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Alternative Algebra for Multiplication and Inverse of Interval Number

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Abstract Recently, there are a lot of arithmetic interval forms. One of them only defines nonnegative interval numbers, whereas another one defines all forms of intervals. However, there are not many differences among the many arithmetic forms that were provided, particularly for addition and subtraction. For multiplication, division or inverse, there are many types of operations offered. But the problem is how to determine inverse of an interval number. There are many alternative offers to determine inverse of an interval number $\tilde{a}$. But only for certain cases and for many cases, we have $\tilde{a}\otimes\frac{1}{\tilde{a}}$ which is not equal to interval number $\tilde{\imath} = [1,1]$. Based on these conditions, in this article an analysis of the issues with several existing interval algebras will be given and based on the analysis an alternative will be proposed to determine the form of multiplication and inverse from an interval number, which begins to define the positivity of an interval number with mid-point $m(\tilde{a})$ and then we construct algebra operations especially for multiplication. From the multiplication operation, we can construct the inverse form of an interval number $\tilde{a}$. Furthermore, it is proven that for numbers of interval $= [a_l, a_r]$ where $\tilde{a} \neq \tilde{0} = [0,0]$, there is an interval number $\frac{1}{\tilde{a}}$, so that it applies $\tilde{a}\otimes\frac{1}{\tilde{a}} = \tilde{\imath}$.

Keywords Interval Arithmetic, Interval Matrix, Inverse of Interval Number

Mathematics Subject Classification: 15A09, 65F05, 65G30

1. Introduction

There are many ways given by various authors to determine algebra of the interval number. For example, two interval numbers $\tilde{a} = [a_l, a_r]$ and $\tilde{b} = [b_l, b_r]$ with $a_l \leq a_r$ and $b_l \leq b_r$, respectively, and defined $\mathbf{IR} = \{\tilde{a} = [a_l, a_r]$ with $a_l \leq a_r$ and a_l, a_r in $\mathbf{R}\}$. One of algebra operation forms that was widely used by authors [1]–[2],[18]–[21] and [23] was defined as follows:

(i) $\tilde{a}\oplus\tilde{b} = [a_l + b_l,\ a_r + b_r]$

(ii) $\tilde{a} \ominus \tilde{b} = [a_l - b_r,\ a_r - b_l]$

(iii) $\tilde{a} \otimes \tilde{b} = \begin{bmatrix} min\{a_l b_l, a_l b_r, a_r b_l, a_r b_r\}, \\ max\{a_l b_l, a_l b_r, a_r b_l, a_r b_r\} \end{bmatrix}$

(iv) $\propto \tilde{a} = \begin{cases} [\propto a_l, \propto a_r], if & \propto \geq 0 \\ [\propto a_r, \propto a_l], if & \propto < 0 \end{cases}$

(v) $\frac{\tilde{a}}{\tilde{b}} = [a_l, a_r]\otimes\left[\frac{1}{b_r}, \frac{1}{b_l}\right]$

In defining the arithmetic above, there are some things that are not in accordance with the rule of the real number, for example, $\tilde{a} - \tilde{a}$ is uncertain equal to $\tilde{0} = [0,0]$. If $0 \in \tilde{a} = [a_l, a_r]$ where $a_l \leq 0 \leq a_r$, then $\frac{1}{\tilde{a}} = \left[\frac{1}{a_r}, \frac{1}{a_l}\right]$ is uncertain $\frac{1}{\tilde{a}} \in \mathbf{IR}$. So that $\frac{\tilde{a}}{\tilde{a}}$ is also uncertain equal to $\tilde{\imath} = [1,1]$.

Some authors [1]–[3], [6]–[9], [20] and [26] define mid-point $m(\tilde{a})$ and width (or half-width) $w(\tilde{a})$ which was mostly used to solve linear systems in the form of interval number. Specifically to calculate the determinant and inverse of coefficient matrix on linear system $\tilde{A}\otimes\tilde{x} =$

$\tilde{b}$. Furthermore, there was also one who used it to see the singularity of the matrix $\tilde{A}$ [24] or other things related to the interval matrix [25].

Although only a few authors used it, the algebra operation of interval number was given by [6]–[9], which was interesting and gave a lot of solutions to interval number problem. Basically, the operation offered by [6]–[9] specifically for addition and subtraction operations could be shown that their operations were the same as a pre-existing operation. But, their division operation did not apply for $0 \in \tilde{a}$.

The other operations that are quite interesting were given by [4], [5] and [24]. However, multiplication operations were shown in 10 cases to be basically the same as the multiplication concept in equation (3). But, the division concept that they offered is still incomplete. For example, from formulas that they have offered, we cannot count the value of $\frac{1}{\tilde{a}}$ if $a_l < 0$ and $a_r = 0$. This means that some of the problems that were raised at the beginning also have not been resolved.

Based on the conditions above, we need to offer a concept of algebra operations of interval numbers, especially multiplication and division so that $\tilde{a} \otimes \frac{1}{\tilde{a}} = \tilde{I} = [1,1]$ and the other properties of interval numbers. The methodology is used by adopting the definition of positivity triangular fuzzy numbers as in [10]–[15], which is applied to the definition of positivity interval number $\tilde{a} = [a_l, a_r]$ based on the value of $m(\tilde{a})$.

2. Analysis of Interval Algebra

For two interval numbers $\tilde{a} = [a_l, a_r]$ and $\tilde{b} = [b_l, b_r]$ with $a_l \leq a_r$ and $b_l \leq b_r$, respectively, we define $IR = \{\tilde{a} = [a_l, a_r] | \ a_l \leq a_r \text{ and } a_l, a_r \in \mathbf{R}\}$. As explained in the algebra operation above, there are some things that are incompatible with the rule of real numbers, for example

1. Based on the algebra of interval numbers given by [4], [19]–[20], and [26] definition $\frac{1}{\tilde{a}}$ also will be undefined if $a_l = 0$ or $a_r = 0$.
2. If $0 \in \tilde{a} = [a_l, a_r]$ where $a_l \leq 0 \leq a_r$, with used algebra of [1]–[2], [4]–[9], [18] and [26]-[28] also be undefined $\frac{1}{\tilde{a}}$.
3. Many authors [4]–[9], [18]–[20], [22]–[23] define

$$\frac{\tilde{a}}{\tilde{a}} = [a_l, a_r] \otimes \left[\frac{1}{a_r}, \frac{1}{a_l}\right] = \left[\min\left\{a_l \frac{1}{b_r}, a_l \frac{1}{b_l}, a_r \frac{1}{b_r}, a_r \frac{1}{b_l}\right\}, \max\left\{a_l \frac{1}{b_r}, a_l \frac{1}{b_l}, a_r \frac{1}{b_r}, a_r \frac{1}{b_l}\right\}\right]$$

which is also most likely the value of $\frac{\tilde{a}}{\tilde{a}}$ not equal to the interval $\tilde{I} = [1,1]$ for example, if $\tilde{a} = [-2, 6]$, then $\frac{\tilde{a}}{\tilde{a}} = [-3, 1] \neq \tilde{\imath} = [1,1]$.

Next for $\tilde{a} = [a_l, a_r]$, other authors [6] – [9],[26] define $m(\tilde{a}) = \frac{a_l + a_r}{2}$, $w(\tilde{a}) = \frac{a_r - a_l}{2}$ and $\text{dual}(\tilde{a}) = \text{dual } [a_l, a_r] = [a_r, a_l]$ so we get $\tilde{a} - \text{dual}(\tilde{a}) = \tilde{0} = [0,0]$, but not $\tilde{a} - \tilde{a} = \tilde{0}$. They also claimed that $\tilde{a} \otimes \frac{1}{dual(\tilde{a})} = [a_l, a_r] \otimes \frac{1}{[a_r, a_l]} = [1,1]$ was not applicable for $a_l = 0$ or $a_r = 0$. On the other hand, they said two interval numbers $\tilde{a} = [a_l, a_r]$ and $\tilde{b} = [b_l, b_r]$ are equal if $m(\tilde{a}) = m(\tilde{b})$. The operations that they gave are:

(i) $\tilde{a} \oplus \tilde{b} = [m(\tilde{a}) + m(\tilde{b}) - k, m(\tilde{a}) + m(\tilde{b}) + k]$

(ii) $\tilde{a} \ominus \tilde{b} = [m(\tilde{a}) - m(\tilde{b}) - k, m(\tilde{a}) - m(\tilde{b}) + k]$ where $k = \left\{\frac{(a_r + b_r) - (a_l + b_l)}{2}\right\}$

(iii) $\tilde{a} \otimes \tilde{b} = [m(\tilde{a}) m(\tilde{b}) - k, m(\tilde{a}) \ m(\tilde{b}) + k]$ where $k = min\{m(\tilde{a}) m(\tilde{b}) - \alpha, \ \beta - m(\tilde{a}) \ m(\tilde{b})\}$, and $\alpha = \min\{a_l b_l, a_l b_r, a_r b_l, a_r b_r\}$, $\beta = \max\{a_l b_l, a_l b_r, a_r b_l, a_r b_r\}$,

(iv) $1 : \tilde{a} = \frac{1}{\tilde{a}} = \frac{1}{[a_l, a_r]} = \left[\frac{1}{m(\tilde{a})} - \delta, \frac{1}{m(\tilde{a})} + \delta\right]$ where $\delta = \min\left\{\frac{1}{a_r}\left(\frac{a_r - a_l}{a_l + a_r}\right), \frac{1}{a_l}\left(\frac{a_r - a_l}{a_l + a_r}\right)\right\}$ and $0 \notin \tilde{a}$.

It was clear from the formula that they offered, we could not calculate $\frac{1}{\tilde{a}}$ if $0 \notin \tilde{a}$, which means we must find an alternative to determine $\frac{1}{\tilde{a}}$ for arbitrary $\tilde{a} \neq \tilde{0} = [0,0]$. On the other hand, the formula (i′) from [6-9] is the same as formula (i) above, and similarly, the formula of (ii′) is the same as formula (ii). We can show it as follows:

Consider the formula (i') that is given in [6] – [9], if we substitute the values of $m(\tilde{a})$ and $w(\tilde{a})$, we will get

$$\tilde{a} \oplus \tilde{b} = \left[\left(\frac{a_l + a_r}{2}\right) + \left(\frac{b_l + b_r}{2}\right) - \frac{(b_r + a_r) - (b_l + a_l)}{2}, \left(\frac{a_l + a_r}{2}\right) + \left(\frac{b_l + b_r}{2}\right) + \frac{(b_r + a_r) - (b_l + a_l)}{2}\right]$$
$$= [a_l + b_l, a_r + b_r]$$

So the formula (i') provided by [6]–[9] was clearly the same as (i), which was given by the various previous authors. In the same way, we can also show that $(\mathbf{ii'})$ is exactly the same as (ii). But, the result of multiplication on $(\mathbf{iii'})$, gives a shorter interval. This can be shown as follows:

***Case* 1**. Let $k = m(\tilde{a})m(\tilde{b}) - \alpha$. If we substitution to (iii′),we get

$$\tilde{a}\otimes\tilde{b} = \left[m(\tilde{a})m(\tilde{b}) - \left(m(\tilde{a})m(\tilde{b}) - \alpha\right), m(\tilde{a})\ m(\tilde{b}) + \left(m(\tilde{a})m(\tilde{b}) - \alpha\right)\right] = \left[\alpha, 2m(\tilde{a})\ m(\tilde{b}) - \alpha\right]$$

If we substitute all possible values of α, then it will be obtained $2m(\tilde{a})\ m(\tilde{b}) - \alpha \le \beta$. It means that

$$\left[\alpha, 2m(\tilde{a})\ m(\tilde{b}) - \alpha\right] \subseteq [\alpha, \beta] \tag{2.1}$$

***Case* 2**. Let $k = \beta - m(\tilde{a})\ m(\tilde{b})$. If we substitute it into (iii′), we get

$$\tilde{a}\otimes\tilde{b} = \left[m(\tilde{a})m(\tilde{b}) - \left(\beta - m(\tilde{a})m(\tilde{b})\right), m(\tilde{a})m(\tilde{b}) + \left(\beta - m(\tilde{a})m(\tilde{b})\right)\right] = \left[2m(\tilde{a})m(\tilde{b}) - \beta, \beta\right]$$

In the same way, if we substitute all possible values of β on equation (3.1), then we will get $2m(\tilde{a})m(\tilde{b}) - \beta \ge \alpha$, which means that

$$\left[2m(\tilde{a})m(\tilde{b}) - \beta, \beta\right] \subseteq [\alpha, \beta] \tag{2.2}$$

From (2.1) and (2.2), it can be observed that in all cases, $\tilde{a}.\tilde{b}$ of (iii′) is shorter than $\tilde{a}\otimes\tilde{b}$ given in (ii). But it is necessary to note that not all shorter intervals are said to be better.

Next, an analysis of algebra operation was provided by Siahlooei and Shahzadeh Fazeli [5] which was also almost similar to what was provided by [17]. These authors provided many alternatives for multiplication and division of two interval numbers, while for addition and subtraction, their approaches were the same as that of most other authors such as [1]–[3], [16]–[20] and [27]. Actually, their multiplication formula was the same as the multiplication provided by many authors above. In fact, concept of division does not cover cases where $a_2 \le 0$ and $b_2 \le 0$. So we can't calculate $\frac{1}{\tilde{a}}$ for $\tilde{a} = [-3, 0]$. However, there are several authors [5]–[9], [18], [22]–[23] who made the rule for division of two interval numbers as follows.

$$\frac{\tilde{a}}{\tilde{b}} = \left[\min\left\{\frac{a_l}{b_l}, \frac{a_l}{b_r}, \frac{a_r}{b_l}, \frac{a_r}{b_r}\right\}, \max\left\{\frac{a_l}{b_l}, \frac{a_l}{b_r}, \frac{a_r}{b_l}, \frac{a_r}{b_r}\right\}\right] \text{ to } b_l \neq 0 \text{ and } b_r \neq 0.$$

This division rule can't solve the problem for many forms of interval number, so an alternative is needed to determine algebra operations of interval number.

3. Materials and Methods

If $\tilde{a} = [a_l, a_r]$ and $\tilde{b} = [b_l, b_r]$, based on the concepts of set then

$$\tilde{a} + \tilde{b} = \left\{x \in \mathbf{R} \middle| x = a + b, \quad \text{for all } a \in \tilde{a} \text{ and } b \in \tilde{b}\right\}$$

and this can be shown that the value will be the same as $\tilde{a} + \tilde{b} = [\inf(\tilde{a} + \tilde{b}), \sup(\tilde{a} + \tilde{b})]$.

Since it is a closed interval, so

$$\tilde{a} + \tilde{b} = [\min(\tilde{a} + \tilde{b}), \max(\tilde{a} + \tilde{b})] = [a_l + b_l, a_r + b_r].$$

It is clear that for every $\tilde{a} \in$ IR there is $\tilde{0} = [0,0]$ such that $\tilde{a} + \tilde{0} = \tilde{0} + \tilde{a}$.

Next,

$$\tilde{a} \ominus \tilde{b} = \left\{x \in \mathbf{R} \middle| x = a - b, \quad \text{for all } a \in \tilde{a} \text{ and } b \in \tilde{b}\right\}$$

This can also be shown $\tilde{a} - \tilde{b} = [\inf(\tilde{a} - \tilde{b}), \sup(\tilde{a} + \tilde{b})]$, and for $\tilde{a}$, $\tilde{b} \in$ IR**,** and also since this is a closed interval, then

$$\tilde{a} - \tilde{b} = [\min(\tilde{a} - \tilde{b}), \max(\tilde{a} + \tilde{b})] = [a_l - b_r, a_r - b_l]$$

What's interesting here is for every $\tilde{a} \in$ IR with $a_l < a_r$ not yet $-\tilde{a} \in$ IR such that $\tilde{a} - \tilde{a} = 0$, but for every $\tilde{a} \in$ IR**,** there is $\tilde{x}^* = [x_l, x_r] = [a_r, a_l]$, with $x_l = a_r$ and $x_r = a_l$, (that means $\tilde{x}^* \notin$IR, since $x_l > x_r$). From these conditions, some authors [4, 5, 16] define IR* as a set of all intervals of real numbers, namely the interval set which is in form of proper and improver interval, so IR $\subseteq$ IR*. Then it applies for every $\tilde{a} \in$ IR* there is $\tilde{a}^* \in$ IR*such that $\tilde{a} - \tilde{a}^* = 0$. While for

$$\tilde{a}\tilde{b} = \left\{x \in \mathbf{R} \middle| x = ab, \quad \text{for all } a \in \tilde{a} \text{ and } b \in \tilde{b}\right\}$$

For this condition, the value of inf $(\tilde{a}\tilde{b})$ and sup$(\tilde{a}\tilde{b})$ can't be specified uniquely, because it depends on $\tilde{a}$ and $\tilde{b}$, if $\tilde{a} = [a_l, a_r]$, where $a_l > 0$ and $\tilde{b} = [b_l, b_r]$ where $b_l > 0$ value of inf $(\tilde{a}\tilde{b})$ will be different if $\tilde{a} =$

$[a_l, a_r]$ with $0 \in \tilde{a}$ and $\tilde{b} = [b_l, b_r]$ where $b_l > 0$. Since $\tilde{a}$ and $\tilde{b}$ are closed interval, then $\tilde{a}\tilde{b}$ is also closed interval with value of $\inf(\tilde{a}\tilde{b}) = \min\{a_l b_l, a_l b_r, a_r b_l, a_r b_r\}$, and $\sup(\tilde{a}\tilde{b}) = \max\{a_l b_l, a_l b_r, a_r b_l, a_r b_r\}$, so that $\tilde{a}\tilde{b}$ can be written in the form

$$\tilde{a}\tilde{b} = [\min\{a_l b_l, a_l b_r, a_r b_l, a_r b_r\}, \max\{a_l b_l, a_l b_r, a_r b_l, a_r b_r\},]$$

It is clear for every $\tilde{a} \in \mathrm{IR}^*$ there is $\tilde{\iota} = [1,1]$ so that $\tilde{a}\tilde{\iota} = \tilde{\iota}\tilde{a} = \tilde{a}$.

Whereas for $\frac{1}{\tilde{a}} = \left\{\frac{1}{a} \in \mathbf{R} | \text{ for all } a \in \tilde{a}\right\}$, the value of $\frac{1}{\tilde{a}}$ is not necessarily limited. Then it is defined

$$\frac{1}{\tilde{a}} = \begin{cases} \left[\frac{1}{a_r}, \frac{1}{a_l}\right], & \text{if } 0 < a_l \le a_r \text{ or } a_l \le a_r < 0 \\ \left[\frac{1}{a_r}, \infty\right), & \text{if } 0 = a_l < a_r \\ (-\infty, \infty), & \text{if } a_l < 0 < a_r \\ \left(-\infty, \frac{1}{a_l}\right], & \text{if } a_l < a_r = 0 \\ (-\infty, \infty), & \text{if } \tilde{a} = (-\infty, \infty) \\ \text{undefined}, & \text{if } \tilde{a} = [0,0] \end{cases}$$

If we use the operation of

$$\tilde{a}\tilde{b} = [\min\{a_l b_l, a_l b_r, a_r b_l, a_r b_r\}, \max\{a_l b_l, a_l b_r, a_r b_l, a_r b_r\}\ ],$$

then for every $\tilde{a} \in \mathbf{IR}^*$, it is not necessarily available that there exists $\frac{1}{\tilde{a}} \in \mathbf{IR}^*$ such that $\tilde{a}\frac{1}{\tilde{a}} = \tilde{\iota} = [1,1]$,

So based on the condition above, we can use the rules of addition and subtraction, but for multiplication and division operations, it is necessary to find the other formula and notations of +, -, x and: like operators that apply to ordinary sets, can't be used. So for this interval operations, we use the other operators, for example, the operators $\tilde{a}\oplus\tilde{b}, \tilde{a}\Theta\tilde{b}, \tilde{a}\otimes\tilde{b}$ and $\frac{1}{\tilde{a}}$ whose operation details are given in Section 4 below.

4. Results and Discussion

Before defining the multiplication and inverse, in order to demonstrate the characteristic multiplication properties of interval numbers, one must first define the positivity of interval number. For arbitrary interval number $\tilde{a} = [a_l, a_r] \in \mathrm{IR}$, defined $m(\tilde{a}) = \frac{a_l + a_r}{2}$ and $w(\tilde{a}) = \frac{a_r - a_l}{2}$. The definitions are used as follows:

***Definition* 1** For an arbitrary interval number $\tilde{a} = [a_l, a_r] \in \mathrm{IR}$, defined

$$\tilde{a} = [a_l, a_r] = \begin{cases} \text{positive}, & \text{if } m(\tilde{a}) > 0 \\ \text{negative}, & \text{if } m(\tilde{a}) < 0 \\ \text{zero}, & \text{if } m(\tilde{a}) = 0 \\ \text{pure zero}, & \text{if } \tilde{a} = [0,0] \\ \text{identity element}, & \text{if } \tilde{\iota} = [1,1] \end{cases}$$

***Definition* 2.** Interval $\tilde{a}$ and $\tilde{b}$ in IR are said to be identical (equivalent) if $m(\tilde{a}) = m(\tilde{b})$ and $w(\tilde{a}) \ne w(\tilde{b})$. It is notated by $\tilde{a} \approx \tilde{b}$, if $m(\tilde{a}) = m(\tilde{b})$ and $w(\tilde{a}) = w(\tilde{b})$, then $\tilde{a}$ is equal to $\tilde{b}$.

For interval number $\tilde{a} = [a_l, a_r]$ and $\tilde{b} = [b_l, b_r]$, almost all authors agree with the operation for subtraction, that is

(1) $\tilde{a}\oplus\tilde{b} = [a_l + b_l, a_r + b_r]$

(2) $\ominus\ \tilde{b} = [a_l - b_r, a_r - b_l]$

So, for arbitrary interval numbers, we can use the operations of addition and subtraction as in (1) and (2), while for scalar multiplication, it still uses the standard rules, that is

(3) $\alpha\tilde{a} = \begin{cases} [\alpha a_l, \alpha a_r], & \text{if } \alpha \ge 0 \\ [\alpha a_r, \alpha a_l], & \text{if } \alpha < 0 \end{cases}$

Next, the multiplication operation for two interval numbers is defined as:

Definition 3. Let $\tilde{a} = [a_l, a_r]$ and $\tilde{b} = [b_l, b_r]$ with $\tilde{a}, \tilde{b} \in IR^*$. Multiplication of $\tilde{a}$ and $\tilde{b}$ is

(4) $\tilde{a} \otimes \tilde{b} = [a_l m(\tilde{b}) + b_l m(\tilde{a}) - m(\tilde{a})\ m(\tilde{b}), a_r m(\tilde{b}) + b_r m(\tilde{a}) - m(\tilde{a})\ m(\tilde{b})]$

where $m(\tilde{a}) = \frac{a_l + a_r}{2}$ and $m(\tilde{b}) = \frac{b_r + b_l}{2}$.

From this definition of multiplication, we get $m(\tilde{a} \otimes \tilde{b}) = m(\tilde{a})m(\tilde{b})$.

***Remark*1**: By definition 3, we have

(i) If $\tilde{a}$ and $\tilde{b}$ are positives, then $\tilde{a} \otimes \tilde{b}$ is positive.
(ii) If $\tilde{a}$ and $\tilde{b}$ are negatives, then $\tilde{a} \otimes \tilde{b}$ is positive.
(iii) If $\tilde{a}$ is negative and $\tilde{b}$ is positive, then $\tilde{a} \otimes \tilde{b}$ is negative.
(iv) If $\tilde{a}$ is positive and $\tilde{b}$ is negative, then $\tilde{a} \otimes \tilde{b}$ is negative.

Based on the definition and remark above, then we construct the inverse form for arbitrary interval number $\tilde{a} = [a_l, a_r] \in IR$ as in the following theorem.

Theorem 1. For an arbitrary interval $\tilde{a} = [a_l, a_r] \in IR$ where $m(\tilde{a}) \neq 0$, there is $\tilde{x}^* = \frac{1}{\tilde{a}} = \left[\frac{2m(\tilde{a}) - a_l}{m^2(\tilde{a})}, \frac{2m(\tilde{a}) - a_r}{m^2(\tilde{a})}\right] \in IR^*$ such that $\tilde{a} \otimes \tilde{x}^* = \tilde{\imath} = [1,1] = \tilde{x}^* \otimes \tilde{a}$

Proof: Consider $\tilde{a} = [a_l, a_r] \in IR$ where $m(\tilde{a}) \neq 0$. Determined $\tilde{x}^* = [x_l, x_r] = \frac{1}{\tilde{a}}$ such that $\tilde{a} \otimes \tilde{x}^* = \tilde{\imath} = [1,1] = \tilde{x}^* \otimes \tilde{a}$. If $\tilde{a} \otimes \tilde{x}^* = \tilde{\imath} = [1,1]$ then

$$m(\tilde{a} \otimes \tilde{x}^*) = m(\tilde{a}) \otimes m(\tilde{x}^*) = 1 \quad \text{or} \quad m(\tilde{x}^*) = \frac{1}{m(\tilde{a})}.$$

Consequently,

$$\tilde{a} \otimes \tilde{x}^* = \left[a_l m(\tilde{x}^*) + x_l m(\tilde{a}) - m(\tilde{a}) m(\tilde{x}^*), a_r m(\tilde{x}^*) + x_r m(\tilde{a}) - m(\tilde{a}) m(\tilde{x}^*)\right]$$
$$= \left[\frac{a_l}{m(\tilde{a})} + x_l m(\tilde{a}) - 1, \frac{a_r}{m(\tilde{a})} + x_r m(\tilde{a}) - 1\right].$$

Then $\frac{a_l}{m(\tilde{a})} + x_l m(\tilde{a}) - 1 = 1$ and $\frac{a_r}{m(\tilde{a})} + x_r m(\tilde{a}) - 1 = 1$. So we get $x_l = \frac{2m(\tilde{a}) - a_l}{m^2(\tilde{a})}$ and $x_r = \frac{2m(\tilde{a}) - a_r}{m^2(\tilde{a})}$. Therefore, for every $\tilde{a} = [a_l, a_r] \in IR$ where $m(\tilde{a}) \neq 0$, there is $\tilde{x}^* = \frac{1}{\tilde{a}} = \left[\frac{2m(\tilde{a}) - a_l}{m^2(\tilde{a})}, \frac{2m(\tilde{a}) - a_r}{m^2(\tilde{a})}\right]$ such that $\tilde{a} \otimes \tilde{x}^* = \tilde{\imath} = [1,1] = \tilde{x}^* \otimes \tilde{a}$

Corollary 1. For arbitrary $\tilde{a} = [a_l, a_r]$ and $\tilde{b} = [b_l, b_r]$, we have

$$\frac{\tilde{a}}{\tilde{b}} = \tilde{a} \otimes \frac{1}{\tilde{b}} = [a_l, a_r]\left[\frac{2m(b) - b_l}{(m(b))^2}, \frac{2m(b) - b_r}{(m(b))^2}\right]$$

$$= \left[a_l \frac{1}{m(b)} + \frac{2m(b) - b_l}{(m(b))^2} m(a) - \frac{m(a)}{m(b)}, a_r \frac{1}{m(b)} + \frac{2m(b) - b_r}{(m(b))^2} m(a)\right]$$

$$= \left[\frac{a_l m(b) - b_l m(a) + m(a) m(b)}{(m(b))^2}, \frac{a_r m(b) - b_r m(a) + m(a) m(b)}{(m(b))^2}\right]$$

***Remark* 2**. $\frac{\tilde{a}}{\tilde{a}} = \left[\frac{a_l m(a) - a_l m(a) + m(a) m(a)}{(m(a))^2}, \frac{a_r m(a) - a_r m(a) + m(a) m(a)}{(m(a))^2}\right] = [1,1]$

In a simple way, such as in the proof of corollary and remark above, it can also be easily shown that other properties of interval numbers, as in the following theorem.

***Theorem* 2.** Let $\tilde{a}, \tilde{b}, \tilde{c} \in IR^*$. we have:

a. $\tilde{a} \otimes \tilde{0} = \tilde{0}$

b. $\tilde{a} \otimes \tilde{1} = \tilde{a}$

c. $\tilde{a} \otimes \tilde{b} = \tilde{b} \otimes \tilde{a}$

d. $(\tilde{a} \otimes \tilde{b}) \otimes \tilde{c} = \tilde{a} \otimes (\tilde{b} \otimes \tilde{c})$

e. $(\tilde{a} \oplus \tilde{b}) \otimes \tilde{c} = (\tilde{a} \otimes \tilde{c}) \oplus (\tilde{b} \otimes \tilde{c})$

f. If $\tilde{a} \otimes \tilde{x} = \tilde{b}$ and $\tilde{a} \not\approx \tilde{0}$, then $\tilde{x} = \frac{\tilde{b}}{\tilde{a}}$.

g. If $\tilde{a} \otimes \tilde{b} = \tilde{0}$ then $\tilde{a} \approx \tilde{0}$ or $\tilde{b} \approx \tilde{0}$.

h. If $\tilde{a} \otimes \tilde{b} = \tilde{a} \otimes \tilde{c}$ and $\tilde{a} \not\approx \tilde{0}$, then $\tilde{b} = \tilde{c}$.

i. If $\tilde{a} \not\approx \tilde{0}$ then $\frac{1}{\tilde{a}} \not\approx \tilde{0}$ and $\frac{1}{\frac{1}{\tilde{a}}} = \tilde{a}$

j. If $\tilde{a} \not\approx \tilde{0}$ and $\tilde{b} \not\approx \tilde{0}$ then $\frac{1}{\tilde{a} \otimes \tilde{b}} = \frac{1}{\tilde{a}} \otimes \frac{1}{\tilde{b}}$

Proof: Let $\tilde{a} = [a_l, a_r], \tilde{b} = [b_l, b_r], \tilde{c} = [c_l, c_r] \in IR^*$

a) $.\tilde{a} \otimes \tilde{0} = [a_l.m(\tilde{0}) + 0m(\tilde{a}) - m(\tilde{a}).m(\tilde{0}), a_r.m(\tilde{0}) + 0.m(\tilde{a}) - m(\tilde{a}).m(\tilde{0})]$

$= [a_l.0 + 0.m(\tilde{a}) - m(\tilde{a}).0, \ a_r.0 + 0.m(\tilde{a}) - m(\tilde{a}).0]$

$= [0,0] = \tilde{0}$

b) $. \tilde{a} \otimes \tilde{1} = [a_l.m(\tilde{1}) + 1.m(\tilde{a}) - m(\tilde{a}).m(1), a_r m(\tilde{1}) + 1.m(\tilde{a}) - m(\tilde{a}).m(\tilde{1})]$

$= [a_l.1 + 1.m(\tilde{a}) - m(\tilde{a}).1, a_r.1 + 1.m(\tilde{a}) - m(\tilde{a}).1]$

$= [a_l, a_r] = \tilde{a}$

c) $.\tilde{a} \otimes \tilde{b} = [a_l.m(\tilde{b}) + b_l.m(\tilde{a}) - m(\tilde{a}).m(b), a_r m(\tilde{b}) + b_r.m(\tilde{a}) - m(\tilde{a})m(\tilde{b})]$

$= [b_l.m(\tilde{a}) + a_l.m(\tilde{b}) - m(\tilde{b}).m(\tilde{a}), b_r m(\tilde{a}) + a_r.m(\tilde{b}) - m(\tilde{b})m(\tilde{a})]$

$= \tilde{b} \otimes \tilde{a}$

d) It is easy to prove that

$$(\tilde{a} \otimes \tilde{b}) \otimes \tilde{c} = ([a_l.m(\tilde{b}) + b_l.m(\tilde{a}) - m(\tilde{a}).m(\tilde{b}), a_r m(\tilde{b}) + b_r.m(\tilde{a}) - m(\tilde{a})m(\tilde{b})]) \otimes \tilde{c}$$

$$= [a_l.m(\tilde{b}).m(\tilde{c}) + b_l.m(\tilde{a}).m(\tilde{c}) + c_l.m(\tilde{a}).m(\tilde{b}) - 2.m(\tilde{a}).m(\tilde{b}).m(\tilde{c}),$$

$$a_r m(\tilde{b}).m(\tilde{c}) + b_r.m(\tilde{a}).m(\tilde{c}) + c_r m(\tilde{a})m(\tilde{b}) - 2.m(\tilde{a}).m(\tilde{b}).m(\tilde{c})].$$

and

$$(\tilde{a} \oplus \tilde{b}) \otimes \tilde{c} = ([a_l + b_l, a_r + b_r]) \otimes [c_l, c_l]$$

$$= [(a_l + b_l).m(\tilde{c}) + c_l.(m(\tilde{a}) + m(\tilde{b})) - (m(\tilde{a})m(\tilde{b})).m(\tilde{c}),$$

$$(a_r + b_r).m(\tilde{c}) + c_r.(m(\tilde{a}) + m(\tilde{b})) - (m(\tilde{a})m(\tilde{b})).m(\tilde{c})]$$

$$= [a_l.m(\tilde{c}) + c_l.m(\tilde{a}) - m(\tilde{a}).m(\tilde{c}), a_r m(\tilde{c}) + c_r.m(\tilde{a}) - m(\tilde{a})m(\tilde{c})] \oplus$$

$$[b_l.m(\tilde{c}) + c_l.m(\tilde{b}) - m(\tilde{b}).m(\tilde{c}), b_r m(\tilde{c}) + c_r.m(\tilde{b}) - m(\tilde{b})m(\tilde{c})]$$

$$= (\tilde{a} \otimes \tilde{b}) \oplus (\tilde{b} \otimes \tilde{c})$$

(a) Let $\tilde{a} \otimes \tilde{x} = \tilde{b}$ and $\tilde{a} \not\approx \tilde{0}$. Then there is $\frac{1}{\tilde{a}} \neq \tilde{0}$ such that $\tilde{a} \cdot \frac{1}{\tilde{a}} = \frac{1}{\tilde{a}} \cdot \tilde{a} = \tilde{\imath}$. Multiplying both sides by $\frac{1}{\tilde{a}}$, we have $\tilde{x} = \frac{\tilde{b}}{\tilde{a}}$.

(b) Let $\tilde{a} \otimes \tilde{b} = \tilde{0}$. If $\tilde{a} \not\approx \tilde{0}$, then there is $\frac{1}{\tilde{a}} \neq \tilde{0}$ such that $\tilde{a} \cdot \frac{1}{\tilde{a}} = \frac{1}{\tilde{a}} \cdot \tilde{a} = \tilde{\imath}$. Multiplying both sides by $\frac{1}{\tilde{a}}$, we have $\tilde{b} = \tilde{0}$.

(c) Let $\tilde{a}\otimes\tilde{b} = \tilde{a}\otimes\tilde{c}$ and $\tilde{a} \not\approx \tilde{0}$. We have $\tilde{a}\otimes\tilde{b} - \tilde{a}\otimes\tilde{c} = \tilde{0}$. Then $\tilde{a}\otimes(\tilde{b} - \tilde{c}) = \tilde{0}$. Since $\tilde{a} \not\approx \tilde{0}$, then $\tilde{b} - \tilde{c} = \tilde{0}$. Therefore, $\tilde{b} = \tilde{c}$.

(d) Let $\tilde{a} \not\approx \tilde{0}$ Then there is $\frac{1}{\tilde{a}} \not\approx \tilde{0}$ such that $\tilde{a} \cdot \frac{1}{\tilde{a}} = \frac{1}{\tilde{a}} \cdot \tilde{a} = \tilde{\imath}$. So, we have $\tilde{a} = \frac{1}{\frac{1}{\tilde{a}}}$.

(e) Let $\tilde{a} \not\approx \tilde{0}$ and $\tilde{b} \not\approx \tilde{0}$. Then there is $\frac{1}{\tilde{a}} \not\approx \tilde{0}$ such that $\tilde{a} \cdot \frac{1}{\tilde{a}} = \frac{1}{\tilde{a}} \cdot \tilde{a} = \tilde{\imath}$ and also there is $\frac{1}{\tilde{b}} \not\approx \tilde{0}$ such that $\tilde{b} \cdot \frac{1}{\tilde{b}} = \frac{1}{\tilde{b}} \cdot \tilde{b} = \tilde{\imath}$. So, we have

$$(\tilde{a}\otimes\tilde{b})\otimes\left(\frac{1}{\tilde{a}}\otimes\frac{1}{\tilde{b}}\right) = (\tilde{b}\otimes\tilde{a})\otimes\left(\frac{1}{\tilde{a}}\otimes\frac{1}{\tilde{b}}\right) = (\tilde{b}\otimes\tilde{a})\otimes\left(\frac{1}{\tilde{a}}\otimes\frac{1}{\tilde{b}}\right) = \tilde{b}\otimes\left(\tilde{a}\otimes\left(\frac{1}{\tilde{a}}\otimes\frac{1}{\tilde{b}}\right)\right) = \tilde{b}\otimes\left(\left(\tilde{a}\otimes\frac{1}{\tilde{a}}\right)\otimes\frac{1}{\tilde{b}}\right) = \tilde{b}\otimes\left(\tilde{\imath}\otimes\frac{1}{\tilde{b}}\right)$$

$$= \tilde{b}\otimes\frac{1}{\tilde{b}} = \tilde{\imath}.$$

5. Conclusions

For arbitrary interval numbers $\tilde{a} = [a_l, a_r]$ and $\tilde{b} = [b_l, b_r]$, we have addition, subtraction, and scalar multiplication operations that are the same as those found by most authors, such as the rule (1), (2) and (3). Multiplication is like a rule (4). While for the inverse, we use the rule of Theorem 1. The rules of algebra operations for this interval number can be said to be better than the existing form of operations, because its multiplication and division operations are more complete and include a wider case.

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