Mathematics and Statistics 11(4): 733-737, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110415

Overtrees and Their Chromatic Polynomials

Iakov M. Erusalimskiy, Vladimir A. Skorokhodov*

Institute of Mathematics, Mechanics and Computer Science, Southern Federal University, Russia

Received April 4, 2023; Revised July 8, 2023; Accepted July 20, 2023

Cite This Paper in the following Citation Styles

***(a)**: [1] Iakov M. Erusalimskiy, Vladimir A. Skorokhodov, "Overtrees and Their Chromatic Polynomials," Mathematics and Statistics, Vol.11, No.4, pp. 733-737, 2023. DOI: 10.13189/ms.2023.110415*

***(b)**: Iakov M. Erusalimskiy, Vladimir A. Skorokhodov, (2023). Overtrees and Their Chromatic Polynomials. Mathematics and Statistics, 11(4), 733-737. DOI: 10.13189/ms.2023.110415*

Abstract In this paper, graphs called overtrees are introduced and studied. These are connected graphs that contain a single simple cycle. Such graphs are connected graphs following the trees in terms of the number of edges. An overtree can be obtained from a tree by adding an edge to connect two non-adjacent vertices of a tree. The same class of graphs can also be defined as a class of graphs obtained from trees by replacing one vertex of the tree with a simple cycle. The main characteristics of overtrees are n, which is the number of vertices, and k, which is the number of vertices of a simple cycle ($3 \le k \le n$). A formula for the chromatic polynomial of an overtree is obtained, which is determined by the characteristics n and k only. As a consequence, it is obtained the formula for the chromatic function of a graph which is built from a tree by replacing some of its vertices (possibly all) with simple cycles of arbitrary length. It follows from these formulas that any overtree with an even-length cycle is two-colored, and with an odd-length cycle is three-colored. The same is true for graphs obtained from trees by replacing some vertices with simple cycles.

Keywords Graph, Tree, Overtree, Cycle, Coloring, Chromatic Polynomial

1 Introduction

The problem of coloring a graph and finding its chromatic polynomial can now be considered completely solved, since for two extreme cases of connected graphs which are trees and complete graphs formulas for their chromatic polynomials have been obtained. Also theorems that allow calculating the chromatic polynomials of the other connected graphs by reducing to complete graphs or trees have been proved. However, these theorems are procedural. They also show that in the general setting, the problem of finding the chromatic polynomial of a graph in terms of its invariants is not solvable. Examples of non-isomorphic graphs with the same chromatic polynomials are constructed.

In this paper, two classes of undirected graphs are defined and studied. These classes are overtrees and graphs that can be obtained from a tree by replacing some of its vertices with simple cycles. It is clear that by the number of cycles, overtrees are the next connected graphs type after trees.

Formulas for the chromatic polynomials of such graphs are obtained. These polynomials are uniquely determined by two parameters of these graphs: the number of vertices and the lengths of the simple cycles they contain.

The result is a formula for the chromatic polynomial of any graph that is obtained from a tree by replacing some its vertices with simple cycles. The chromatic polynomials of such graphs are also uniquely determined by the number of vertices and the lengths of simple cycles.

It follows from the obtained formulas that if the lengths of all cycles are even, these graphs are two-colored, and otherwise they are three-colored.

2 Overtrees. Definition, examples and the simplest properties

It is known that a tree is a connected graph with the minimum number of arcs. The trees are well studied. Their properties in [1, 2] were formulated in the form of the main theorem on trees (Theorem 4.1. in [1]. This theorem allows us to give several equivalent definitions of a tree. According with [1], we consider the following as the main definition of a tree:

Definition 1. *A tree is a finite connected graph without cycles.*

Now define a class of graphs that we call overtrees. This, in the point of the number of arcs, overtrees is the class of connected graphs that next to the trees class.

Definition 2. *An overtree is a finite connected graph that contains a unique simple cycle.*

In the figure 1 two overtrees with four vertices are shown.

It is clear that the following theorem holds.

Theorem 1. *Let $G(X, U)$ be an overtree and $|X| = n$, then $|U| = n$.*

The proof of this theorem is trivial. Indeed, if we remove an arbitrary edge of a simple cycle, we get a finite connected graph with n vertices which contains no cycles, that is a tree. By the main theorem on trees, such a graph contains $n - 1$ edges. This means that the original overtree contains n arcs.

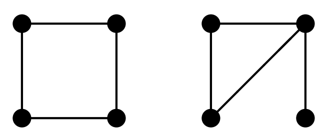

Figure 1. Two overtrees with four vertices.

It follows from this theorem and the main theorem on trees that any overtree can be considered to be obtained from a tree by adding one arc. As a result of such adding a unique simple cycle is formed.

Figure 2 shows a tree and two overtrees, which are obtained from the original tree by adding one edge. For each overtree, the added edge is shown with a dotted line.

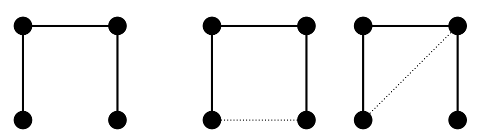

Figure 2. Tree and two its overtrees.

It is clear that in addition to the "number of vertices" characteristic, the overtree has one more characteristic, that is "the length of a simple cycle".

We denote by $ST_{n,k}$ an overtree with n vertices and a cycle of length k. It's obvious that $ST_{n,n}$. Here C_n is a simple cycle of length n.

In the case, when $3 \leq k < n$, an overtree can be obtained as a simple cycle, in one or several vertices of which trees are "planted" (see figure 3, which illustrates that has been said).

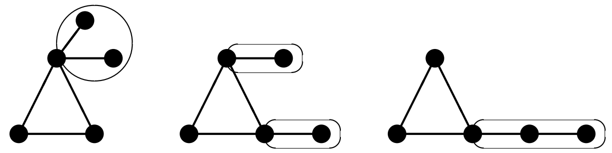

Figure 3. Three overtrees $ST_{5,3}$.

It is clear that in the case when $k = n - 1$, there is a unique overtree that has the form of a simple cycle C_{n-1}, one of the vertices of which is connected by an edge to unique vertex not from the cycle. Therefore, in this case (as in the case of $k = n$) there is only one overtree, the construction of which we just have described. In other cases, the characteristics n and k do not define a overtree up to isomorphism (see figure 3).

It is clear that following theorems hold

Theorem 2. *For any overtree $ST_{n,k}$, if $3 \leq k < n$, then there exists at least one vertex x for which $\deg x = 1$ (such a vertex is called a pendant vertex).*

Theorem 3. *The blocks graph of the overtree $ST_{n,k}$ is a tree with $n - k + 1$ vertices. This graph can be obtained from the overtree $ST_{n,k}$ by removing the edges of the cycle C_k and combining of all the vertices of this cycle into one vertex.*

Remind (see [2]) that a block is a maximal in embedding connected subgraph which does not contain articulation points. A block graph is obtained from the original graph by combining all vertices of each of the blocks into one vertex and removing the inner arcs of each of the blocks.

It is clear that all the overtrees, which are shown in figure 3, have the same blocks graph (see figure 4).

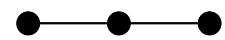

Figure 4. Blocks graph for any overtree $ST_{5,3}$.

3 Overtrees chromatic polynomials

We now turn to finding the chromatic polynomials of overtrees. It will be shown that they are uniquely determined by the characteristics of n and k.

Remind (see [1]) that a coloring of the vertices set X of a graph G by m colors is a mapping $p : X \to [1; m]_N$, which satisfies the condition: if the vertices x and y are connected by an edge, then $p(x) \neq p(y)$. The chromatic number of a graph is the smallest m for which a coloring of the vertices of the graph exists. The chromatic number of the graph is denoted by $\chi(G)$.

We now recall the definition of the chromatic function of a graph:

Definition 3. *The chromatic function of a graph G is a function f_G which is defined on the set N as follows:*

$$f_G(x) = \text{"the number of ways to color the vertices of the graph G by } x \text{ colors"}. \quad (1)$$

Thus, by equality (1) a graph chromatic function is defined combinatorially. Its behavior is clearly defined by following:

1. $f_G(x) = 0$ if $0 \leq x \leq \chi(G) - 1$;
2. $0 < f_G(\chi(G)) < f_G(\chi(G)+1) < f_G(\chi(G)+2) < \dots$.

If for a specific graph we can go from the combinatorial specification of the chromatic function (equality (1)) to its specification using a formula, then this allow us to calculate the chromatic number of the graph.

The chromatic function has been studied well enough (see, for example, [1, 2]). It turned out that for any graph with n vertices it is a polynomial of degree n with integer coefficients, therefore it is called the chromatic polynomial of the graph. The history of chromatic polynomials goes back to G.D. Birkhoff, H. Whitney and W.T. Tutt. Over the past 50 years, chromatic polynomials and their generalizations have been actively studied by many authors (see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20]), including in connection with the four-color conjecture (before it has been proved).

The chromatic number of a graph is an important characteristic of a graph, in particular, it is used by chemists involved in the design of chemical compounds (see, for example, [18, 19]). In graph theory itself, the coloring of vertices can be used to find internally stable sets of a graph, and the chromatic polynomial can be used to recognize that graphs are non-isomorphic.

There are two classes of graphs, for which such a polynomial is known and depends only on the number of vertices of the graph. They are complete graphs K_n with n vertices and trees T_n with n vertices:

$$f_{K_n}(x) = x(x-1)(x-2)\cdot\ldots\cdot(x-(n-1));$$
$$f_{T_n}(x) = x(x-1)^{n-1}.$$

Finding the chromatic polynomials of arbitrary graphs is based on these two facts. The following theorem allows one to find the chromatic polynomial of an incomplete graph using the chromatic polynomials of complete graphs.

Theorem 4. (see, for example, [3, Theorem 5.6]). *Let G be a graph and its vertices y and z are non-adjacent (not connected by an edge), then the following equality holds:*

$$f_G(x) = f_{G_{y-z}}(x) + f_{G_{y=z}}(x). \quad (2)$$

Here G_{y-z} is the graph obtained by adding an edge that connect the vertices y and z, and $G_{y=z}$ is the graph obtained by combining the vertices y and z into one vertex.

It is clear that in a finite number of steps of applying this theorem, one can obtain on the right-hand side of (2) a linear combination of complete graphs chromatic polynomials.

Theorem 4 gives us the quickly method to obtain the chromatic polynomial of a graph that close to a complete graph in the number of arcs.

Let us give this theorem a slightly different form:

Theorem 5. (see, for example, [4, Theorem 5.1.1]). *Let G be a graph and its vertices y and z are adjacent, that is connected by an edge u, then the following equality holds:*

$$f_G(x) = f_{G_{-u}}(x) + f_{G_{y=z}}(x). \quad (3)$$

Here G_{-u} is the graph obtained from the graph G by deleting the edge u.

There is one more theorem that speeds up the finding of the chromatic function in some cases.

Theorem 6. (see, for example, [3, Theorem 5.8]). *Let G be a graph for which $G = G_1 \cup G_2$ and $G_1 \cap G_2 = K_s$, then the following equality holds*

$$f_G(x) = \frac{f_{G_1}\cdot f_{G_2}}{f_{K_s}}. \quad (4)$$

It is known that any tree that contains at least two vertices is two-colored. Any simple cycle with an even number of vertices is two-colored, and with an odd number of vertices is three-colored. Since an overtree $ST_{n,k}$ is represented in the form of a simple cycle C_k, at one or several vertices of which trees are "planted", therefore, the following equality holds

$$\chi(ST_{n,k}) = \begin{cases} 2, & k \equiv 0 \,(\mathrm{mod}\, 2); \\ 3, & otherwise. \end{cases} .$$

Consider an overtree $ST_{n,3}$. Let the edge u is an edge of the cycle C_3. Denote by y and z its end vertices. We now apply the Theorem 5 (equality (3)).

$$f_{ST_{n,3}}(x) = f_{T_n}(x) - f_{(ST_{n,3})_{y=z}}(x) =$$
$$= f_{T_n}(x) - f_{T_{n-1}}(x) = x\cdot(x-1)^{n-1} - x\cdot(x-1)^{n-2} =$$
$$= x\cdot(x-1)^{n-2}\cdot(x-2).$$

Thus, all overtrees $ST_{n,3}$ (some of them are non-isomorphic) have the same chromatic polynomial

$$f_{ST_{n,3}}(x) = x\cdot(x-1)^{n-2}\cdot(x-2). \quad (5)$$

Consider an overtree $ST_{n,k}$ for which $k > 3$. We take an arbitrary arc of the cycle C_k on it and apply Theorem 5, then

$$f_{ST_{n,k}}(x) = f_{T_n}(x) - f_{ST_{n-1,k-1}}(x). \quad (6)$$

The recurrence relation (6) is obtained, which one can now apply together with the formula (5). Consider its application

$$f_{ST_{n,4}}(x) = f_{T_n}(x) - f_{ST_{n-1,3}}(x) =$$
$$= x\cdot(x-1)^{n-1} - x\cdot(x-1)^{n-3}\cdot(x-2) =$$
$$= x\cdot(x-1)^{n-3}\cdot(x^2-2x+1-x+2) =$$
$$= x\cdot(x-1)^{n-3}\cdot(x^2-3x+3).$$

Thus, the following formula is obtained:

$$f_{ST_{n,4}}(x) = x\cdot(x-1)^{n-3}\cdot(x^2-3x+3). \quad (7)$$

It is clear that the recurrence relation (6) and formula (5) allow to get the formula for $f_{ST_{n,k}}(x)$. In this case, such a formula will depend only on n and k.

However, to obtain it, we do not use the recurrence relation (6), but a different technique and the fact that the formula for the chromatic polynomial of a simple cycle is known.

We start with the overtree $ST_{n,n}$. It is known that in this case $ST_{n,n} = C_n$ and we can use the formula for the chromatic polynomial of a simple cycle:

$$f_{ST_{n,n}}(x) = f_{C_n}(x) = (x-1)^n - (-1)^{n-1}\cdot(x-1). \quad (8)$$

Consider now the overtree $ST_{n,k}$ for $3 \le k < n$. By Theorem 2, it has at least one pendant vertex y. Consider as graph G_1 the graph which is generated by incident to vertex y edge u, and as graph G_2 the graph obtained from $ST_{n,k}$ by deleting pendant vertex y. It is clear that intersection of two these graphs consists of only one vertex, that is, it is a complete graph K_1. Therefore, we can use Theorem 6 (the equality (4):

$$f_{ST_{n,k}} = \frac{f_{T_2}(x)\cdot f_{ST_{n-1,k}}(x)}{f_{K_1}(x)} =$$
$$= \frac{x\cdot(x-1)\cdot f_{ST_{n-1,k}}(x)}{x} = (x-1)\cdot f_{ST_{n-1,k}}(x).$$

Thus, we have obtained one more recurrence relation for the chromatic functions of overtrees. Let's apply it sequentially until we cut off all the vertices of the trees which are "planted" in vertices of the simple cycle C_k. As a result, we get the following formula:

$$f_{ST_{n,k}}(x) = (x-1)^{n-k} \cdot f_{C_k}(x).$$

Taking into account that the chromatic polynomial of a simple cycle is known, we obtain a formula for the chromatic polynomial of any overtree $ST_{n,k}$ with $3 \leq k < n$:

$$f_{ST_{n,k}}(x) = (x-1)^{n-k} \cdot \left((x-1)^k - (-1)^{k-1} \cdot (x-1)\right). \tag{9}$$

Note that in case $k = n$ it coincides with the formula (8) we obtained earlier. Hence the formula (9) is valid for all overtrees $ST_{n,k}$ with $3 \leq k \leq n$.

Let's check that it gives the same result for $ST_{n,3}$ and for $ST_{n,4}$, which we obtained earlier.

$$ST_{n,3}(x) = (x-1)^{n-3} \cdot \left((x-1)^3 - (x-1)\right) =$$
$$= (x-1)^{n-2} \cdot (x^2 - 2x) = x \cdot (x-1)^{n-2} \cdot (x-2).$$

This result coincides with the formula (5).

$$ST_{n,4}(x) = (x-1)^{n-4} \cdot \left((x-1)^4 + (x-1)\right) =$$
$$= (x-1)^{n-3} \cdot (x^3 - 3x^2 + 3x) = x \cdot (x-1)^{n-3} \cdot (x^2 - 3x + 3).$$

The last coincides with the formula (7).

Let's find now the chromatic of graph G_c in the figure 5. It

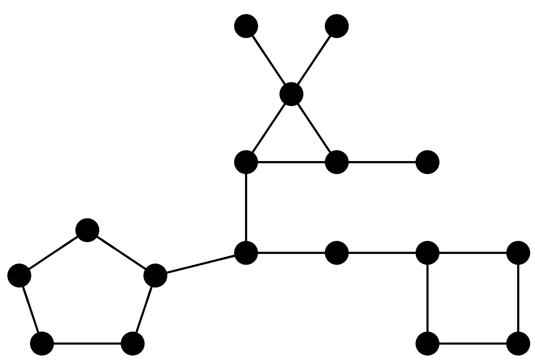

Figure 5. Graph, which contains three cycles.

is clear graph G_c is obtained from tree, which is shown in the figure 6, replacing three its vertices by cycles C_5, C_5 and C_5.

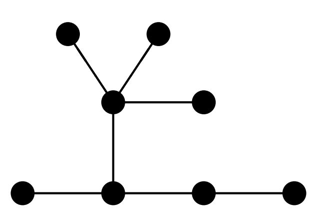

Figure 6. Tree for the graph G_c.

Select the overtree $ST_{6,5}$ part and the other part on the graph G_c (see figure 7).

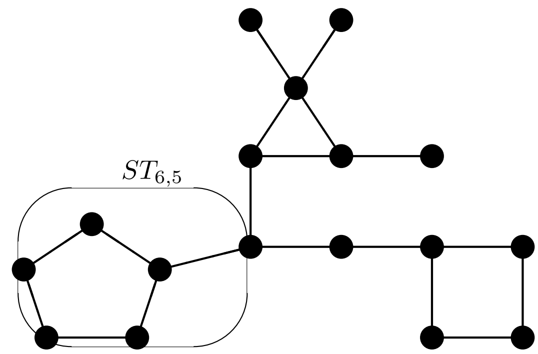

Figure 7. Overtree $ST_{6,5}$ and the other part of the graph G_c.

The other part of G_c can be represented as a union of two overtrees $ST_{7,3}$ and $ST_{5,4}$, which have a common vertex. Thus we can apply theorem 6 (equality (4)) twice. As a result it is obtained the following equality:

$$f_{G_c}(x) = \frac{g(x)}{x^2}, \tag{10}$$

where $g(x) = (x-1)^{10} \cdot \left((x-1)^2 - 1\right) \cdot \left((x-1)^3 + 1\right) \times$ $\times \left((x-1)^4 - 1\right)$.

Remark 1. *The numerator of the fraction is divisible by the denominator, that is, after cancellation, the right-hand side of equality* (10) *is a polynomial with integer coefficients, as befits a chromatic polynomial.*

Remark 2. *It is clear that the representation of the last graph in the form of a union of overtrees is not unique, but the right-hand side of equality* (10) *does not depend on this.*

Now we describe the general construction of graphs that are similar to the one we have just considered. Take a tree T_n and a sequence of natural numbers $\{k_i\}_{i=1}^s$ where $1 \leq s \leq n$ and $k_i \geq 3 \; \forall \, i \in [1; s]_Z$. Select s different vertices $\{x_1, \ldots, x_s\}$ on the tree T_n and replace each of the selected vertices x_i with a simple cycle C_{k_i}.

Obviously, the resulting graph, that we denote as $T_n \leftarrow \{C_{k_i}\}_{i=1}^s$, is such that the procedure for calculating the chromatic function which described in the last example is applicable to it. Then the following formula for the chromatic polynomial is obtained:

$$f_{T_n \leftarrow \{C_{k_i}\}_{i=1}^s}(x) = \frac{(x-1)^{n-1} \cdot h(x)}{x^{s-1}}, \tag{11}$$

where

$$h(x) = \prod_{i=1}^{s} \left((x-1)^{k_i} - (-1)^{k_i} \cdot (x-1)\right).$$

4 Conclusions

Probably, the last described class of graphs $T_n \leftarrow \{C_{k_i}\}_{i=1}^s$ could be differently defined as a class of connected graphs that contain a finite set of cycles without common vertices. But in last case it would be necessary to prove the theorem that the graph obtained from such a graph by replacing each cycle with a vertex is a tree.

It is also interesting that the polynomial in the numerator of the fraction in the formula (11) is divisible by x^{s-1} where $1 \leq s \leq n$. This follows from the fact that the chromatic function of any graph is a polynomial.

REFERENCES

[1] I. M. Erusalimskiy. "Discrete Mathematics: Theory, Problems, Applications", Vuzovskaya kniga, Moscow, 2008.

[2] A. A. Zykov. "Fundamentals of graph theory", Vuzovskaya kniga, Moscow, 2004.

[3] I. M. Erusalimskiy. "This 'Simple' Mathematics", Southern Federal University, Rostov-on-Don – Taganrog, 2020.

[4] C.-Y. Chao, E. G. Whitehead. "On Chromatic Equivalence of Graphs, Theory and Applications of Graphs", Lecture Notes in Mathematics, Vol.642, 121-131, 1978.

[5] F. M Dong, K. M. Koh, K. L. Teo. "Chromatic polynomials and chromaticity of graphs", World Scientific Publishing Company, 2020, DOI: 10.1142/5814.

[6] O. Giménez, P. Hliněný, M. Noy. "Computing the Tutte polynomial on graphs of bounded clique-width", Proc. 31st Int. Worksh. Graph-Theoretic Concepts in Computer Science (WG 2005), Lecture Notes in Computer Science, Vol.3787, 59-68, 2005.

[7] L. Helme-Guizon, R. Yongwu. "A categorification of the chromatic polynomial", Algebraic & Geometric Topology, Vol. 5, Issue 4, 1365-1368, 2005, DOI: 10.2140/agt.2005.5.1365.

[8] J. Huh. "Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs", 2012, arXiv:1008.4749v3.

[9] B. Jackson. "Zero-Free Interval for Chromatic Polynomials of Graphs", Combinatorics, Probability and Computing, Vol. 2(3), 325-336, 1993, DOI: 10.1017/S0963548300000705.

[10] F. Jaeger, D. L. Vertigan, D. J. A. Welsh. "On the computational complexity of the Jones and Tutte polynomials", Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 108, Issue 1, 35-53, 1990.

[11] N. Linial. "Hard enumeration problems in geometry and combinatorics", SIAM J. Algebr. Discrete Methods, Vol. 7, Issue 2, 331-335, 1986, DOI: 10.1137/0607036.

[12] J. A.Makowsky, U. Rotics, I. Averbouch, B. Godlin. "Computing graph polynomials on graphs of bounded clique-width", Proc. 32nd Int. Worksh. Graph-Theoretic Concepts in Computer Science (WG 2006), Lecture Notes in Computer Science, Vol. 4271, 191-204, 2006, DOI: 10.1007/11917496_18.

[13] J. G. Oxley, D. J. A. Welsh. "Chromatic, flow and reliability polynomials: The complexity of their coefficients", Combinatorics, Probability and Computing, Vol. 11, Issue 4, 403-426, 2002, DOI: 10.1017/S0963548302005175.

[14] S. Pemmaraju, S. Skiena. "Computational Discrete Mathematics: Combinatorics and Graph Theory with Mathematica", Cambridge University Press, 2003.

[15] K. Sekine, H. Imai, S. Tani. "Computing the Tutte polynomial of a graph of moderate size", Algorithms and Computation, 6th International Symposium, Lecture Notes in Computer Science, Vol. 1004, 224-233, 1995.

[16] A. D. Sokal. "Chromatic Roots are Dense in the Whole Complex Plane", Combinatorics, Probability and Computing, Vol. 13, Issue 2, 221-261, 2004, DOI: 10.1017/S0963548303006023.

[17] R. P. Stanley. "Acyclic orientations of graphs", Discrete Math., Vol. 5, Issue 2, 171-178, 1973, DOI: 10.1016/0012-365X(73)90108-8.

[18] J. Gasteiger, T. Engel. "Chemoinformatics : A Textbook", John Wiley & Sons, 2003.

[19] A. R. Leach, V. J. Gillet. "An Introduction to Chemoinformatics", Springer, 2003.

[20] Veninstine Vivik J , Sheeba Merlin G , P. Xavier , Nila Prem JL , "On Tensor Product and Colorability of Graphs", Mathematics and Statistics, Vol. 10, No. 2, 329-334, 2022, DOI: 10.13189/ms.2022.100207.