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A Piecewise Linear Collocation with Closed Newton Cotes Scheme for Solving Second Kind Fredholm Integral Equation (FIE) via Half-Sweep SOR Iteration

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Abstract In this paper, an efficient and reliable algorithm has been established to solve the second kind of FIE based on the lower-order piecewise polynomial and the lower-order quadrature method, namely Half-sweep Composite Trapezoidal (HSCT), which was used to discretize any integral term. Furthermore, due to the benefit of the complexity reduction technique via the half-sweep iteration concept presented from previous studies based on the cell-centered approach, this paper attempts to derive an HSCT piecewise linear collocation approximation equation generated from the discretization process of the proposed problem by considering the distribution of node points with vertex-centered type. Using half-sweep collocation node points over the linear collocation approximation equation, we could construct a system of HSCT linear collocation approximation equations, whose coefficient matrix is huge-scale and dense. Furthermore, to attain the piecewise linear collocation solution of this linear system, we considered the efficient algorithm of the Half-Sweep Successive Over-Relaxation (HSSOR) iterative method. Therefore, several numerical experiments of the proposed iterative methods have been implemented by solving three tested examples, and the obtained results that were based on three parameters, namely iteration quantity, accomplished time, and maximum absolute error, were recorded and compared against other two iterations, namely Full-Sweep Gauss-Seidel (FSGS) and Half-Sweep Gauss-Seidel (HSGS).

Keywords Piecewise Linear Polynomial, Closed Newton Cotes Scheme, Gauss-Seidel (GS), Collocation Approach, Successive over Relaxation (SOR)

1. Introduction

This section makes a quick review of several numerical solutions for solving integral equations. Some methods have been regularly used as a combination approach to the integral equation: Chebyshev, Galerkin, collocation, and Lagrange interpolation [1-4]. Interpolation is one of the standard methods that have been implemented in many areas of studies to derive the function from a given discrete data set. The purpose of interpolation is to find a formula over the node points in the data function itself [5]. Nevertheless, the collocation method is the easiest to handle and is less complex than others [6]. Besides, one of the most appealing, recently popular methods is the quadrature methods of Newton-cotes rules [7, 8]. We shall briefly explain some methods that are commonly used by recent studies. Based on one of the studies, a quadrature scheme, the trapezoidal method has especially been used in solving the integration equation by generating the linear system equation [9]. Based on the outcome of the study, by operating the equation of a linear system by using a selected iterative method, the efficiency of the tested parameters over the problems was in good agreement.

Moreover, the quadrature scheme of modified trapezoidal and modified Simpson methods was also introduced [10]. These methods were implemented on the integral equations in order to obtain the corresponding approximation equations. Apart from the quadrature scheme, the arithmetic mean and geometric mean methods have also been widely and recently used to solve the integral equation [11]. Since we have briefly explained FIE of the second kind, many applications have been frequently applied in the related study areas, mainly in science and engineering. Besides, given the good physical scenarios, the integral equations are, therefore, beneficial for application. The application invented initially came from the expansion of the ideas of integral equations, and it has various approaches prior to the data analysis on the combination of other methods and the integral equation functions. As mentioned earlier, this paper focuses on integral equations, where a few types of integral equations were involved. Generally, the integral equations can be Fredholm, Volterra, Voltera-integro, and Volterra-Fredholm [12-15]. However, on the other hand, they can also have a variation kind of FIE [16-19]. The following equations entail the three types of Fredholm integral equations of the first, second, and third kinds of Fredholm integral equation as the functions applied in this paper.

$$\int_a^b k(x,t)U(t)dt = g(x), x \in [a,b], \quad (1)$$

$$U(x) + \lambda \int_a^b k(x,t)U(t)dt = g(x), x \in [a,b], \quad (2)$$

$$U(x) + \lambda \int_a^b k(x,t)U(t)dt = A(\mu)g(x), x \in [a,b], \quad (3)$$

Before we focus on achieving the purpose of the study which is to get the numerical method by applying the lower-degree polynomial piecewise collocation with the lower-degree quadrature method on the Fredholm integral equation of the second kind, we need to know the characteristic of the FIE of the second kind. By referring to Eq. (2), the FIE of the second kind is a smooth kernel. Meanwhile, the $U(t)$ is an unknown function and the $g(x)$ is a provided function with the lambda parameter included in the real number element [20].

2. Derivative of the Lower-degree Piecewise Polynomial FIE of Second Kind

This section will cover the lower-degree piecewise polynomial with collocation with lower-degree quadrature on the FIE of the second kind. This section will discuss the quadrature method's definition. Prior to constructing the approximation equation for the proposed problem, its solution domain, $I = [a,b]$ requires the division of n subintervals into equal halves. Whereas the solution domain's every node points x_i, $i = 0,2,4,\ldots,n-2,n$ can be expressed as

$$a = x_0 < x_2 < x_4 < \cdots < x_n = b$$

Let's define the h term on the solution domain as the length size of its subinterval on the interval, $[a,b]$ so that we can build a grid network of these node point. The subinterval the $I = [a,b]$ can be expressed as

$$h = \frac{b-a}{n} \quad (4)$$

Half-sweep Composite Trapezoidal (HSCT), a lower-order half-sweep quadrature method that is primarily used to discretize integral terms of problems, is also discussed in this subsection (2). First, let's discuss how to go about this, $x_0, x_2, x_4, \ldots x_{n-2}, x_n$ be $(\frac{n}{2}+1)$ real abscissas and $U(x_0), U(x_2), U(x_4)\ldots U(x_n)$ represent their corresponding values, respectively. Fig. 1 provides an illustration of the type of node points' grid network. A set of the node points of type is the main part that needs to be highlighted because it makes the derivation of the lower-degree quadrature approximation equation easier by identifying these node points as we derived the lower-degree polynomial piecewise, collocation with lower-degree quadrature approximation equations of a problem (2).

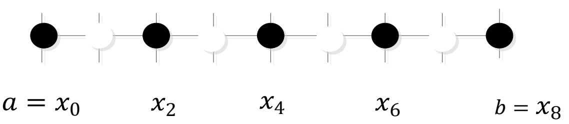

Figure 1. Half-sweep interval domain of $I = [a,b]$ on FIE of second kind

All node points of type ● on the whole interval $I = [a,b]$ that have been handled as collocation node points for the half-sweep case have been displayed in the half-sweep grid network in Fig. 1. We can demonstrate the formulation of the Half-Sweep Trapezoidal scheme at any half-sweep subinterval $[x_i, x_{i+2}]$, which can be stated as follows:

$$\int_{x_i}^{x_{i+2}} f(x)dx = \frac{h}{2}[f_i + f_{i+2}], \; i = 0,2,4,\cdots n-2,n. \quad (5)$$

In general, we can demonstrate the Full- and Half-Sweep Trapezoidal schemes' formulation at any subinterval, $[x_i, x_{i+\sigma}]$

$$\int_{x_i}^{x_{i+\sigma}} f(x)dx = \frac{(\sigma h)}{2}[f_i + f_{i+\sigma}], \; i = 0,1,\sigma,\cdots,n-\sigma,n \quad (6)$$

where the value of $\sigma = 1,2$ represents the full-, and half-sweep cases, respectively.

Clearly, we can see from Eq. (5) that is a subinterval of a finite sum for the solution domain. The half-sweep Trapezoidal scheme (5) over the interval is expanded and integrated into the discussion that follows. It was wisely explained in the most recent part how the collocation approach applies to the FIE of the second kind. Let's begin by defining the lower-order piecewise polynomial approximation function, which may be written as follows, based on the half-sweep case over the solution domain $[a,b]$.

$$U(x)=\sum_{i=0}^{n} H_i(x).\delta_i(x),\quad x\in[a,b], \qquad (7)$$

where the function of $H_i(x)$, and $\delta_i(x)$, $i=0,2,4,\ldots,n-2,n$ are specifically defined as

$$H_i(x)=\begin{cases} N_1(x)U_{i-2}+N_2(x)U_i,\, x_{i-2}<x<x_i,\, i=2,4,...,n \\ 0, otherwise \end{cases}$$

and

$$\delta_i(x)=\begin{cases} 1, x_{i-2}<x<x_i,\, i=2,4,...,n \\ 0, others \end{cases}$$

In fact, the function, $H_i(x)$, $i=0,2,4,\ldots,n-2,n$ is known as a piecewise linear function and the function $N_p(x)$, is a linear function that can be defined as

$$N_1(x)=\frac{(x-x_{i-2})}{2h},$$

and

$$N_2(x)=\frac{(x_i-x)}{2h},$$

respectively.

Now that the discretization process has begun, we construct the relevant approximation equation by substituting Eq. (7) into Eq. (2) as follows:

$$U(x)+\lambda\int_a^b k(x,t)\sum_{j=0}^{n} H_j(x).\delta_i(x)dt=g(x), \qquad (8)$$

We simplify Eq. (9) such that it is clearly understandable as

$$U(x)+\lambda\sum_{j=0}^{n}\int_a^b k(x,t).H_j(x)\delta_i(x)dt=g(x), \qquad (9)$$

The collocation scheme is one of the crucial steps in obtaining the approximation equation, as was stated in the preceding section. Using the edge-vertex technique, we will now impose all the collocation node points into Eq. (9) to obtain

$$U_i+\lambda\sum_{j=0}^{n} H_{ji}\int_a^b k(x_i,t).\delta_{ji}dt=g_i \qquad (10)$$

for

$$i=0,2,4,...,n-2,n.$$

where $U_i=U(x_i)$, $H_j(x_i)=H_{ji,}$ $\delta_j(x_i)=\delta_{ji}$, $g(x_i)=g_i$.

By referring to Eq. (5), we can apply the lower-degree of quadrature method, namely HSCT in the discretization process of an integral term in Eq. (10).

Following the current discretization process, the following is how the evolution of the linear system was generated:

$$F\underline{U}=\underline{g} \qquad (11)$$

where,

$$F=\begin{bmatrix} 1+A_0G_{0,0} & A_2G_{0,2} & \cdots & A_nG_{0,n} \\ A_0G_{2,0} & 1+A_2G_{2,2} & \cdots & A_nG_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_0G_{n,0} & A_2G_{n,2} & \cdots & 1+A_nG_{n,n} \end{bmatrix}$$

$$\underline{U}=[U_0\quad U_2\quad \cdots\quad U_n],$$

$$\underline{g}=[g_0\quad g_2\quad \cdots\quad g_n].$$

It is obvious that a large-scale and dense matrix is the primary characteristic of the coefficient matrix, F, of the linear system (11). Although using a lower-degree piecewise polynomial with a collocation scheme, the main characteristic of the dense coefficient matrix is due to the HSCT discretization strategy for the integral term.

3. Derivative of HSSOR

In this section, we focus our attention to get the numerical solution of the linear system (11) produced from the result of the discretization process over the proposed problem. Generally, we have direct methods and iterative methods that can be considered to solve the linear system. Nevertheless, it can be observed that the critical features of the coefficient matrix are huge-scale and dense. Therefore, an appropriate method for such a linear system needs to consider a family of iterative methods. As we know, SOR iterative method is one of the most efficient point iterative methods for solving a linear system of HSCT linear collocation approximation equations [21]. In fact, the primary features of the SOR iterative method are performed by using a weighted parameter, in which this weight parameter can increase the convergence rate and efficiency for solving a linear system. [22]. In contrast with the standard SOR iterative method, FSSOR, we attempt to combine the half-sweep iteration and the SOR iterative method to get the HSSOR method based on the HSCT piecewise linear collocation approximation equation to develop a fast and reliable algorithm in finding the numerical solution of problem (2) [23-26].

Moreover, according to the development of studies, various types of SOR iteration families that have been introduced play a significant role in the convergence rate where it seeks the range of values between the set of $\omega = C\{0\}$, which we called the convergence area [27]. The values of the convergence area then will be adjustable between the ranges based on its proposed problems. To begin constructing the formulation of the HSSOR iterative method based on the HSCT piecewise linear collocation approximation equation, we need to rewrite the coefficient matrix, F as the summation of the following three matrices

$$F = D + C + V \tag{12}$$

where D, C and V are diagonal, lower triangular and upper triangular matrices, respectively. Substitute Eq. (12) into Eq. (11), Therefore, the formula in general for HSSOR iterative method can be shown [28-31]

$$\underline{U}^{(k+1)} = (1-\omega)\underline{U}^{(k)} + \omega(D+C)^{-1}(-V\underline{U}^{(k)} + g). \tag{13}$$

where ω is relaxation factor, k is the number of iterations and $\underline{U}^{(k+1)} = [U_0^{(k)}, U_2^{(k)}, U_n^{(k}]$ As we assign $\omega = 1$ the HSSOR iterative method can be said as the HSGS iterative method. By referring considering to general form of the HSCT piecewise linear collocation, we can rewrite the HSSOR iterative method in the matrix equation (13) in the context of point iteration as follows

$$\underline{U}_i^{(k+1)} = (1-\omega)\underline{U}_i^{(k)} + \omega\underline{U}_i * \tag{14}$$

where

$$\underline{U}_i^* = \frac{\sum_{j=0}^{i=2} A_j G_{i,j} \underline{U}_j^{(k+1)} - b\sum_{j=i+2}^{n} A_j G_{i,j} \underline{U}_j^{(k)}}{1 + A_i G_{i,i}}$$

For $i = 0,,2,4,\ldots,n-2,n$. To find the numerical solution of the linear system (11), the iteration process of the HSSOR iterative method continued until it matched with the predetermined error. By establishing the values of matrices D, C and V in Eq. (12), the general implementation for the HSSOR method to find the piecewise linear collocation solution over the linear system (11) may be described as in

Algorithm 1: HSSOR Iteration

1. Set up the initial value of $U^{(0)} \leftarrow 0, \varepsilon \leftarrow 10^{-10}$.
2. Assign the optimum value of ω.
3. Execute

a. For $i = 0,2,\cdots,n$, calculate

$$\underline{U}_i^* = \frac{\sum_{j=0}^{i=2} A_j G_{i,j} \underline{U}_j^{(k+1)} - b\sum_{j=i+2}^{n} A_j G_{i,j} \underline{U}_j^{(k)}}{1 + A_i G_{i,i}}$$

$$U_i^{(k+1)} = (1-\omega)U_i^{(k)} + \omega U_i *$$

b. Examine the convergence test $|\underline{U}_i^{(k+1)} - \underline{U}_i^{(k)}| \le \varepsilon = 10^{-10}$. If it fits the convergence test, proceed to step 4. Otherwise, redo step 3.

4. Show the approximate solutions of $|\underline{U}_i^{(k+1)}|$.

4. Computational Experiment on FIE of Second Kind

This section discusses the computational experiments over three examples of FIE of the second kind to test the efficiency of the presented methods such as FSGS, HSGS, and HSSOR by considering their corresponding piecewise linear collocation approximation equations. To do this, we must compare the results from three parameters: iteration quantity (Iter), accomplished time (s) in seconds, and maximum absolute error obtained from the application of these methods. Consequently, all results have been recorded based on a few variations in mesh size of $N = 512, 1024, 2048, 4096, 8192$. The following equations entail the three selected examples in this paper:

Example 1 [32]

FIE of the second kind was taken from

$$U(x) = x - \int_0^1 (4xt - x^t)U(t)dt \tag{15}$$

while the exact solution is stated as

$$U(x) = 24x - 9x^2 \tag{16}$$

Example 2 [33]

The second example was introduced by

$$U(x) = e^{3x} - \frac{1}{9}(2e^3 + 1)x + \int_0^1 xtU(t)dt, \quad 0 < x < 1 \tag{17}$$

while its approximation solution is given by

$$U(x) = e^{3x}. \tag{18}$$

Example 3 [34]

The third example of linear FIE of the second kind was mentioned as follows:

$$U(x) = \sin(2\pi x) + \int_0^1 \cos(x)U(t)dt \tag{19}$$

while its exact solution is stated as

$$U(x) = \sin(2\pi x) \tag{20}$$

The reported data are displayed in Tables 1 to Table 3. By observing the iteration quantity (Iter), accomplished time (s), and maximum absolute error, we can develop valid assumptions on the outcomes of the three proposed methods over the problem (2). As can be observed in Table 1 to Table 3, the iteration quantity and accomplished time of the HSSOR iterative method were shorter than those of the HSGS and FSGS iterative methods. However, the maximum absolute error of these iterative methods increased with the increase in grid size, n.

Table 1. Comparison the iteration quantity of Examples 1, 2 and 3 on FIE of second kind

Example	n	Iteration quantity (Iter)		
		FSGS	HSGS	HSSOR
1	512	194	193	40
	1024	194	194	40
	2048	195	194	40
	4096	195	195	40
	8192	195	195	40
2	512	15	15	13
	1024	15	15	13
	2048	15	15	13
	4096	15	15	13
	8192	15	15	13
3	512	64	64	25
	1024	64	64	25
	2048	64	64	25
	4096	64	64	25
	8192	64	64	25

Table 2. Comparison the accomplished time of Examples 1, 2 and 3 on FIE of second kind.

Example	n	Accomplished time (s)		
		FSGS	HSGS	HSSOR
1	512	12.17	3.12	0.72
	1024	48.81	13.74	2.77
	2048	196.08	50.63	10.83
	4096	782.24	202.2	43.12
	8192	3131.13	849.36	172.63
2	512	0.93	0.25	0.22
	1024	3.66	0.95	0.80
	2048	14.06	3.72	3.13
	4096	56.52	14.29	12.41
	8192	225.17	57.86	50.16
3	512	41.79	10.94	4.25
	1024	167.08	43.17	16.86
	2048	668.25	171.85	67.27
	4096	2672.55	688.72	269.05
	8192	11359.94	2755.22	1080.94

Table 3. Comparison the maximum absolute error of Examples 1, 2 and 3 on FIE of second kind.

Example	n	Maximum absolute error		
		FSGS	HSGS	HSSOR
1	512	9.90E-05	3.96E-04	3.96E-04
	1024	2.48E-05	9.90E-05	9.90E-05
	2048	6.19E-06	2.48E-05	2.48E-05
	4096	1.55E-06	6.19E-06	6.19E-06
	8192	3.88E-07	1.55E-06	1.55E-06
2	512	1.69E-05	6.95E-05	6.95E-05
	1024	4.22E-06	1.71E-05	1.71E-05
	2048	1.06E-06	4.25E-06	4.25E-06
	4096	2.64E-07	1.06E-06	1.06E-06
	8192	6.60E-08	2.64E-07	2.64E-07
3	512	3.21E-06	4.13E-06	4.13E-06
	1024	3.21E-06	3.33E-06	3.32E-06
	2048	3.21E-06	3.22E-06	3.22E-06
	4096	3.21E-06	3.21E-06	3.21E-06
	8192	3.21E-06	3.21E-06	3.21E-06

Table 4. Reduction percentage of the HSSOR method when compare to the FSGS method of example 1, 2 and 3.

Example	HSSOR	
	Iter	Time
1	79.38-79.48	94.08 -94.4
2	13.33	76.34 -78.14
3	60.93	89.83 -90.48

5. Conclusions

Based on the values observed in Table 4, there were significant differences in iteration quantity (Iter) and accomplished time (s). Based on the previous study, the formula used finally recorded reduction percentages of iteration quantity obtained from the HSSOR method with 79.38-79.48%, 13.33%, and 60.93%, respectively. In terms of accomplished time, huge significant differences were observed with reduction percentages of 94.08-94.49%, 76.34%-78.14%, and 89.83-90.48%, respectively. The results of the numerical computational on the family of GS and SOR methods in Table 1 to Table 3 show that the SOR iterative family needs a small iteration quantity and accomplished time compared to the GS iterative family, as the SOR iterative family uses a one-weighted parameter that increases the convergence rate of the iteration process.

List of Abbreviations

GS	Gauss-Seidel
SOR	Successive over Relaxation
FSGS	Full-Sweep Gauss-Seidel
HSGS	Half-Sweep Gauss-Seidel
HSSOR	Half-Sweep Successive over Relaxation
ω	Relaxation factor
(Iter)	Iteration quantity
FIE	Fredholm Integral Equation
HSCT	Half-Sweep Composite Trapezoidal

Greek Symbols

n	Size grid
λ	Lambda parameter
$\lvert U_i^{(k+1)} \rvert$	Approximation solution of $\underline{U}$ at $k+1$
$U_i^{(k)}$	Approximation solution of $\underline{U}$ at k
D	Diagonal matrix
V	Upper triangular matrix
C	Lower triangular matrix

The conclusion based on the numerical computational on FIE of the second kind with lower-degree polynomial piecewise with a combination of lower-degree quadrature with the half-sweep case of the SOR iterative method is better than HSGS and FSGS in terms of iteration quantity and accomplished time due to less operational complexity. Due to the benefit of the SOR iteration family such as FSSOR and HSSOR, which is classified as the one-weighted parameter iteration family, the findings of this work should be expanded and improved by considering other linear solvers through the implementation of two-weighted parameter iteration families, MSOR [35] and AOR [36], as well as the two-step level iteration families, Alternating Group Explicit [37] and weighted mean [38].

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