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Fractional Differential Equations and Matrix Bicomplex Two-parameter Mittag-Leffler Functions

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Abstract The skew field of Quaternions is the best known extension of the field of Complex numbers. The beauty of the Quaternions is that they form a field but the handicap is loss of commutativity. Thus the four- dimensional algebra called Bicomplex numbers with the set of all Complex numbers as a subalgebra preserving commutativity came into existence, by considering two imaginary units. The conventional calculus is generalized using Fractional calculus which is useful to extend derivatives of integer order to fractional order. Due to their vast applications to various disciplines of Science and Engineering, Mittag- Leffler functions have become prominent. Our contribution here is a combination of all the three streams mentioned above. In our research findings, bicomplex two-parameter Mittag- Leffler functions have been obtained as the solutions for the set of fractional differential equations that are linear in bicomplex space. A block diagonal of a square matrix A is a diagonal matrix whose Principal diagonal elements are square matrices A_i and the diagonal elements of A_i lie along the diagonal of A. A Jordan block is a matrix that is upper triangular with $\lambda \in C$ in the Principal diagonal, 1s just above the Principal diagonal and all other entries as 0. A Jordan Canonical form is a block diagonal matrix where each block is Jordan. A minimal polynomial of a matrix A is a polynomial which is monic in A with least degree. By using the methods of the minimal polynomial (eigenpolynomial) and Jordan canonical matrix, we have computed matrix Mittag–Leffler functions. The solutions obtained for the numerical examples have been visualized and interpreted using MATLAB.

Keywords Bicomplex Mittag – Leffler Function, Bicomplex Laplace Transform, Fractional Calculus, Fractional Derivative, Fractional Differential Equation

1 Introduction and Preliminaries

Fractional calculus generalizes the conventional integral and differential operators. The term Fractional calculus appeared in History as an answer to the question of whether the derivatives of integer order can be extended to fractional order.

The Riemann-Liouville, the Grünwald-Letnikov fractional integral and derivative, and the Liouville-Caputo fractional derivative are often used in this context [1, 2, 3, 4, 5, 6, 7]. To know about the applications of fractional calculus one can refer [8, 9, 10, 11, 12, 13].

Let $f(t)$ be defined on the positive Real line. The Riemann–Liouville integral of order η is defined as

$$\mathbb{J}_t^\eta f(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} f(\tau) d\tau \quad for \quad \eta > 0 \tag{1}$$

and $\mathbb{J}_t^\eta f(t) = f(t)$ for $\eta = 0$. Giving (1) a convolution structure we have

$$\mathbb{J}_t^\eta f(t) = \frac{t^{\eta-1}}{\Gamma(\eta)} * f(t) \tag{2}$$

The Riemann-Liouville derivative of order γ and Liouville-Caputo fractional derivative are respectively defined to be

$${}^R\mathbb{D}_t^\gamma f(t) = \frac{d^n}{dt^n}(\mathbb{J}_t^{n-\gamma} f(t)), \quad 0 < n-1 < \gamma \leq n, \tag{3}$$

$$\mathbb{D}_t^\gamma f(t) = \mathbb{J}_t^{n-\gamma} f^{(n)}(t), \quad 0 < n-1 < \gamma \leq n. \tag{4}$$

One can refer to [14, 15, 16, 17] for the consistency of the system, uniqueness of solutions and stability. For analytical and numerical methods, one can see [18, 19, 20, 21].

In October, 1843, Sir William Rowan Hamilton, came across a very special structure namely Quaternions, which allows the existence of multiplicative inverse for every element other than the additive identity. It is one of the generalizations of the complex numbers.

The handicap of Quaternions was that commutativity was not satisfied. To overcome this handicap, in 1892, Corrado Segre (1860-1924)[22], inspired by the work of Hamilton and Clifford, introduced a new algebric structure, called Bicomplex numbers.

Bicomplex numbers are defined as

$$C_2 = \{\xi : \xi = x_0 + i_1x_1 + i_2x_2 + jx_3 | x_0, x_1, x_2, x_3 \in C_0\}, \tag{5}$$

or

$$C_2 = \{\xi : \xi = z_1 + i_2z_2 | z_1, z_2 \in C_1\}, \tag{6}$$

where i_1 and i_1 are imaginary units such that $i_1^2 = i_2^2 = -1$, $i_1i_2 = i_2i_1 = j, j^2 = 1$ and C_0, C_1 and C_2 denote the sets of real numbers, complex numbers and bicomplex numbers respectively.

Every bicomplex number can be uniquely written as a combination of the complex numbers e_1 and e_2, that is,

$$\xi = z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2, \tag{7}$$

where $e_1 = \frac{1+j}{2}, e_2 = \frac{1-j}{2}; e_1 + e_2 = 1$ and $e_1e_2 = e_2e_1 = 0$.

Such a representation is termed as Idempotent representation of ξ. The coefficients $(z_1 - i_1z_2)$ and $(z_1 + i_1z_2)$ are said to be the Idempotent components of the bicomplex number $\xi = z_1 + i_2z_2$ and $\{e_1, e_2\}$, the Idempotent Basis.

The complex Mittag-Leffler function was introduced and analysed by Mittag- Leffler [23] and is defined by

$$\mathbb{E}_\gamma(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(\gamma k + 1)}, \gamma > 0, z \in \mathbb{C}. \tag{8}$$

It has been extended to bicomplex one and two-parameter Mittag-Leffler functions by Agarwal et.al.[24, 25]. They are given by

$$\mathbb{E}_\gamma(\xi) = \sum_{k=o}^{\infty} \frac{\xi^k}{\Gamma(\gamma k + 1)}, \gamma > 0, \xi \in \mathbb{C}_2. \tag{9}$$

$$\mathbb{E}_{\gamma,\eta}(\xi) = \sum_{k=o}^{\infty} \frac{\xi^k}{\Gamma(\gamma k + \eta)}, \gamma > 0, \xi \in \mathbb{C}_2. \tag{10}$$

There exist various methods to extend the definition of the Mittag-Leffler function to matrix arguments. Matrix Mittag-Leffler function, Jordan canonical form, Cauchy integral and Hermite interpolation are to name a few [26]. If M is a diagonal matrix with eigenvalues $\vartheta_1, ..., \vartheta_n$, then $\mathbb{E}_{\gamma,\eta}(M)$ is also a diagonal matrix, namely $\mathbb{E}_{\gamma,\eta}(M) = diag(\mathbb{E}_{\gamma,\eta}(\vartheta_1), ..., \mathbb{E}_{\gamma,\eta}(\vartheta_n))$ and the diagonal arguments can be only Mittag- Leffler functions with scalar arguments. The matrix Mittag-Leffler function is given by

$$\mathbb{E}_{\gamma,\eta}(M) = \sum_{k=o}^{\infty} \frac{M^k}{\Gamma(\gamma k + \eta)}, \gamma > 0, \eta > 0 \tag{11}$$

The matrix $\mathbb{E}_{\gamma,\eta}(M)$ is well-defined for any n^{th} order matrix M. Here $M^0 = I$, the unit matrix of order n.

Srivastava [27] and Tomovski et.al [28] have discussed about a set of fractional differential operators which are generalized and their association with Mittag–Leffler functions. The authors [29] have suggested multi-variable complex Mittag-Leffler functions as another analytic technique to find the solutions.

The solution of the linear fractional system can be represented as the matrix Mittag–Leffler function. In [30], Krylov subspace methods have been applied to compute the matrix Mittag-Leffler functions and results on convergence have been discussed. Matychyn and Onyshchenko have described the Jordan canonical form [31] and MATLAB code has been used to solve the Bagley–Torvik equation.

The original scalar Mittag–Leffler function and its derivatives are essential to compute the matrix Mittag–Leffler function and the process of evaluation was discussed by Garrappa, R. and Popolizio, M. in [32].

The matrix Mittag–Leffler functions have been useful in generalizing exponential time differencing methods for fractional differential equations and to find the solutions of multi-term fractional differential equations. For details see [33, 34, 35]. The solution of a linear fractional system can also be represented by using the matrix bicomplex Mittag-Leffler functions.

Agarwal et.al. [25], have introduced the bicomplex two-parameter Mittag-Leffler function and analysed it for its properties. Inspired by Duan et.al. [12], we have obtained the solutions for fractional differential equations in terms of bicomplex two-parameter Mittag-Leffler function using Jordan canonical matrix and minimal polynomial (eigen-polynomial).

2 Computation of Matrix bicomplex Mittag - Leffler Functions

In this section, we have computed the matrix bicomplex Mittag–Leffler functions $\mathbb{E}_{\gamma,\eta}(Mt^\gamma)$, using Jordan canonical matrix and minimal polynomials.

2.1 Computation of $\mathbb{E}_{\gamma,\eta}(Mt^\gamma)$ using Jordan Canonical Matrix

Let M be a matrix of order n, $J = diag(J_1, J_2, ..., J_s)$, the Jordan canonical form of M and $A = PJP^{-1}$. Here the

Jordan blocks $J_i, i = 1, 2, ..., s$ are given by

$$\mathbb{J}_i = \begin{bmatrix} \vartheta_i & 1 & & & \\ & \vartheta_i & 1 & & \\ & & ... & ... & \\ & & & ... & 1 \\ & & & & \vartheta_i \end{bmatrix}_{n_i \times n_i}, \sum_{i=1}^{s} n_i = n.$$

From matrix theory, we have

$$\begin{aligned} \mathbb{E}_{\gamma,\eta}(Mt^\gamma) &= P\mathbb{E}_{\gamma,\eta}(\mathbb{J}t^\gamma)P^{-1} \\ &= Pdiag(\mathbb{E}_{\gamma,\eta}(\mathbb{J}_1 t^\gamma), ..., \mathbb{E}_{\gamma,\eta}(\mathbb{J}_s t^\gamma))P^{-1} \end{aligned} \quad (12)$$

A simple computation leads us to

$$f(\mathbb{J}_i) = \begin{bmatrix} f(\vartheta_i) & f'(\vartheta_i) & \frac{1}{2!}f''(\vartheta_i) & ... & \frac{1}{(n-1)!}f^{(n_i-1)}(\vartheta_i) \\ & f(\vartheta_i) & f'(\vartheta_i) & ... & \frac{1}{(n-2)!}f^{(n_i-2)}(\vartheta_i) \\ & & f(\vartheta_i) & ... & ... \\ & & & ... & f'(\vartheta_i) \\ & & & & f(\vartheta_i) \end{bmatrix} \quad (13)$$

By assigning $f(\xi) = \mathbb{E}_{\gamma,\eta}(\xi t^\gamma)$, we obtain

$$\mathbb{E}_{\gamma,\eta}(\mathbb{J}_i t^\gamma) = \begin{bmatrix} \mathbb{E}_{\gamma,\eta}(\vartheta_i t^\gamma) & t^\gamma \mathbb{E}'_{\gamma,\eta}(\vartheta_i t^\gamma) & ... & \frac{t^{(n_i-1)\gamma}\mathbb{E}^{(n_i-1)}_{\gamma,\eta}(\vartheta_i t^\gamma)}{(n-1)!} \\ & \mathbb{E}_{\gamma,\eta}(\vartheta_i t^\gamma) & ... & \frac{t^{(n_i-2)\gamma}\mathbb{E}^{(n_i-2)}_{\gamma,\eta}(\vartheta_i t^\gamma)}{(n-2)!} \\ & & ... & ... \\ & & ... & t^\gamma \mathbb{E}'_{\gamma,\eta}(\vartheta_i t^\gamma) \\ & & & \mathbb{E}_{\gamma,\eta}(\vartheta_i t^\gamma) \end{bmatrix}$$

where $\mathbb{E}^{(k)}_{\gamma,\eta}(\vartheta_i t^\gamma) = \frac{d^k \mathbb{E}_{\gamma,\eta}(\xi)}{d\xi^k}|_{\xi=\vartheta_i t^\gamma}$

2.2 Estimation of $\mathbb{E}_{\gamma,\eta}(Mt^\gamma)$ using Minimal Polynomial (Eigenpolynomial)

The minimal polynomial of a matrix A of order n over a field F is a polynomial $m_A(\vartheta)$, in a single-variable with the coefficients of highest-degree term as 1, and the least degree term $m_A(M)$ as 0.
According to matrix theory, for any matrix of order n, there exists a unique minimal polynomial $m_A(\vartheta)$. For any square matrix M of order n, the minimal and eigenpolynomials will have the same roots. The minimal polynomial of the matrix M cannot have a degree greater than n, according to the Cayley-Hamilton theorem.
Let $\vartheta_1, \vartheta_2, ..., \vartheta_s$ be the distinct eigenvalues of an $n \times n$ matrix M and let

$$\begin{aligned} m_A(\vartheta) = (\vartheta - \vartheta_1)^{e_1}(\vartheta - \vartheta_2)^{e_2}...(\vartheta - \vartheta_s)^{e_s}, \\ \sum_{j=1}^{s} e_j = m \leq n \end{aligned} \quad (14)$$

be the minimal polynomial of M. According to Wu et.al [9], $\mathbb{E}_{\gamma,\eta}(Mt^\gamma)$ can be written as a polynomial of degree $m-1$ in matrix uniquely and it is termed as the Lagrange–Sylvester interpolation polynomial and we let this polynomial

$$\psi(\vartheta; t) = b_0(t) + b_1(t)\vartheta + ... + b_{m-1}(t)\vartheta^{m-1} \quad (15)$$

Then $\psi(M;t) = \mathbb{E}_{\gamma,\eta}(Mt^\gamma)$ holds $\iff$ $\psi(M;t)$ and $\mathbb{E}_{\gamma,\eta}(Mt^\gamma)$ are consistent over the matrix's spectrum, that is

$$\frac{d^k\psi(\vartheta;t)}{d\vartheta^k}|_{\vartheta=\vartheta_j} = t^{k\gamma}\mathbb{E}^k_{\gamma,\eta}(\vartheta_j t^\gamma). \quad (16)$$

where j varies from 1 to s and k from 1 to $e_j - 1$. The coefficients in (15) are determined using (16). The matrix Mittag-Leffler function $\mathbb{E}_{\gamma,\eta}(Mt^\gamma)$ is thus obtained, by a finite sum as

$$b_0(t) + b_1(t)M + ... + b_{m-1}(t)M^{m-1} = \mathbb{E}_{\gamma,\eta}(Mt^\gamma) \quad (17)$$

We can use the eigen polynomial in place of the minimal polynomial $m_A(\vartheta)$ if it is difficult to determine the minimal polynomial.

$$\begin{aligned} det(\vartheta I - M) = (\vartheta - \vartheta_1)^{m_1}(\vartheta - \vartheta_2)^{m_2}...(\vartheta - \vartheta_s)^{m_s}, \\ \sum_{j=1}^{s} m_j = n \end{aligned} \quad (18)$$

A polynomial of degree $n-1$

$$\pi(\vartheta; t) = b_0(t) + b_1(t)\vartheta + ... + b_{n-1}(t)\vartheta^{n-1} \quad (19)$$

is determined such that $\pi(\vartheta; t) = \mathbb{E}_{\gamma,\eta}(Mt^\gamma)$. This seems to be true and is given by

$$\frac{d^k\pi(\vartheta;t)}{d\vartheta^k}|_{\vartheta=\vartheta_i} = t^{k\gamma}\mathbb{E}^k_{\gamma,\eta}(\vartheta_i t^\gamma), i = 1, ..., s, k = 0, 1, .., m_i - 1. \quad (20)$$

3 Illustrations

Here we have considered three examples and computed matrix Mittag-Leffler function using the methods discussed in section 3. We also have used MATLAB code to visualize and interpret the results

Example 1 *Let*

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad (21)$$

By utilising the canonical form $M = AJA^{-1}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

we obtain the matrix Mittag-Leffler function $\mathbb{E}_{\gamma,\eta}(Mt^\gamma)$

$$\mathbb{E}_{\gamma,\eta}(Mt^\gamma) = A \begin{bmatrix} \mathbb{E}_{\gamma,\eta}(-t^\gamma) & 0 & 0 \\ 0 & \mathbb{E}_{\gamma,\eta}(-t^\gamma) & 0 \\ 0 & 0 & \mathbb{E}_{\gamma,\eta}(2t^\gamma) \end{bmatrix} A^{-1} \quad (22)$$

As an alternate, using the matrix's eigenpolynomial $det(\vartheta I - M) = (\vartheta - 2)(\vartheta + 1)^2$ and we allow the polynomial $b_0(t) +$

$b_1(t)\vartheta + b_2(t)\vartheta^2$ and the function $\mathbb{E}_{\gamma,\eta}(\vartheta t^\gamma)$ to be the same in the spectrum of matrix M. As a result we attain

$$\begin{aligned} b_0(t) - b_1(t) + b_2(t) &= \mathbb{E}_{\gamma,\eta}(-t^\gamma) \\ b_1(t) - 2b_2(t) &= t^\gamma \mathbb{E}'_{\gamma,\eta}(-t^\gamma) \\ b_0(t) + 2b_1(t) + 4b_2(t) &= \mathbb{E}_{\gamma,\eta}(2t^\gamma) \end{aligned}$$

Solving the system we have

$$\begin{aligned} b_0(t) &= \frac{8}{9}\mathbb{E}_{\gamma,\eta}(-t^\gamma) + \frac{1}{9}\mathbb{E}_{\gamma,\eta}(2t^\gamma) + \frac{2}{3}t^\gamma \mathbb{E}'_{\gamma,\eta}(-t^\gamma) \\ b_1(t) &= \frac{1}{3}t^\gamma \mathbb{E}_{\gamma,\eta}(-t^\gamma) + \frac{2}{9}(\mathbb{E}_{\gamma,\eta}(2t^\gamma) - \mathbb{E}_{\gamma,\eta}(-t^\gamma)) \\ b_2(t) &= \frac{1}{9}\mathbb{E}_{\gamma,\eta}(2t^\gamma) - \frac{1}{9}\mathbb{E}_{\gamma,\eta}(-t^\gamma) - \frac{1}{3}t^\gamma \mathbb{E}_{\gamma,\eta}(-t^\gamma) \end{aligned}$$

Consequently, the matrix Bicomplex Mittag-Leffler function changes to

$$\begin{aligned} \mathbb{E}_{\gamma,\eta}(Mt^\gamma) = {}& [\frac{8}{9}\mathbb{E}_{\gamma,\eta}(-t^\gamma) + \frac{1}{9}\mathbb{E}_{\gamma,\eta}(2t^\gamma) + \frac{2}{3}t^\gamma \mathbb{E}'_{\gamma,\eta}(-t^\gamma)]I \\ & + [\frac{1}{3}t^\gamma \mathbb{E}_{\gamma,\eta}(-t^\gamma) + \frac{2}{9}(\mathbb{E}_{\gamma,\eta}(2t^\gamma) - \mathbb{E}_{\gamma,\eta}(-t^\gamma))]M \\ & + [\frac{1}{9}\mathbb{E}_{\gamma,\eta}(2t^\gamma) - \frac{1}{9}\mathbb{E}_{\gamma,\eta}(-t^\gamma) - \frac{1}{3}t^\gamma \mathbb{E}_{\gamma,\eta}(-t^\gamma)]M^2 \end{aligned} \quad (23)$$

We found that the Equations (22) and (23) are consistent and the same for the minimal polynomial and eigenpolynomial in this case.

Example 2 *[36] Let us consider*

$$\begin{aligned} D_*^\gamma x_1(t) &= x_1(t) + x_2(t) \\ D_*^\eta x_2(t) &= -x_1(t) + x_2(t) \end{aligned}$$

with initial condionts $x_1(0) = 0$ and $x_2(0) = 1$.
Consider the matrix Mittag-Leffler function $\mathbb{E}_{\gamma,\eta}(Mt^\gamma)$, where

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad (24)$$

In order to solve the initial value problem, we need to do the following:

$$\mathbb{D}_t^\gamma x(t) = Mx(t), x(0) = (0,1)^T \quad (25)$$

$M = AJA^{-1}$, with

$$A = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \text{ and } \quad J = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$$

leads to

$$\mathbb{E}_{\gamma,\eta}(Mt^\gamma) = A \begin{bmatrix} \mathbb{E}_{\gamma,\eta}((1+i)t^\gamma) & 0 \\ 0 & \mathbb{E}_{\gamma,\eta}((1-i)t^\gamma) \end{bmatrix} A^{-1} \quad (26)$$

The eigenpolynomials of the matrix M can be used to solve the problem, $det(\vartheta I - M) = (\vartheta + (-1-i))(\vartheta + (-1+i))$ and by letting the polynomial $b_0 t + b_1(t)\vartheta$ and the function $\mathbb{E}_{\gamma,\eta}(\vartheta t^\gamma)$ in terms of matrix spectrum, to be identical, we find

$$\begin{aligned} b_0(t) + b_1(t)(1+i) &= \mathbb{E}_{\gamma,\eta}((1+i)t^\gamma) \\ b_0(t) + b_1(t)(1-i) &= \mathbb{E}_{\gamma,\eta}((1-i)t^\gamma) \end{aligned}$$

The system yields

$$\begin{aligned} b_0(t) &= \frac{1+i}{2}\mathbb{E}_{\gamma,\eta}((1+i)t^\gamma) + \frac{1-i}{2}\mathbb{E}_{\gamma,\eta}((1-i)t^\gamma) \\ b_1(t) &= \frac{-i}{2}\mathbb{E}_{\gamma,\eta}((1+i)t^\gamma) + \frac{i}{2}\mathbb{E}_{\gamma,\eta}((1-i)t^\gamma) \end{aligned}$$

as the solutions. Thus, the matrix Bicomplex Mittag-Leffler function also is of the form

$$\begin{aligned} \mathbb{E}_{\gamma,\eta}(Mt^\gamma) = {}& [\frac{1+i}{2}\mathbb{E}_{\gamma,\eta}((1+i)t^\gamma) + \frac{1-i}{2}\mathbb{E}_{\gamma,\eta}((1-i)t^\gamma)]I \\ & + [\frac{-i}{2}\mathbb{E}_{\gamma,\eta}((1+i)t^\gamma) + \frac{i}{2}\mathbb{E}_{\gamma,\eta}((1-i)t^\gamma)]M \end{aligned} \quad (27)$$

We checked and observed that the Equations (27) and (26) are consistent and the solution of (25) parameterized by the order γ is

$$\begin{aligned} x(t) &= \mathbb{E}_{\gamma,1}(Mt^\gamma)x(0) \\ &= \frac{1}{2}\begin{bmatrix} -i \\ 1 \end{bmatrix} \mathbb{E}_{\gamma,1}((1+i)t^\gamma) + \frac{1}{2}\begin{bmatrix} i \\ 1 \end{bmatrix} \mathbb{E}_{\gamma,1}((1-i)t^\gamma) \end{aligned} \quad (28)$$

If $\gamma = 1$, then the solution is

$$x(t) = e^t \begin{bmatrix} sint \\ cost \end{bmatrix} \quad (29)$$

By fixing η at one and varying γ we obtain Figures 1 and 2 that represent the nature of the solutions.

Example 3 *[37] Consider*

$$\begin{aligned} D_*^\gamma x_1(t) &= 2x_1(t) + x_2(t) \\ D_*^\eta x_2(t) &= -x_1(t) + 2x_2(t), \end{aligned}$$

a system of linear fractional differential equations, with initial conditions $x_1(0) = 2$ and $x_2(0) = 1$.
Consider the matrix Mittag-Leffler function $\mathbb{E}_{\gamma,\eta}(Mt^\gamma)$, where

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (30)$$

As a result, the IVP can be modeled as follows:

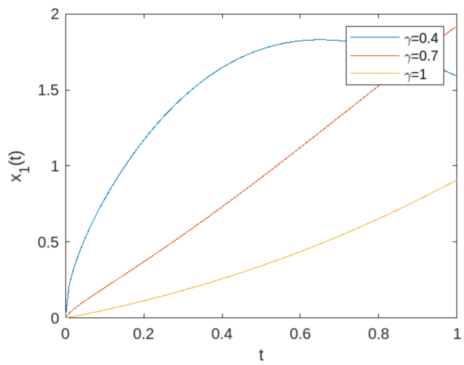
$$\mathbb{D}_t^\gamma x(t) = Mx(t), x(0) = (2,1)^T \quad (31)$$

Using the canonical form of Jordan $M = AJA^{-1}$, where

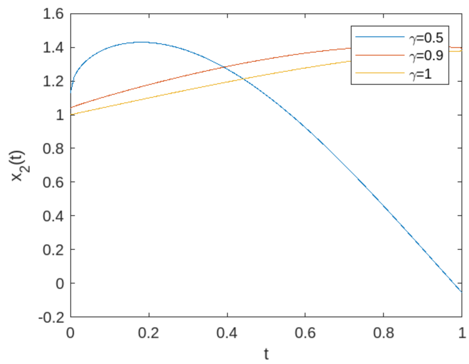
$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, J = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

we obtain

$$\mathbb{E}_{\gamma,\eta}(Mt^\gamma) = A \begin{bmatrix} \mathbb{E}_{\gamma,\eta}(3t^\gamma) & 0 \\ 0 & \mathbb{E}_{\gamma,\eta}(t^\gamma) \end{bmatrix} A^{-1} \quad (32)$$



(a) Figure 1. The graphical representation of $x_1(t)$



(b) Figure 2. The graphical representation of $x_2(t)$

On the other hand by using the eigenpolynomial of the matrix $M, det(\vartheta I - M) = (\vartheta - 3)(\vartheta - 1)$ *and the assumption of the polynomial* $b_0(t) + b_1(t)\vartheta$ *and the function* $\mathbb{E}_{\gamma,\eta}(\vartheta t^\gamma)$ *being the same in the spectrum of matrix* M*, we obtain*

$$b_0(t) + 3b_1(t) = \mathbb{E}_{\gamma,\eta}(3t^\gamma)$$
$$b_0(t) + b_1(t) = \mathbb{E}_{\gamma,\eta}(t^\gamma)$$

Solving the system

$$b_0(t) = \frac{-1}{2}\mathbb{E}_{\gamma,\eta}(3t^\gamma) + \frac{3}{2}\mathbb{E}_{\gamma,\eta}(t^\gamma)$$
$$b_1(t) = \frac{1}{2}\mathbb{E}_{\gamma,\eta}(3t^\gamma) - \frac{1}{2}\mathbb{E}_{\gamma,\eta}(t^\gamma)$$

This implies that the matrices Bicomplex Mittag-Leffler function also have representations

$$\mathbb{E}_{\gamma,\eta}(Mt^\gamma) = [\frac{-1}{2}\mathbb{E}_{\gamma,\eta}(3t^\gamma) + \frac{3}{2}\mathbb{E}_{\gamma,\eta}(t^\gamma)]I + [\frac{1}{2}\mathbb{E}_{\gamma,\eta}(3t^\gamma) - \frac{1}{2}\mathbb{E}_{\gamma,\eta}(t^\gamma)]M \quad (33)$$

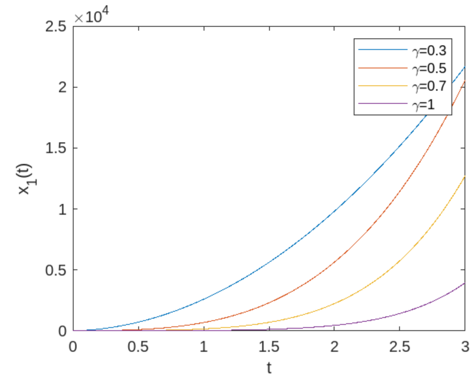
Equations (32) and (33) are consistent. The solution from (31) parameterized by the order γ *is*

$$x(t) = \mathbb{E}_{\gamma,1}(Mt^\gamma)x(0) = \frac{1}{2}\begin{bmatrix}3\\3\end{bmatrix}\mathbb{E}_{\gamma,1}((3)t^\gamma) + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\mathbb{E}_{\gamma,1}((1)t^\gamma) \quad (34)$$

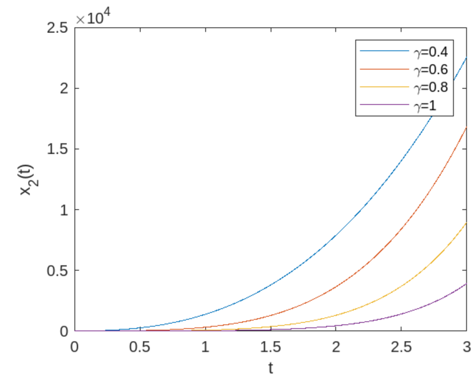
If $\gamma = 1$*, we find the solution to be*

$$x(t) = \frac{e^t}{2}\begin{bmatrix}3e^{2t}+1\\3e^{2t}-1\end{bmatrix} \quad (35)$$

By assigning η *the value one and varying* γ*, we obtain Figures 3 and 4 and they depict the nature of the solutions obtained.*



(a) Figure 3. The graphical representation of $x_1(t)$



(b) Figure 4. The graphical representation of $x_2(t)$

4 Conclusions

By using Jordan canonical form and minimal polynomials, we have found the solutions in terms of bicomplex two-parameter Mittag-Leffler functions to a system containing linear fractional differential equations. Visualization and interpretation of the solutions obtained have been done using MATLAB coding.

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