{NL}(\mathcal{R})$ and $\delta(aq) = (aq)\delta$. As aq is nilpotent, then $aq = \delta_1 - \delta_2$, where $\delta_1, \delta_2 \in \mathcal{ID}(\mathcal{R})$ and $\delta_1\delta_2 = \delta_2\delta_1$. This implies $aq + \delta_2 = \delta_1$. Thus, $(aq + \delta_2)^2 = aq + \delta_2$. Clearly, $(aq)\delta_2 = \delta_2(aq)$, then we have $(aq)^2 + 2aq\delta_2 + \delta_2^2 = aq + \delta_2$, this yield $(aq)^2 + 2aq\delta_2 = aq$, so $(aq)^2 = aq - 2aq\delta_2$, gives $(aq)^2 = aq(1 - 2\delta_2)$, but $1 - 2\delta_2 = u \in \mathcal{UN}(\mathcal{R})$, and $(1 - 2\delta_2)^2 = 1$. Thus, $aq = (aq)^2u$. As aq is nilpotent, then $aq = 0$. Therefore, $a^2 = \delta$. □

Corollary 3.11. *Suppose the ring $\mathcal{R}$ is $\mathcal{NE} - \mathcal{NC}$ in which every nilpotent element is a difference of two idempotents that commute, and $2 \in \mathcal{R}$ is nilpotent. Then $\mathcal{R}$ is a Boolean.*

Proof. Take a in $\mathcal{R}$. Then $a^2 = \delta + aq$ where $\delta \in \mathcal{ID}(\mathcal{R})$, $aq \in \mathcal{NL}(\mathcal{R})$, and $\delta(aq) = (aq)\delta$. Applying Proposition 3.10, we get $a^2 = \delta$, and $2 = 0$. Hence, $\mathcal{R}$ is of characteristic 2. Now consider $(a + \delta)^2 = a^2 + 2a\delta + \delta = a^2 + \delta = \delta + \delta = 0$. By Proposition 3.10. $a + \delta = 0$, this gives $a + \delta = 2\delta$. Then $a = \delta$. Whence, $\mathcal{R}$ is a Boolean ring. □

4 $\mathcal{NE} - \mathcal{NC}$ and related rings

The objective of the present section is to establish the relationships between $\mathcal{NE} - \mathcal{NC}$ rings, strongly clean, $\mathcal{SNC}$ rings, strongly 2-$\mathcal{NC}$-rings and weak* nil clean rings.

Theorem 4.1. *Every $\mathcal{SNC}$ ring is $\mathcal{NE} - \mathcal{NC}$.*

Proof. For any $a \in \mathcal{R}$, $a = \delta + q$ and $\delta q = q\delta$, whenever, $\delta \in \mathcal{ID}(\mathcal{R})$ and $q \in \mathcal{NL}(\mathcal{R})$. Since q is a nilpotent element, then there is a positive integer r such that $q^r = 0$.
Set $a_1 = (\delta - \delta q + \delta q^2 - \delta q^3 + \cdots + (-1)^{r-1}q^{r-1})$, then $aa_1 = (\delta + q)(\delta - \delta q + \delta q^2 - \delta q^3 + \cdots (-1)^{r-1}q^{r-1})$, clearly, $aa_1 = \delta$. Observe that,
$a - a_1 = (\delta + q) - (\delta - \delta q + \delta q^2 - \cdots (-1)^{r-1}q^{r-1}) = (1 + \delta - \delta q + \cdots (-1)^{r-1}q^{r-2})q$. Since $q\delta = \delta q$, then $a - a_1 \in \mathcal{NL}(\mathcal{R})$, and $qa_1 = a_1q$. Therefore, $\mathcal{R}$ is a $\mathcal{NE} - \mathcal{NC}$ ring. □

Below we shall provide a counter example to Theorem 4.1.

Example 4.2. *The ring $M_2(Z_3)$ is $\mathcal{NE} - \mathcal{NC}$, but it is not considered nil clean, since the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is not $\mathcal{SNC}$ element.*

Proposition 4.3. *Let $\mathcal{R}$ be $\mathcal{NE} - \mathcal{NC}$ ring, and let $2 \in \mathcal{R}$ is nilpotent. Then $\mathcal{R}$ is an $\mathcal{SNC}$ ring.*

Proof. Take $a \in \mathcal{R}$, there is $a_1 \in \mathcal{R}$, such that $aa_1 = \delta$ with $a - a_1 = q$ and $qa_1 = a_1q$ where $\delta \in \mathcal{ID}(\mathcal{R}), q \in \mathcal{NL}(\mathcal{R})$. Using Lemma 3.3, the element a^2 will be $\mathcal{SNC}$. Applying [11, Theorem 2.11], we get a is $\mathcal{SNC}$ element. Therefore, $\mathcal{R}$ is a $\mathcal{SNC}$ ring. □

Our next, theorem illustrates the relationship between $\mathcal{NE} - \mathcal{NC}$ with strongly 2-$\mathcal{NC}$ rings.

Theorem 4.4. *A ring $\mathcal{R}$ is strongly 2-$\mathcal{NC}$ if and only if $\mathcal{R}$ is $\mathcal{NE} - \mathcal{NC}$.*

Proof. Assume $\mathcal{R}$ is strongly 2-$\mathcal{NC}$, and suppose $a \in \mathcal{R}$. According to Lemma 2.5, $a = \delta - \delta_1 + q$, whenever, $\delta, \delta_1 \in \mathcal{ID}(\mathcal{R})$ and $q \in \mathcal{NL}(\mathcal{R})$. Here δ, δ_1 and q commute with one another. As q nilpotent, then $q^r = 0$ for some positive integer r. Clearly, $(\delta - \delta_1)^2$ is idempotent and $(\delta - \delta_1)$ is a tripotent.

Set $a_1 = (\delta-\delta_1)-(\delta-\delta_1)^2q+(\delta-\delta_1)q^2-\cdots+(-1)^{r+1}(\delta-\delta_1)^rq^{r-1}$.
Then
$aa_1 = (\delta-\delta_1+q)((\delta-\delta_1)-(\delta-\delta_1)^2q+(\delta-\delta_1)q^2-\cdots+(-1)^{r+1}(\delta-\delta_1)^rq^{r-1})$
$= (\delta-\delta_1)^2-(\delta-\delta_1)q+(\delta-\delta_1)^2q^2-\cdots+(-1)^{r+1}(\delta-\delta_1)^{r+1}q^{r-1}+(\delta-\delta_1)q-(\delta-\delta_1)^2q^2+(\delta-\delta_1)q^3-\cdots+0 = (\delta-\delta_1)^2$.
Furthermore,
$a-a_1 = (\delta-\delta_1+q)-((\delta-\delta_1)-(\delta-\delta_1)^2q+(\delta-\delta_1)q^2-\cdots+(-1)^{r+1}(\delta-\delta_1)^rq^{r-1})$
$= q(1+(\delta-\delta_1)-\cdots+(-1)^{r+1}(\delta-\delta_1)^rq^{r-2}) \in \mathcal{NL}(\mathcal{R})$.
Clearly, $qa_1 = a_1q$. Therefore, $\mathcal{R}$ is $\mathcal{NE}-\mathcal{NC}$. □

Conversely, let $a \in \mathcal{R}$. Then $aa_1 = \delta$, for some $a_1 \in \mathcal{R}$ with $a - a_1 = q$, $qa_1 = a_1q$ where $\delta \in \mathcal{ID}(\mathcal{R})$, $q \in \mathcal{NL}(\mathcal{R})$. Using Lemma 3.3, a^2 will be a $\mathcal{SNC}$ element. Applying [11, Theorem 2.3], we have, a is a strongly 2-$\mathcal{NC}$ element. Thus $\mathcal{R}$ is a strongly 2-$\mathcal{NC}$ ring.

Next, we shall establish the relationship between $\mathcal{NE}-\mathcal{NC}$ and strongly clean rings.

Theorem 4.5. *Every $\mathcal{NE}-\mathcal{NC}$ ring is strongly clean.*

Proof. Take $a \in \mathcal{R}$. Then $a^2 = \delta + aq$ where $\delta \in \mathcal{ID}(\mathcal{R}), q \in \mathcal{NL}(\mathcal{R}), \delta(aq) = (aq)\delta$. Set $a^2 = (1-\delta)+(2\delta-1+aq)$. Since $(2\delta-1)^2 = 1$, then $(2\delta-1) \in \mathcal{UN}(\mathcal{R})$. Observe that $aq \in \mathcal{NL}(\mathcal{R})$, and $(2\delta-1)aq = aq(2\delta-1)$, by Lemma 2.9, we have $2\delta-1+aq = u \in \mathcal{UN}(\mathcal{R})$. This implies $a^2-(1-\delta) = u$, and hence, $(a-(1-\delta))(a+(1-\delta)) = u$, so $a-(1-\delta) = v \in \mathcal{UN}(\mathcal{R})$, then $a = (1-\delta)+v$ but $(1-\delta)$ is idempotent, v is unit and $v(1-\delta) = (1-\delta)v$. Therefore, $\mathcal{R}$ is a strongly clean ring. □

Observe that the reverse of Theorem 4.5 is false, as it will be shown in the next example.

Example 4.6. *In the ring Z_{10}. Clearly, Z_{10} is strongly clean, while Z_{10} is not $\mathcal{NE}-\mathcal{NC}$ since the element $\bar{2}$ is not $\mathcal{NE}-\mathcal{NC}$.*

Proposition 4.7. *Assume $\mathcal{R}$ is a strongly clean ring and $2 \in \mathcal{NL}(\mathcal{R})$, with every unit $u, u^2 = 1$. Then $\mathcal{R}$ is a $\mathcal{NE}-\mathcal{NC}$ ring.*

Proof. Letting $a \in \mathcal{R}$, then $a = \delta + u$, with $\delta u = u\delta$ where $\delta \in \mathcal{ID}(\mathcal{R})$, $u \in \mathcal{UN}(\mathcal{R})$, let $a_1 = u-\delta$, then $aa_1 = (u+\delta)(u-\delta) = u^2+\delta u-u\delta-\delta^2 = 1-\delta$. Furthermore, $a-a_1 = \delta+u-u+\delta = 2\delta \in \mathcal{NL}(\mathcal{R})$, but $q(u-\delta) = (u-\delta)q, qa_1 = a_1q$. Therefore, $\mathcal{R}$ is $\mathcal{NE}-\mathcal{NC}$ ring. □

Theorem 4.8. *Every weak* nil clean ring is $\mathcal{NE}-\mathcal{NC}$.*

Proof. Given $a \in \mathcal{R}$ be a weak* nil clean, then $a = q+\delta$ or $a = q-\delta$ and $q\delta = \delta q$, where $\delta \in \mathcal{ID}(\mathcal{R}), q \in \mathcal{NL}(\mathcal{R})$. Then there is a positive integer r, such that $q^r = 0$. If $a = q+\delta$, then according to Theorem 4.1, $\mathcal{R}$ is $\mathcal{NE}-\mathcal{NC}$ ring. If
$a = q-\delta$, write
$a_1 = (-\delta-\delta q-\delta q^2-\delta q^3-\cdots-q^{r-1})$, then
$aa_1 = (q-\delta)(-\delta-\delta q-\delta q^2-\delta q^3-\cdots-q^{r-1})$. Clearly, $aa_1 = \delta$ and
$a-a_1 = (q-\delta)-(-\delta-\delta q-\delta q^2-\delta q^3-\cdots-q^{r-1})$
$= (1+\delta+\delta q+\cdots+q^{r-2})q$. So $q\delta = \delta q$, then $a-a_1 \in \mathcal{NL}(\mathcal{R})$, and $qa_1 = a_1q$. Therefore, $\mathcal{R}$ is $\mathcal{NE}-\mathcal{NC}$ ring. □

Observe that the converse of the above theorem is untrue. Below we shall provide counter example. In Example 3.2(1), it is clear $M_2(Z_3)$ is a $\mathcal{NE}-\mathcal{NC}$ ring, which is not weak* nil clean. Because the elements $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}2 & 2\\0 & 1\end{bmatrix}$ are in $M_2(Z_3)$, which are not weak* nil clean element. Hence $M_2(Z_3)$ is not weak* nil clean ring.

5 NT-nil clean rings

The purpose of this section is to show the concept of the so-called NT-nil clean rings that generalize the idea of $\mathcal{NE}-\mathcal{NC}$ rings. Our aim is to explore its fundamental properties and its relationship with other related rings.

Definition 5.1. *An element a in a ring $\mathcal{R}$ is assumed to be NT-nil clean if for each $a \in \mathcal{R}$ there exists $a_1 \in \mathcal{R}$, such that $aa_1 = t$ with $a-a_1 = q$ and $a_1q = qa_1$, where $t \in Tr(\mathcal{R})$ and $q \in \mathcal{NL}(\mathcal{R})$. A ring $\mathcal{R}$ is NT-nil clean ($\mathcal{NT}-\mathcal{NC}$ for short) if each element of $\mathcal{R}$ is $\mathcal{NT}-\mathcal{NC}$.*

Note that: Every $\mathcal{NE}-\mathcal{NC}$ ring is $\mathcal{NT}-\mathcal{NC}$ since every idempotent element is tripotent. The reverse implication is false, as presented in the below example.

Example 5.2. *In ring Z_{10}, we have*
$\mathcal{ID}(Z_{10}) = \{\bar{0},\bar{1},\bar{5},\bar{6}\}$, $Tr(Z_{10}) = \{\bar{0},\bar{1},\bar{4},\bar{5},\bar{6},\bar{9}\}$ and $\mathcal{NL}(Z_{10}) = \{\bar{0}\}$, we can easily verify that Z_{10} is $\mathcal{NT}-\mathcal{NC}$ which is not $\mathcal{NE}-\mathcal{NC}$, since the elements $\bar{2},\bar{3},\bar{7},\bar{8}$ are not $\mathcal{NE}-\mathcal{NC}$ elements.

As a starting point, we use the following lemma in several of our proofs.

Lemma 5.3. *Assume $\mathcal{R}$ is an $\mathcal{NT}-\mathcal{NC}$ ring, then for each a in $\mathcal{R}$, we have:*

1. *a^2 is strongly Trinil clean.*
2. *a^2 is strongly 2-$\mathcal{NC}$.*
3. *a^4 is $\mathcal{SNC}$.*
4. *a is clean.*

Proof. 1. Let a in $\mathcal{R}$, there is $a_1 \in \mathcal{R}$ such that $aa_1 = t$ with $a-a_1 = q$ and $qa_1 = a_1q$, where $t \in Tr(\mathcal{R})$ and $q \in \mathcal{NL}(\mathcal{R})$, So $a^2-aa_1 = aq$, gives $a^2-t = aq$. Clearly, $aq = qa$, then $aq \in \mathcal{NL}(\mathcal{R})$, say q_1. Thus, $a^2 = t+q_1$, and $tq_1 = q_1t$. Therefore, a^2 is a strongly Trinil clean element.

2. Follows from [11, Theorem 2.8].

3. From part (1), $a^2 = t+q_1$. Thus, $a^4 = t^2+2tq_1+q_1^2 = t^2+(2t+q_1)q_1$. Since $(2t+q_1)q_1 = q_1(2t+q_1)$, this gives $(2t+q_1)q_1 \in \mathcal{NL}(\mathcal{R})$, say q_2. So $a^4 = t^2+q_2$ and $t^2q_2 = q_2t^2$. Therefore, a^4 is an $\mathcal{SNC}$ element.

4. From (3), $a^4 = t^2 + q_2$. We may write $a^4 = 1-t^2+(2t^2-1)+q_2$. Clearly, $1-t^2$ is idempotent and $(2t^2-1)^2 = 4t^2-4t^2+1 = 1$ and $q_2(2t^2-1) = (2t^2-1)q_2$. This mean that $(2t^2-1)+q_2 \in \mathcal{UN}(\mathcal{R})$, say u_1. So $a^4 = 1-t^2+u_1$, implies $a^4-(1-t^2) = u_1$. Thus, $((a-(1-t^2))(a^3-3a^2(1-t^2))+3a(1-t^2)^2-(1-t^2)^3) = u_1$. Hence $a-(1-t^2) \in \mathcal{UN}(\mathcal{R})$.

□

Proposition 5.4. *If $\mathcal{R}$ is an $\mathcal{NT}-\mathcal{NC}$ ring, then $\mathcal{J}(\mathcal{R}) \subseteq \mathcal{NL}(\mathcal{R})$.*

Proof. Letting $j \in \mathcal{J}(\mathcal{R})$, then there is $a_1 \in \mathcal{R}$ such that $ja_1 = t$ with $j-a_1 = q$, and $qa_1 = a_1q$, where $q \in \mathcal{NL}(\mathcal{R})$ and $t \in Tr(\mathcal{R})$. But $j \in \mathcal{J}(\mathcal{R})$ shows that, $ja_1 = t \in \mathcal{J}(\mathcal{R})$. Thus, $t = 0$. Hence, $ja_1 = 0$, so $j^2 - 0 = jq \in \mathcal{NL}(\mathcal{R})$, this yield $j \in \mathcal{NL}(\mathcal{R})$. □

Proposition 5.5. *Let $\mathcal{R}$ be an $\mathcal{NT}-\mathcal{NC}$ ring, for which 2 is nilpotent. Then $\mathcal{R}$ is $\mathcal{NE}-\mathcal{NC}$ ring.*

Proof. Given $a \in \mathcal{R}$, then by Lemma 5.3 part (1), $a^2 = t + q_1$, where $t \in Tr(\mathcal{R}), q_1$ is nilpotent, and $q_1t = tq_1$, from Lemma 5.3 part (3), $a^4 = t^2+q_2$. Hence, $a^4-a^2 = t^2+q_2-t-q_1$. So $a^4-a^2 = t^2-t+(q_2-q_1)$. Consider $(t^2-t)^2 = t^4-2t^3+t^2 = t^2-2t+t^2 = 2(t^2-t)$. Since $2 \in \mathcal{NL}(\mathcal{R})$, then $t^2-t \in \mathcal{NL}(\mathcal{R})$. So, $a^4-a^2 \in \mathcal{NL}(\mathcal{R})$, this means $a(a^3-a) \in \mathcal{NL}(\mathcal{R})$, so $(a^2-1)a(a^3-a) \in \mathcal{NL}(\mathcal{R})$. Then $(a^3-a)^2 \in \mathcal{NL}(\mathcal{R})$, yielding $a^3-a \in \mathcal{NL}(\mathcal{R})$. Applying Theorem 2.8, $\mathcal{R}$ is a strongly 2-$\mathcal{NC}$ ring. Using Theorem 4.4, $\mathcal{R}$ is $\mathcal{NE}-\mathcal{NC}$. □

Theorem 5.6. *Assume $\mathcal{R}$ is a right uniform $\mathcal{NT}-\mathcal{NC}$ ring. Then $R-\{1\} = \mathcal{NL}(\mathcal{R})$.*

Proof. Letting $1 \neq a \in \mathcal{R}$, then by Lemma 5.3 part(1), $a^2 = t+q$, where $t = t^3$, q is a nilpotent and $qt = tq$. Let $r(a^2)$ be a non-trivial right ideal of $\mathcal{R}$. We shall show that $tR \cap r(a^2) = 0$. Let $c \in tR \cap r(a^2)$. Then $c = t\mathcal{R}$ and $a^2c = 0$ for some $r \in \mathcal{R}$, this implies $(t+q)tr = 0, (1+qt)t^2r = 0$, but $1+qt \in \mathcal{UN}(\mathcal{R})$. Hence, $t^2r = 0$, gives $tr = c = 0$. Since $\mathcal{R}$ is uniform by an assumption, then $t\mathcal{R} = 0$, so $t = 0$. Whence, it follows that $a^2 = 0+q$. Hence, $a \in \mathcal{NL}(\mathcal{R})$. □

The following result explores the relationship between $\mathcal{NT}-\mathcal{NC}$ ring and Zhou ring.

Theorem 5.7. *Suppose $\mathcal{R}$ is $\mathcal{NT}-\mathcal{NC}$ ring. Then $\mathcal{R}$ is a Zhou ring.*

Proof. Given a in $\mathcal{R}$, then by Lemma 5.3 part(2), a^2 is a strongly 2-$\mathcal{NC}$ element. According to Theorem 2.8, $(a^2)^3 - a^2 \in \mathcal{NL}(\mathcal{R})$. So $a(a^5-a) \in \mathcal{NL}(\mathcal{R})$, and $(a^4-1)a(a^5-a) = (a^5-a)^2 \in \mathcal{NL}(\mathcal{R})$. Therefore, $a^5-a \in \mathcal{NL}(\mathcal{R})$. Hence, $\mathcal{R}$ is Zhou according to [20, Theorem 2.11]. □

Corollary 5.8. *If $\mathcal{R}$ is $\mathcal{NT}-\mathcal{NC}$ for which 2 is nilpotent, then each element of $\mathcal{R}$ is a sum of two idempotents and two units.*

Proof. Suppose a in $\mathcal{R}$, then by Theorem 5.7, $a = t_1+t_2+q$, where $t_1, t_2 \in Tr(\mathcal{R}), q \in \mathcal{NL}(\mathcal{R})$ that commute with one another. Now we may write $a = (1-t_1^2)+(t_1^2+t_1-1)+(1-t_2^2)+(t_2^2+t_2-1)+q$. Note that $(t_1^2+t_1-1)^2 = t_1^4+t_1^2+1+2t_1^3-2t_1^2-2t_1$ $= t_1^2+t_1^2+1+2t_1-2t_1^2-2t_1 = 1$. Thus, $t_1^2+t_1-1 = u_1 \in \mathcal{UN}(\mathcal{R})$. Similarly $t_2^2+t_2-1 = u_2 \in \mathcal{UN}(\mathcal{R})$. Whence, it follows that $a = (1-t_1^2)+u_1+(1-t_2^2)+u_2+q$, as $u_2q = qu_2$, then $u_2+q = u_3 \in \mathcal{UN}(\mathcal{R}), (1-t_1^2), (1-t_2^2) \in \mathcal{ID}(\mathcal{R})$. This implies $a = (1-t_1^2)+(1-t_2^2)+u_1+u_3$. □

6 Conclusions

In this work, the notion of $\mathcal{NE}-\mathcal{NC}$ rings is presented. We give the fundamental properties of these rings and show that these rings generalize $\mathcal{SNC}$ rings and equivalent to a strongly 2-SN ring. Furthermore, we introduce the concept of $\mathcal{NT}-\mathcal{NC}$ rings. Indeed, these class of rings generalize $\mathcal{NE}-\mathcal{NC}$ rings, and show that every $\mathcal{NT}-\mathcal{NC}$ ring is a Zhou rings.

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