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Self-Adjoint Operators in Bilinear Spaces

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Abstract In this research, it was agreed that a bilinear form is an extension of the inner product since a symmetry bilinear form will be equivalent to the inner product over a field of real numbers. Concepts in bilinear space, such as the concept of orthogonality of two vectors, the concept of orthogonal subspace of a subspace, the concept of adjoint operators of a linear operator and the concept of closed subspace are defined according to those prevailing in the inner product space fact assumed to be extensions of the concepts applicable in the inner product space. In the context of a cap subspace, we can identify the necessary and sufficient conditions for any linear operator in a continuous Hilbert space. These results open up opportunities to introduce the concept of pseudo-continuous linear mapping in bilinear spaces. We have obtained the result that pseudo-continuous linear mapping spaces in bilinear spaces have a relationship with linear mapping spaces that have adjoint mapping. We have also obtained the result that the structure of linear operators limited to Hilbert spaces can be extended to pseudo-continuous operator structures in bilinearal spaces. In this study, we have generalized the properties of self-adjoint operators in product spaces in infinite dimensions to bilinear, including closed properties of addition operations, and scalar multiplication, commutative properties, properties of inverse operators, properties of zero operators, properties of polynomial operators over real fields, and orthogonal properties of eigenspaces of different eigenvalues.

Keywords Self-Adjoint Operator, Non-Degenerated Bilinear Forms, Pseudo-Continuity

1. Introduction

The bilinear forms are very closely related to the inner products. In any vector space over a field where a bilinear form is defined, it is called a bilinear space. It generally holds that a symmetry bilinear form will be equivalent to the inner product over a field of real numbers. According to this nature, in this research, it was agreed that a bilinear form is an extension of the inner product. Since in the inner product space, the concept of orthogonality of two vectors, the concept of orthogonal subspace of a subspace, the concept of adjoint operator of a linear operator and the concept of closed subspace have been known, then in bilinear space the concept is defined in the same way and can be viewed as an extension of the concept that applies in the inner product space.

2. Materials and Methods

We were able to identify the necessary and sufficient condition for linear operators in Hilbert spaces being continuous in terms of closed subspaces [1]. This fact gave us the opportunity to introduce the pseudo-continuous notion of linear mappings on bilinear spaces. We obtained that the class of pseudo-continuous linear mappings on bilinear spaces is nonetheless the class of linear mappings that have adjoint mappings [2]. As a result, a class of pseudo-continuous linear operators in a bilinear space forms a subalgebra. It is interesting to see how far the structure of the subalgebra of bounded linear operators on a Hilbert space can be extended to the subalgebra of pseudo-continuous operators in a bilinear space.

2.1 Study of Literature

The literature review shows that the research topic related to the bilinear form is not a new topic, because many researchers have studied this research topic, and some of them can be listed as follows.

1 Dokovic [3] studied the behavior of an odd-dimensional V bilinear space on a field with characteristics ≠2 which is algebraically closed. If space V can be composed as a degenerate orthogonal direct sum, then space V and isometric groups of space V can be composed into a principal component via canonical orthogonal decomposition. This composition is obtained by isomorphism of the direct product of the isometric groups of the prime components.

2 Alnajjar [4] showed the existence and uniqueness of a bilinear form so that the tridiagonal pairs of related linear operators are self-adjoint.

3 Gow [5] studied an n-dimensional V swinging bilinear space, which can be composed into as many as (n−1)/2 subspaces, each of which has n-dimensional (odd n) over K with K cyclic Galois expansion fields of degree-n. According to the Galois automorphism, it can be shown that the non-zero elements of each of these subspaces have a constant rank.

2.2. Bilinear Spaces

Suppose that V and W are vector spaces over the field F. The bilinear form $[-,-]$ is an operator on VxW, i.e.

$$[-,-]:VxW \to \mathrm{F}$$

that satisfies the following linear conditions

1 $$[\alpha v_1+\beta v_2,w]=\alpha[v_1,w]+\beta[v_2,w]$$
$$\forall v_1,v_2\in V;\quad \forall w\in W;\quad \forall\alpha,\beta\in\mathrm{F}$$

2 $$[v,\gamma w_1+\delta w_2]=\gamma[v,w_1]+\delta[v,w_2]$$
$$\forall v\in V;\quad \forall w_1,w_2\in W;\quad \forall\gamma,\delta\in\mathrm{F}$$

These two linear conditions state that each bilinear form applies the property of linearity in the first and second arguments [6]. A vector space over a field, in which a bilinear shape is defined, and it is called a bilinear space. Because in the inner products applying positivity properties, it is necessary to add a property that is equivalent to these properties, namely non-degenerated. The property is defined as follows:

Definition 2.2.1. The bilinear form on VxW is said to be non-degenerated if only the null vector in the bilinear space V is orthogonal bilinear to the bilinear space W and also only the zero vector in the bilinear space W which is orthogonal bilinear to the bilinear space V [6].

Here are some examples of bilinear spaces

1 Sequence Spaces over $\mathbf{R}$

$$V=\left\{(x_n)\mid\sum_{n=1}^{\infty}x_n^2<\infty\right\}$$

with a bilinear form

$$[-,-]:VxW\to\mathbf{R}$$

that is

$$[(x_n),(y_n)]=\sum_{n=1}^{\infty}x_ny_n$$

$$\forall(x_n),(y_n)\in V$$

that satisfies

$$[\alpha(x_n)+\beta(y_n),(z_n)]$$
$$=\sum_{n=1}^{\infty}(\alpha x_n+\beta y_n)z_n$$
$$=\alpha\sum_{n=1}^{\infty}x_nz_n+\beta\sum_{n=1}^{\infty}y_nz_n$$
$$=\alpha[(x_n),(z_n)]+\beta[(y_n),(z_n)]$$

and

$$[(x_n),\gamma(y_n)+\delta(z_n)]$$
$$=\sum_{n=1}^{\infty}x_n(\gamma y_n+\delta z_n)$$
$$=\gamma\sum_{n=1}^{\infty}x_ny_n+\delta\sum_{n=1}^{\infty}x_nz_n$$
$$=\gamma[(x_n),(y_n)]+\delta[(x_n),(z_n)]$$

2 Laurent series space truncated with coefficients in the $\mathbb{R}$ field

$$\mathbf{R}((z^{-1}))=\left\{\sum_{j=-\infty}^{N_f}f_jz^j\mid f_j\in\mathbf{R};N_f\in\mathbf{Z}\right\}$$

with a bilinear form

$$[-,-]:\mathbf{R}((z^{-1}))x\mathbf{R}((z^{-1}))\to\mathbf{R}$$

that is

$$[f(z),g(z)]=\sum_{j=-N_g-1}^{N_f}g_{-j-1}f_j$$

$$\forall f(z), g(z) \in \mathbf{R}\left(\left(z^{-1}\right)\right)$$

that satisfies

$$[\alpha f(z)+\beta g(z), h(z)] = \sum_{j=-N_h-1}^{N_{f+g}} h_{-j-1}(\alpha f_j + \beta g_j)$$

$$= \alpha \sum_{j=-N_h-1}^{N_f} h_{-j-1} f_j + \beta \sum_{j=-N_h-1}^{N_g} h_{-j-1} g_j$$

$$= \alpha[f(z), h(z)] + \beta[g(z), h(z)]$$

and

$$[f(z), \gamma g(z) + \delta h(z)]$$

$$= \sum_{j=-N_{g+h}-1}^{N_f} (\gamma g_{-j-1} + \delta h_{-j-1}) f_j$$

$$= \gamma \sum_{j=-N_g-1}^{N_f} g_{-j-1} f_j + \delta \sum_{j=-N_h-1}^{N_f} h_{-j-1} f_j$$

$$= \gamma[f(z), g(z)] + \delta[f(z), h(z)]$$

2.3. Closed Subspaces

The concept of closed subspaces in the bilinear spaces is defined as follows:

Definition 2.3.1 Suppose V is a bilinear space over the field F. Subset S of space V is said to be a closed subspace if

$$S^{\perp\perp} = S$$

where $S^{\perp\perp}$ is the orthogonal subspace of

$$S^{\perp} = \{x \in V \mid [x, s] = 0; \forall s \in S\} \text{ [7]}$$

Base on Definition 2.3.1, it is concluded that the subspace that is orthogonal with the orthogonal subspace S is all contained in the subspace S. The following describes some of the continuity properties of a linear operator in the inner product spaces (Hilbert) that can be extended to the bilinear spaces.

Theorem 2.3.2 Suppose H_1 and H_2 are Hilbert spaces where the inner product succession is $\langle -,- \rangle_1$ and $\langle -,- \rangle_2$. Suppose also τ is a linear operator in space $L(H_1, H_2)$. Hence the following statements are the equivalent

1 The operator τ is continuous.

2 There is only one adjoint operator τ^* in the $L(H_2, H_1)$ space that satisfies

$$\langle \tau(v), w \rangle_2 = \langle v, \tau^*(w) \rangle_1 \quad ; \quad \forall v \in H_1, w \in H_2$$

3 For each closed subspace S of the H_2 then $\tau^{-1}(S)$ is the closed subspace of the H_1 [1].

The above theorem inspired to extend the concept of continuity in the inner product spaces to bilinear spaces by utilizing the concept of closed subspaces.

Definition 2.3.3 Suppose V_1 and V_2 are bilinear spaces with successive bilinear forms $[-,-]_1$ and $[-,-]_2$. Suppose also τ is a linear operator in the $L(V_1, V_2)$. The adjoint operator of τ $(\equiv \tau^*)$ is a linear operator in the $L(V_2, V_1)$ that satisfies

$$[\tau(v), w]_2 = [v, \tau^*(w)]_1; \quad \forall v \in V_1, w \in V_2; \quad (1)$$

The definition of adjoint operator in the bilinear spaces is defined similarly to the definition of adjoint operator in Hilbert space. An extension of Theorem 2.3.2 on the bilinear spaces is presented in the following definition.

Definition 2.3.4 Suppose V_1 and V_2 are bilinear spaces with successive bilinear forms $[-,-]_1$ and $[-,-]_2$. The linear operator τ in $L(V_1, V_2)$ is said to be pseudo-continuous if for each closed subspace S of V_2 then $\tau^{-1}(S)$ is the closed subspace of V_1 [2].

The above definition said that the pseudo-continuity condition of a linear operator in the bilinear spaces can be related to the condition of the serendipity of premap of any closed subspace of the bilinear spaces.

2.4. Pseudo-Continuity Operators

As an implication of the Definition 2.3.4, the following theorem will be obtained.

Theorem 2.4.1 Suppose V_1, V_2 and V_3 are bilinear spaces over F with successive bilinear forms $[-,-]_1$, $[-,-]_2$ and $[-,-]_3$. Suppose that τ and σ are successively pseudo-continuous linear operators in $L(V_1, V_2)$ and $L(V_2, V_3)$. Then the following applies

1 The operator $(\alpha\tau)$ is the pseudo-continuous for each $\alpha \in \mathrm{F}$.

2 The operator $(\tau \circ \sigma)$ is the pseudo-continuous [1].

So the set of pseudo-continuous linear operators in bilinear space is closed to the composition operation and scalar multiplication.

Theorem 2.4.2 Suppose V_1 and V_2 are bilinear spaces with successive bilinear forms $[-,-]_1$ and $[-,-]_2$. Suppose τ is a linear operator in $L(V_1,V_2)$. Hence the following statements are equivalent

Operator τ is pseudo-continuous.

There is only one adjoint operator τ^* in $L(V_2,V_1)$ that satisfies equation (1) [1].

So a pseudo-continuous linear operator in the bilinear spaces always has an adjoint operator. Based on the Theorem 2.3.2 defined the following set.

Definition 2.4.3 Suppose V_1 and V_2 are bilinear spaces with successive bilinear forms $[-,-]_1$ and $[-,-]_2$. The set of pseudo-continuous linear operators in $L(V_1,V_2)$ $(\equiv B(V_1,V_2))$ is defined as

$$B(V_1,V_2)=\{\tau\in L(V_1,V_2)\,|\,\tau : pseudo-continuous\}$$

[2]. In $B(V_1,V_2)$, the following properties apply.

Theorem 2.4.4 Suppose V_1 and V_2 are bilinear spaces with successive bilinear forms $[-,-]_1$ and $[-,-]_2$, α scalar. For each operator f and g in $B(V_1,V_2)$ then the operators $(f+g)$ and (αf) are contained in $B(V_1,V_2)$ [1].

It should be noted that the scalar addition and multiplication operations in $B(V_1,V_2)$ correspond to operations in $L(V_1,V_2)$ so to observe $B(V_1,V_2)$ as a vector space over field F. It is sufficiently proved that space $B(V_1,V_2)$ is a subspace of space $L(V_1,V_2)$. This is provided by the following Corollary.

Corollary 2.4.5 Suppose V_1, V_2 are bilinear spaces, then $B(V_1,V_2)$ is a subspace of $L(V_1,V_2)$ [1].

Given that $L(V_1,V_2)$ is an algebra, then $B(V_1,V_2)$ is a subalgebra of $L(V_1,V_2)$.

Theorem 2.4.6 Suppose that V is a bilinear space, then $B(V)$ with addition and composition operations corresponding to algebra $L(V)$ forms a subalgebra of $L(V)$ [1].

Theorem 2.4.2 mentions a link between pseudo-continuous linear mapping in bilinear space and the existence of adjoining mapping, to expand the results of this finding. It is necessary to present the following definition.

2.5. Adjoint Operator

The following describes the properties of adjoint operator in the inner product spaces.

Definition 2.5.1 Suppose V is an inner product space, and τ is a linear operator of the space $L(V)$.

1 The operator τ is said to be self-adjoint, if $\tau^*=\tau$.

2 The operator τ is a uniter if τ is bijective and $\tau^*=\tau^{-1}$.

3 The operator τ is normal if $\tau\tau^*=\tau^*\tau$ [7].

3. Results

The following are presented the results of the study

Theorem 3.1 Suppose that V is a bilinear space, and σ,τ are linear operator in $B(V)$ space.

1 If the operators σ and τ are self-adjoint, then $(\sigma+\tau)$ is also self-adjoint.

2 If the operator τ is self-adjoint and α is scalar, then $(\alpha\tau)$ is also self-adjoint.

3 If the operators σ and τ are self-adjoint, then $\sigma\tau$ is self-adjoint $\Leftrightarrow \sigma\tau=\tau\sigma$.

4 If the τ operator is self-adjoint and invertible, then τ^{-1} is also self-adjoint.

5 If the operator τ is self-adjoint and $[\tau(v),v]=0 \quad ; \quad \forall v\in V \Rightarrow \tau=0$.

6 Suppose that $p(x)=\alpha_0+\alpha_1x+\alpha_2x^2+\cdots+\alpha_nx^n\in\mathbf{R}[x]$, then $p(\tau)$ is self-adjoint.

7 The eigenspaces of the different eigenvalues of the self-adjoint operator τ in $B(V)$ are mutually orthogonal.

Proof:

1 Suppose that σ and τ are operators self-adjoint in space $B(V)$ and v,w are any vectors in space V

$$\left[v,(\sigma+\tau)^*(w)\right]=\left[(\sigma+\tau)(v),w\right]$$

$$=\left[\sigma(v),w\right]+\left[\tau(v),w\right]$$

$$=\left[v,\sigma(w)\right]+\left[v,\tau(w)\right]$$

$$=\left[v,(\sigma+\tau)(w)\right]$$

Since the bilinear form $[-,-]$ is non-degenerated it is obtained

$$\left(\sigma+\tau\right)^{*}=\left(\sigma+\tau\right)$$

So $\left(\sigma+\tau\right)$ is self-adjoint operator.

2 Suppose that σ and τ are self-adjoint operators in space $B\left(V\right)$ and let v,w be any vectors in space V

$$\left[v,\left(\alpha\tau\right)^{*}\left(w\right)\right]=\left[\left(\alpha\tau\right)\left(v\right),w\right]$$
$$=\alpha\left[\tau\left(v\right),w\right]$$
$$=\alpha\left[v,\tau\left(w\right)\right]$$
$$=\left[v,\left(\alpha\tau\right)\left(w\right)\right]$$

Since the bilinear form $\left[-,-\right]$ is non-degenerated it is obtained

$$\left(\alpha\tau\right)^{*}=\left(\alpha\tau\right)$$

So $\left(\alpha\tau\right)$ is self-adjoint operator.

3 $(\Rightarrow)$ Note that

$$\left[\left(\sigma\tau\right)\left(v\right),w\right]=\left[v,\left(\sigma\tau\right)\left(w\right)\right]$$
$$=\left[\sigma\left(v\right),\tau\left(w\right)\right]$$
$$=\left[\left(\tau\sigma\right)\left(v\right),w\right]$$

Since the bilinear form $\left[-,-\right]$ is non-degenerated it is obtained

$$\sigma\tau=\tau\sigma$$

$(\Leftarrow)$ Note that

$$\left[\left(\sigma\tau\right)^{*}\left(v\right),w\right]=\left[v,\left(\sigma\tau\right)\left(w\right)\right]$$
$$=\left[\sigma\left(v\right),\tau\left(w\right)\right]$$
$$=\left[\left(\tau\sigma\right)\left(v\right),w\right]$$
$$=\left[\left(\sigma\tau\right)\left(v\right),w\right]$$

Since the bilinear form $\left[-,-\right]$ is non-degenerated, it is obtained

$$\left(\sigma\tau\right)^{*}=\left(\sigma\tau\right)$$

So $\left(\sigma\tau\right)$ is self-adjoint operator.

4 Suppose that the operator τ is self-adjoint invertible in space $B\left(V\right)$ and vectors v and w are any vectors in space V.

$$\left[v,w\right]=\left[\left(\tau\tau^{-1}\right)\left(v\right),w\right]$$
$$=\left[\left(\tau\tau^{-1}\right)^{*}\left(v\right),w\right]$$
$$=\left[\left(\tau^{-1}\right)^{*}\tau\left(v\right),w\right]$$

Since the bilinear form $\left[-,-\right]$ is non-degenerated, it is obtained

$$\left(\tau^{-1}\right)^{*}\tau=i$$

where i is the identity operator in space V. So

$$\left(\tau^{-1}\right)^{*}=\tau^{-1}$$

Thus τ^{-1} is self-adjoint operator.

5 Suppose that the operator τ is self-adjoint in space $B\left(V\right)$ and let v,w be any vectors in space V

$$0=\left[\tau\left(v+w\right),\left(v+w\right)\right]$$
$$=\left[\tau\left(v\right),w\right]+\left[\tau\left(w\right),v\right]$$
$$=\left[\tau\left(v\right),w\right]+\left[v,\tau\left(w\right)\right];\mathrm{F}=\mathbf{R}$$
$$=\left[\tau\left(v\right),w\right]+\left[\tau\left(v\right),w\right]$$
$$=2\left[\tau\left(v\right),w\right]$$

obtained

$$\left[\tau\left(v\right),w\right]=0\quad;\quad\forall v,w\in V$$

Since the bilinear form $\left[-,-\right]$ is non-degenerated it is obtained

$$\tau\left(v\right)=0\quad;\quad\forall v\in V$$

Thus

$$\tau=0$$

6 The first it will be proved by mathematical induction that

$$\left(\tau^{n}\right)^{*}=\tau^{n}\quad;\quad\forall n\geq 2 \tag{2}$$

Will be substantiated statement (2). It is true for $n=2$. Note that

$$\left[\tau^2(v),w\right]=\left[\tau(\tau(v)),w\right]$$
$$=\left[\tau(v),\tau(w)\right]$$
$$=\left[v,\tau(\tau(w))\right]$$
$$=\left[v,\tau^2(w)\right]$$

Since $[-,-]$ is non-degenerated bilinear form it is obtained

$$\left(\tau^2\right)^*=\tau^2$$

Suppose the statement (2) is true for $n=k$

Will be substantiated statement (2) is true for $n=k+1$

$$\left[\tau^{k+1}(v),w\right]=\left[\tau\left(\tau^k(v)\right),w\right]$$
$$=\left[\tau^k(v),\tau(w)\right]$$
$$=\left[v,\tau^k(\tau(w))\right]$$
$$=\left[v,\tau^{k+1}(w)\right]$$

Since $[-,-]$ is the non-degenerated bilinear form it is obtained

$$\left(\tau^{k+1}\right)^*=\tau^{k+1}$$

So (2) statement has been proved. The next

$$\left[p(\tau)(v),w\right]$$
$$=\left[\left(\alpha_0+\alpha_1\tau+\alpha_2\tau^2+\cdots+\alpha_2\tau^2\right)(v),w\right]$$
$$=\alpha_0[v,w]+\alpha_1\left[\tau(v),w\right]+\alpha_2\left[\tau^2(v),w\right]$$
$$+\cdots+\alpha_n\left[\tau^n(v),w\right]$$
$$=\alpha_0[v,w]+\alpha_1\left[v,\tau(w)\right]+\alpha_2\left[v,\tau^2(w)\right]$$
$$+\cdots+\alpha_n\left[v,\tau^n(w)\right]$$
$$=\left[v,\left(\alpha_0+\alpha_1\tau+\alpha_2\tau^2+\cdots+\alpha_n\tau^n\right)(w)\right]$$
$$=\left[v,p(\tau)(w)\right]$$

Since $[-,-]$ is the non-degenerated bilinear form, it is obtained

$$p(\tau)^*=p(\tau)$$

7 Suppose that $E(\lambda)$ and $E(\mu)$ are respectively eigenspaces of the self-adjoint operator τ in space $B(V)$ corresponding to different eigenvalues λ and μ. Let. v,w be any vectors in $E(\lambda)$ and $E(\mu)$

$$\lambda[v,w]=[\lambda v,w]$$
$$=\left[\tau(v),w\right]$$
$$=\left[v,\tau(w)\right]$$
$$=[v,\mu w]$$
$$=\mu[v,w]$$

Thus

$$=(\lambda-\mu)[v,w]=0$$

Since $\lambda\neq\mu$ is obtained

$$[v,w]=0$$

Thus

$$E(\lambda)\perp E(\mu)$$

4. Conclusions

According to the results obtained above, it is concluded as follows:

1 There are several properties of linear mapping in the inner product space that also apply in bilinear space, including closed properties to added operations, and scalar multiplication, commutative properties, inverse mapping properties, zero mapping properties, polynomial properties over real fields, and keortogonalant properties of eigen-space of different eigenvalues.

2 The results of this study open insights in the future to reduce the properties of the inner product to be simpler.

3 There are still properties of linear mapping in the inner product space that are not yet applicable in the bilinear space, this opens up opportunities for further research.

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