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Some Results on Sequences in Banach Spaces

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Abstract In this work, we prove in a very particular way the theorems of Dvoretzky-Roger's, Shur's, Orcliz's and Theorem 14.2 in their versions presented in the text [3]. The demonstrations of these Theorems carried out by us consist in establishing an appropriate link between the object of study and the relation that affirms that, for any n real numbers $a_1, a_2, ..., a_n$, there exists a unique real number λ such that $\sum_{k=1}^{n} a_k = \lambda \sum_{k=1}^{n} a_k^2$. Once the nexus is established, we use the definition of weak or strong convergence together with the Hahn-Banach Theorem to obtain the desired results. The relation $\sum_{k=1}^{n} a_k = \lambda \left(\sum_{k=1}^{n} a_k^2\right)$ is obtained by decomposing the Hilbert space $\mathbb{R}^n$ as the direct sum of a closed subspace and its orthogonal complement. Since the dimension of the space $\mathbb{R}^n$ is finite, this guarantees that any linear functional defined on the space $\mathbb{R}^n$ is continuous, and this guarantees that the kernel of said linear functional is closed in the space $\mathbb{R}^n$. Therefore we have that the space $\mathbb{R}^n$ breaks down, as the direct sum of the kernel of the continuous linear functional f and its orthogonal complement, that is: $\mathbb{R}^n = \ker f \oplus [\ker f]^{\perp}$, where the dimension of $\ker f = n-1$ and the dimension of $[\ker f]^{\perp} = 1$.

Keywords Functional Analysis, Numerical Analysis, Dvoretzky-Rogers Theorem, Orlicz's Theorem, Shur's Theorem

1. Introduction

In Book [3] the Dvoretzky - Rogers Theorem is proved using Lemma (1.3). This Lemma uses sophisticated machinery to achieve its purposes. In this work, the Dvoretzky - Rogers Theorem is proved in a different way and we believe that it is much simpler.

Analogously we approach Theorems (14.2), Shur'z Theorem and Orlic'z Theorem of [3] making use of the technique described in the articles [6], [5]. This technique consists of decomposing as a direct sum the space $\mathbb{R}^n = \ker f \oplus \{\ker f\}^{\perp}$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a conveniently chosen continuous linear functional, obtaining the relation $\sum\limits_{k=1}^{n} a_k = \lambda \sum\limits_{k=1}^{n} a_k^2\,,\ a_k \in \mathbb{R}\,,\ k \in \{1, ..., n\}$.

Schur's Theorem tells us that in the space ℓ_1, the weak convergence and the convergence in norm of sequences are the same. For the proof of this result, the Baire Category Theorem is used. An alternative proof is proposed here, using the aforementioned technique .

Finally within our study of the important results of functional analysis, we address the [3] Theorem (1.11), which is called the Orcliz Theorem. To prove this theorem, Khinchin's and Minkowski's inequality is used. Here, as in the other cases, an alternative demonstration of our previously mentioned technique is proposed.

2. Seccion 1

Theorem 2.1. *Let X be an infinite-dimensional Banach space. For any $(\lambda_n)_{n=1}^{\infty} \in \ell_2$ there is always a summable unconditional sequence $(x_n)_{n=1}^{\infty} \in X$ such that $\|x_n\| = |\lambda_n|$, $\forall n \in \mathbb{N}$.*

Proof. Let $\sum_{k\leq n} x_k$, then

$$\|\sum_{k\leq n} x_k\| = \sup_{\|\varphi\|=1,\ \varphi\in X^*} \left|\varphi\left(\sum_{k\leq n} x_k\right)\right| \tag{1}$$

From relation (1) we get

$$\|\sum_{k\leq n} x_k\| \leq (1+\epsilon)\sum_{k=1}^{n} |\varphi(x_k)| \leq (1+\epsilon)\sum_{k=1}^{n} \|x_k\| \tag{2}$$

$$\sum_{k=1}^{n} |\varphi(x_k)| = \lambda \left[\sum_{k=1}^{n} |\varphi(x_k)|^2\right] \tag{3}$$

From the relations (3) and (2) we obtain:

$$\left\|\sum_{k\leq n} x_k\right\| \leq (1+\epsilon)\sum_{k=1}^{n} |\varphi(x_k)| = \lambda(1+\epsilon)\left(\sum_{k=1}^{n} |\varphi(x_k)|^2\right) \leq \lambda(1+\epsilon)\sum_{k=1}^{n} \|x_k\|^2 \tag{4}$$

The essential fact is in (4) to bound the value of λ. Note that (4) is true for all $\varphi \in X^*$ with $\|\varphi\| = 1$. From the relation (4) we obtain:

$$\left(|\varphi(x_1)| - \frac{1}{2\lambda}\right)^2 + ... + \left(|\varphi(x_n)| - \frac{1}{2\lambda}\right)^2 = \frac{n}{4\lambda^2} \tag{5}$$

From the relation (5) we obtain:

$$|\varphi(x_k)| - \frac{1}{2\lambda} = \frac{\sqrt{n}}{2\lambda} a_k, \tag{6}$$

where

$$a_1^2 + a_2^2 + ... + a_n^2 = 1 \tag{7}$$

Let

$$k_0 \in \{1, ..., n\} \text{ such that } |a_{k_0}| = \min_{k=1,...,n} |a_k| \tag{8}$$

Of relations (7) and (8): $n|a_{k_0}|^2 \leq 1$, where do you have

$$|a_{k_0}| \leq \frac{1}{\sqrt{n}} \tag{9}$$

From the relations (6) and (9) we have

$$|\varphi(x_{k_0})| = \frac{1}{2\lambda}(\sqrt{n} a_{k_0} + 1) \leq \frac{1}{\lambda}\ ,\ \forall\, \varphi \in X^* \text{ with } \|\varphi\| = 1. \tag{10}$$

Then there exists by the Hahn-Banach theorem a $\varphi_{k_0} \in X^*$ with $\|\varphi_{k_0}\| = 1$ and

$$|\varphi_{k_0}(x_{k_0})| = \|x_{k_0}\| \tag{11}$$

From the relation (2) we have

$$\left\|\sum_{k\leq n} x_k\right\| \quad < \quad (1+\epsilon)\left|\varphi_{k_0}\left(\sum_{k\leq n} x_k\right)\right| \tag{12}$$

$$< (1+\epsilon) \sum_{k \le n} \left| \varphi_{k_0}(x_k) \right| \tag{13}$$

From the relations (3) and (13) we obtain:

$$\left\| \sum_{k \le n} x_k \right\| < (1+\epsilon)\lambda \sum_{k \le n} \left| \varphi_{k_0}(x_k) \right|^2 \tag{14}$$

From the relations (14) and (10) we obtain:

$$\left\| \sum_{k \le n} x_k \right\| < (1+\epsilon) \frac{1}{\|x_{k_0}\|} \sum_{k \le n} \|x_k\|^2 \, , \forall \epsilon > 0 \tag{15}$$

For $\|x_n\| = |\lambda_n| \, , \ \forall \, n \in \mathbb{N}$ we get in (15)

$$\left\| \sum_{k \le n} x_k \right\| < \frac{1}{\|x_{k_0}\|} \sum_{k \le n} |\lambda_k|^2 \, , \forall \, n \in \mathbb{N}$$

□

Theorem 2.2. *Let X be an infinite-dimensional Banach space. For each $m \in \mathbb{N}$, we can find vectors $z_1, ..., z_n$ such that for any $a \in \mathbb{R}^m$*

$$\frac{1}{\sqrt{3}} \|a\|_{\ell^n_\infty} \le \left\| \sum_{j \le n} a_j z_j \right\| \le \|a\|_{\ell^n_2}$$

Proof. We know that:

$$\begin{aligned} \left\| \sum_{j \le n} a_j z_j \right\| &= \sup_{\|\varphi\|=1, \ \varphi \in X^*} \left| \varphi \left(\sum_{j \le n} a_j z_j \right) \right| \\ &\le \sup_{\|\varphi\|=1, \ \varphi \in X^*} \left| \sum_{j \le n} \varphi(a_j z_j) \right| \\ &\le \sup_{\|\varphi\|=1, \ \varphi \in X^*} \sum_{j \le n} |\varphi(a_j z_j)| \end{aligned} \tag{16}$$

$$\sum_{j \le n} |\varphi(a_j z_j)| = \lambda \sum_{j \le n} |a_j|^2 |\varphi(z_j)|^2 \tag{17}$$

From the relation (17) we obtain

$$|a_j \varphi(z_j)| - \frac{1}{2\lambda} = \frac{\sqrt{n}}{2\lambda} b_k \, , \ \lambda = \lambda(a_j, \varphi, z_j) \, , \, j \in 1, ..., n \tag{18}$$

where $b_1^2 + ... + b_n^2 = 1$.
Let

$$|b_{k_0}| = \min_{k=\overline{1,n}} |b_k|. \tag{19}$$

From the relations (18) and (19) we obtain

$$|\varphi(a_j z_j)| \le \frac{1}{\lambda} \, , \ \forall \, \varphi \text{ with } \|\varphi\| = 1 \tag{20}$$

Let $\|z_{j_0}\| = \max_{k=1,\ldots,n} \|z_j\|$, then there exists $\varphi_{j_0} \in X^*$, $\|\varphi_{j_0}\| = 1$ such that $\varphi_{j_0}(a_{j_0} z_{j_0}) = \|a_{j_0} z_{j_0}\| = |a_{j_0}| \|z_{j_0}\| \le \dfrac{1}{\lambda}$, that is, the following relationship is obtained:

$$\lambda \le \frac{1}{|a_{j_0}| \|z_{j_0}\|} \tag{21}$$

From (17)

$$\sum_{j\leq n}|\varphi_{j_0}(a_jz_j)| \leq \frac{1}{|a_{j_0}|\|z_{j_0}\|}\sum_{j=1}^{n}|a_j|^2|\varphi_{j_0}(z_j)|^2 \tag{22}$$

From (16), given $\epsilon > 0$,

$$\sup_{\|\varphi\|=1,\ \varphi\in X^*}|\sum_{j\leq n}\varphi(a_jz_j)| < (1+\epsilon)\sum_{j\leq n}|\varphi_{j_0}(a_jz_j)| \tag{23}$$

From (22) and (23)

$$\begin{aligned}\sup_{\|\varphi\|=1,\ \varphi\in X^*}|\sum_{j\leq n}\varphi(a_jz_j)| &< \frac{(1+\epsilon)}{|a_{j_0}|\|z_{j_0}\|}\sum_{j=1}^{n}|a_j|^2|\varphi_{j_0}(z_j)|^2\\ &< \frac{(1+\epsilon)}{|a_{j_0}|\|z_{j_0}\|}\sum_{j=1}^{n}|a_j|^2\|z_j\|^2\\ &< \frac{(1+\epsilon)\|z_{j_0}\|}{|a_{j_0}|}\sum_{j=1}^{n}|a_j|^2\ ,\ \forall\,\epsilon>0\end{aligned}$$

Then, setting $\epsilon \longrightarrow 0$ we get

$$\sup_{\|\varphi\|=1,\ \varphi\in X^*}|\sum_{j\leq n}\varphi(a_jz_j)| \leq \frac{\|z_{j_0}\|}{|a_{j_0}|}\sum_{j=1}^{n}|a_j|^2 \tag{24}$$

From the relations (24) and (16) we obtain

$$\left\|\sum_{j\leq n}a_jz_j\right\| \leq \frac{\|z_{j_0}\|}{|a_{j_0}|}\sum_{j=1}^{n}|a_j|^2 \tag{25}$$

If in (25) we choose $z_{j_0}\in X$ such that for a given with $a\in\mathbb{R}^n$ tengamos $\frac{\|z_{j_0}\|}{|a_{j_0}|}<1$, we will have the first part of the inequality satisfied.
From the relation (16)

$$\left|\sum_{j\leq n}\varphi(a_jz_j)\right| \leq \left\|\sum_{j\leq n}a_jz_j\right\|\ ,\ \forall\,\varphi\in X^*\ \text{with}\ \|\varphi\|=1 \tag{26}$$

Let $\varphi(z_j)=x_j+iy_j$, then

$$a_j\varphi(z_j)=a_jx_j+ia_jy_j \tag{27}$$

where $x_j, y_j\in\mathbb{R}$ $,\forall\, j=1,...,n$.
Then, from the relations (27) and (26) we obtain

$$\left|\sum_{j\leq n}a_jx_j+i\sum_{j\leq n}a_jy_j\right| \leq \left\|\sum_{j\leq n}a_jz_j\right\| \tag{28}$$

From the relation (28) we obtain

$$\left|\sum_{j\leq n}a_jx_j\right| \leq \left\|\sum_{j\leq n}a_jz_j\right\| \tag{29}$$

In (29) we obtain applying the already known technique.

$$\sum_{j\leq n}a_jx_j=\tau\sum_{j\leq n}|a_j|^2|x_j|^2 \tag{30}$$

From the relation (30) we obtain

$$\left(a_jx_j - \frac{1}{2\tau}\right) = \frac{\sqrt{n}}{2|\tau|}c_j \ , \ \ \tau = \tau(a_j, x_j) \, , \, j = \overline{1,n} \tag{31}$$

where $c_1^2 + ... + c_n^2 = 1$

$$a_jx_j = \frac{1}{2\tau}(\sqrt{n}c_j + 1) \tag{32}$$

(a) If $\forall j \in \{1, ..., n\} \ \ a_jx_j = 0 \Rightarrow \sqrt{n}c_j = -1, \forall j = \overline{1,n} \ , \ c_j = \dfrac{-1}{\sqrt{n}}$, since $a \in \mathbb{R}^n$ is any element, taking $a = (a_1, ..., a_n)$ all are non-null elements we will have $x_j = 0, \forall j = 1, ..., n$ with which $\varphi(z_j) = iy_j, j = 1, ..., n$

From (28) would also hold

$$\sum_{j\le n} a_jy_j = \tilde{\tau}\sum_{j\le n}|a_j|^2|y_j|^2 \tag{33}$$

and additionally we would have.

$$a_jy_j - \frac{1}{2\tilde{\tau}} = \frac{\sqrt{n}}{2\tilde{\tau}}\tilde{c_j}, \quad \tilde{\tau} = \tilde{\tau}(a_j, y_j) \tag{34}$$

(b) If in (34) $a_jy_j = 0 \ , \ \forall j = 1, ..., n$, we would have that $\varphi(z_j) = 0 \, , \ \forall \, \varphi \in X^*$, which is false by the Hanh - Banach theorem, since we could choose being $\{z_j\}_{j=1}^n$ a base, even without being a base it is enough that $z_j \neq 0$, then $\varphi_j(z_j) = \|z_j\| \neq 0 \ , \ \|\varphi_j\| = 1$.

Then in (a) or (b) $a_jx_j \neq 0$ for at least one j or $a_jy_j \neq 0$ for some $j \in \{1, ..., n\}$
If we choose the following in the parameterization:

$$|a_{k_0}| = \max_{j=\overline{1,n}}|a_j| \, , \ c_{k_0} = \frac{1}{\sqrt{n}}$$

We have in (31) the following: $a_{k_0}x_{k_0} - \dfrac{1}{2\tau} = \dfrac{1}{2\tau}$, from this relation it follows that:

$$a_{k_0}x_{k_0} = \frac{1}{\tau} \tag{35}$$

From the relation (35) in (30) we obtain:
$|\tau||x_{k_0}||a_{k_0}|^2 = |a_{k_0}x_{k_0}|$ which implies that

$$|\tau||a_{k_0}| = 1 \tag{36}$$

From (36) and (35) we get: $|x_{k_0}| = 1$, therefore in (28) we have

$$\ell_\infty^n(a) \le \|\sum_{j\le 1} a_jz_j\| \tag{37}$$

For fixed k_0, there exists $\varphi \in X^*$ such that
$\varphi(z_{k_0}) = \|z_{k_0}\| = x_{k_0} + iy_{k_0} \ , \ y_{k_0} = 0 \ , \quad x_{k_0} = \|z_{k_0}\| = 1$

Then $\ell_\infty^n(a) \le \left|\left|\sum\limits_{j\le 1} a_jz_j\right|\right| < \ell_2^n(a)$ for each $n \in \mathbb{N}$. It suffices that for some $\varphi : X \to \mathbb{K}, \ \varphi(z_{k_0}) = ||z_{k_0}|| = 1$ and $\varphi(z_k) = 0, \ \forall k = 1, ..., n; \ \ k \neq k_0$.
This is feasible by the Hahn-Banach theorem if $z_{k_0} \notin L = span\{z_k\}_{k=1}^n$; then it exists $\tilde{\varphi} \in X^*$ such that $\tilde{\varphi}(z) = 0, \ \forall z \in L, \ \tilde{\varphi}(z_{k_0}) = 1$ and $\|\tilde{\varphi}\| = \dfrac{1}{d}$, where $d = P(z_{k_0}; L), P_0(z_{k_0}; L)$ is the distance from z_{k_0} to L. Therefore, in the relation (26) with $\tilde{\tilde{\varphi}} = d\tilde{\varphi}$ we observe that $\|\tilde{\tilde{\varphi}}\| = 1$, and under these conditions we have.

$$\big|\sum_{j\le n} d\tilde{\varphi}(a_jz_j)\big| \le \|\sum_{j\le n} a_jz_j\| \tag{38}$$

$$d|a_{k_0}| \leq \| \sum_{j\leq n} a_j z_j \| \tag{39}$$

$$d\ \ \ell_\infty^n(a) \leq \| \sum_{j\leq n} a_j z_j \| \tag{40}$$

□

Theorem 2.3. *(Orlicz Theorem). If (f_n) is an unconditionally summable sequence in $L_1[0,1]$, then $\sum_n ||f_n||^2 < \ infty$*

Proof. As usual, by considering real and imaginary parts separately, it is enough to work with real-valued functions. Let $\sum_{n\leq m} f_n$, we know that

$$\Big\| \sum_{n\leq m} f_n \Big\| = \sup_{\|\varphi\|=1,\ \varphi\in L_1^*[0,1]} \Big| \varphi\Big(\sum_{n\leq m} f_n \Big) \Big| \tag{41}$$

from the relation (41) we have:

$$\Big| \sum_{n\leq m} \varphi(f_n) \Big| \leq \Big\| \sum_{n\leq m} f_n \Big\|,\ \forall\ \ \varphi \in L_1^*[0,1] \tag{42}$$

Let $F : \mathbb{R}^m \longrightarrow \mathbb{R}$, defined by $F(x_1, ..., x_m) = \varphi(f_1)x_1 + ... + \varphi(f_m)x_m$, then F is continuous. Then $\mathbb{R}^m = kerF \oplus [kerF]^\perp$, where

$$[kerF]^\perp = \{\varphi(f_1), ..., \varphi(f_m)\} \tag{43}$$

Since $(1, ..., 1) \in \mathbb{R}^m$ we have that

$$(1, 1, ..., 1) = x + \lambda y\ , \text{ where } y \in \{kerF\}^\perp \tag{44}$$

After the relation (44) we have

$$F(1, 1, ..., 1) = \lambda F\big(\varphi(f_1), ..., \varphi(f_n)\big)$$

$$\sum_{n\leq m} \varphi(f_n) = \lambda \sum_{n\leq m} |\varphi(f_n)|^2 \tag{45}$$

From the relation (45) and (42) we obtain

$$|\lambda| \sum_{n\leq m} |\varphi(f_n)|^2 \leq \| \sum_{n\leq m} f_n \|,\ \forall m \in \mathbb{N}\ and\ \forall\, \varphi \in L_1^*[0,1] \tag{46}$$

From the relation (46) we have that as $\| \sum_{n\leq m} f_n \| < \infty$ then there exists $L > 0$ such that

$$\sum_{n\leq m} |\varphi(f_n)|^2 \leq L, \forall m \in \mathbb{N}\ \ and\ \ \forall \varphi \in L_1^*[0,1] \tag{47}$$

Since

$$\|f_i\|^2 = \sup_{\|\varphi\|=1,\ \varphi\in L_1^*[0,1]} \Big| \varphi(f_i) \Big|^2 \tag{48}$$

From (48) given $\frac{\varepsilon}{2^i} > 0,\ \ \exists\, \varphi \in L_1^*[0,1]$ witht $\|\varphi\| = 1$ such that

$$|\varphi(f_i)|^2 + \frac{\varepsilon}{2^i} > \|f_i\|^2 \tag{49}$$

From the relations (47) and (49) we obtain

$$\sum_{n\leq m} \|f_n\|^2 < L + \sum_{n\leq m} \frac{\varepsilon}{2^n}, \forall\, m \in \mathbb{N} \tag{50}$$

From the relation (50), we obtain

$$\sum_{n\leq m}\|f_n\|^2 \leq L,\ \forall\, m\in\mathbb{N}.$$

We must justify that $\lambda = \lambda(m)$ is bounded $\forall\, m \in \mathbb{N}$. From the relation (46) we have

$$|\lambda||\varphi(f_j)|^2 \leq \|\sum_{n\leq m} f_n\| \tag{51}$$

By the Hahn - Banach theorem, there exists $\varphi \in L1^*[0,1]$ such that

$$|\varphi(f_j)| = \|f_j\| \tag{52}$$

From the relations (51) and (52) it follows that $\lambda = \lambda(m)$ is bounded $\forall\, m \in \mathbb{N}$.

□

Theorem 2.4. *(Shur's Theorem) In ℓ_1, weak convergence and norm convergence of a sequence are the same.*

Proof. Let $\varphi\big(x^{(n)}\big) = \sum\limits_{n=1}^{\infty} b_k^{(n)} x_k^{(n)}$, where $\varphi \in \ell_1^* = \ell_\infty$, $\|(b_k^{(n)})_{k=1}^{\infty}\|_{\ell_\infty}$ = $\|\varphi\|$. Suppose

$$\lim_{n\longrightarrow\infty}\varphi\big(x^{(n)}\big) = 0, \forall\ \varphi \in \ell_1^* \tag{53}$$

Let us define $F : \mathbb{R}^m \longrightarrow \mathbb{R}$ as

$$F(y_1, ..., y_m) = \big(b_1^{(n)})x_1^{(n)}\big)y_1 + ... + \big(b_m^{(n)} x_m^{(n)}\big)y_m \tag{54}$$

It is clear that F is continuous and linear, so since $\mathbb{R}^m$ with the usual metric is a Hilbert space we have that

$$\mathbb{R}^m = kerF \bigoplus [kerF]^\perp \tag{55}$$

It is easy to get $[kerF]^\perp = \big(b_1^{(n)} x_1^{(n)}\ , ...\ , b_m^{(n)} x_m^{(n)}\big)$, then

$$(1, ..., 1) = x + \lambda y, \text{where } x \in kerF \ \text{ and } \ y \in [kerF]^\perp \tag{56}$$

From the relation (56) we obtain:

$$\sum_{i=1}^{m} b_i^{(n)} x_i^{(n)} = \lambda \sum_{i=1}^{m} \left(b_i^{(n)} x_i^{(n)}\right)^2 \tag{57}$$

From (57) we get that $\lambda = \lambda(b_1, ..., b_m, x_1, ..., x_m, m, n)$

$$\left|\sum_{i=1}^{m} b_i^{(n)} x_i^{(n)}\right| = |\lambda| \sum_{i=1}^{m} \left(b_i^{(n)} x_i^{(n)}\right)^2 \geq |\lambda| \left(b_i^{(n)} x_i^{(n)}\right)^2 \tag{58}$$

From the relation (58) we obtain that there is $M > 0$ such that:

$$|\lambda| \leq M, \forall\, m \in \mathbb{N} \tag{59}$$

Since $\varphi(x^{(n)}) = \sum\limits_{k=1}^{\infty} b_k^{(n)} x_k^{(n)}$, given $\dfrac{\epsilon}{2} > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\left|\sum_{k=1}^{m} b_k^{(n)} x_k^{(n)}\right| < \frac{\epsilon}{2} + |\varphi(x^{(n)})|\ ,\ \forall\, m \geq N_0\ and\ n \text{ fixed and for all } \varphi \in \ell_1^* \tag{60}$$

Also

$$|\varphi(x^{(n)})| < \frac{\epsilon}{2}\ ,\ n \geq N_1 \text{ and} \,\forall\, \varphi \in \ell_1^* \tag{61}$$

From the relations (60) and (61) we have:

$$\left|\sum_{k=1}^{m} b_k^{(n)} x_k^{(n)}\right| < \epsilon, \forall\, n \geq N_1\ ,\ m \geq N_0 \text{ and } \forall\, \varphi \in \ell_1^* \tag{62}$$

Applying the same process performed in (ii) we have:

$$\sum_{k=1}^{m}\left|b_i^{(n)}x_i^{(n)}\right| = \tau\sum_{k=1}^{m}\left(b_i^{(n)}x_i^{(n)}\right)^2 \tag{63}$$

From the relations (63), (62) and (58) we have

$$\frac{|\lambda|}{|\tau|}\sum_{k=1}^{m}\left|b_k^{(n)}x_k^{(n)}\right| < \epsilon\,,\, \forall\, n\geq N_1\,,\; m\geq N_0 \text{ and } \forall\,\varphi\in\ell_1^* \tag{64}$$

Let $b_{k_0}^{(n)}) = \min\limits_{1\leq k\leq m}|b_k^{(n)}|$, since $\tau>0$, we see that from the relation (63) τ is bounded, then in (64) we have

$$\frac{|b_{k_0}^{(n)}||\lambda|}{|\tau|}\sum_{k=1}^{m}\left|x_k^{(n)}\right| < \epsilon\,,\, \forall\, n\geq N_1\,,\; m\geq N_0 \text{ and } \forall\,\varphi\in\ell_1^* \tag{65}$$

Since $|b_{k_0}^{(n)}|, |\lambda|$ and $|\tau|$ are bounded $\forall\, n, m$, we have from the relation (65) that the series

$$\sum_{k=1}^{m}\left|x_k^{(n)}\right| < \epsilon\,,\, \forall\, n\geq N_1\,,\; m\geq N_0 \text{ y } \varphi\in\ell_1^*$$

□

Shur's Theorem is also proved in [1] differently from ours and from what is presented by [3].

3. Conclusions

Using the aforementioned technique, demonstrations different from the traditional ones were obtained. We are convinced that this technique can be used to deal with economics, measure theory, Convex analysis, optimization and others.

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