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Properties of Classes of Analytic Functions of Fractional Order

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Abstract The study of *Univalent Function Theory* is very vast and complicated, so simplifying assumptions were necessary. In view of the Riemann Mapping theorem, the most apt thing would be to replace an analytic function defined on an arbitrary domain with an analytic function defined in the unit disc and having a Taylor's series expansion of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. The powers of the series are usually integers, so all the prerequisite results also support the study of analytic functions having a series expansion with integers powers. The main deviation presented here is that we have defined a subclass of analytic functions using a Taylor's series whose powers are non-integers. To make this study more comprehensive, Janowski function which maps the unit disc onto a right half plane has been used in conjunction with two primary tools namely *Subordination* and *Hadamard product*. Motivated by the well-known class of λ-convex functions, here we have defined a fractional differential operator which is a convex combination of two analytic functions. Using the defined fractional differential operator, we introduce and study a new class of analytic functions involving a conic region impacted by the Janowski function. Necessary and sufficient conditions, coefficient estimates, growth and distortion bounds have been obtained for the defined function class. Since studies of various subclasses of analytic functions with fractional powers are rare, here we have pointed out several closely related studies by various authors. However, the superordinate function is a familiar function which has lots of applications.

Keywords Fractional Calculus, Differential Operator, *Subordination*, Starlike Function, Convex Function

1. Introduction, Definitions and Preliminaries

Srivastava et al. in [15] defined a differential operator using the class $\mathcal{A}_\alpha (0 \leq \alpha < 1)$, a class of analytic functions with fractional powers. Typically, it consists of analytic functions defined in $\mathcal{U} = \{z : |z| < 1\}$ and has a series representation of the form

$$f(z) = z^{\alpha+1} + \sum_{n=2}^{\infty} a_n z^{n+\alpha}, (0 \leq \alpha < 1) \quad (1.1)$$

We observe that the analytic function $f \in \mathcal{A}_\alpha$ has the usual normalization if $\alpha = 0$ and $\mathcal{A}_0 = \mathcal{A}$. Moreover, $f(z) \in \mathcal{A} \Longrightarrow z^\alpha f(z) \in \mathcal{A}_\alpha$. Abdulnaby et al. [2] (also see [1], [9]) defined various classes of analytic functions using the function F_μ, which has a power series expansion of the form

$$F_\mu(z) = z + \sum_{n=2}^{\infty} a_n z^{n\mu}, (\mu \geq 1; z \in \mathcal{U}) \quad (1.2)$$

where $\mu = \frac{n+m-1}{m}, m, n \in \mathbb{N}$. We let Π_μ to denote the class of analytic functions having a power series expansion of the form (1.2). We let $*$ and $\prec$ to denote the *Hadamard product* and *subordination* respectively, refer to the standard text for its definition and properties. For a function, $g_\mu(z) = z + \sum_{n=2}^{\infty} \Gamma_n z^{n\mu} \in \Pi_\mu$, we now define an operator $\mathcal{H}_{\lambda,g}^m F_\mu$: $\mathcal{U} \longrightarrow \mathcal{U}$ as follows:

$$\mathcal{H}_{\lambda,g}^{0}F_{\mu}(z)=F_{\mu}(z),$$
$$\mathcal{H}_{\lambda,g}^{1}F_{\mu}(z)=(1-\lambda)\big(F_{\mu}(z)*g_{\mu}(z)\big)\ +\lambda z\big(F_{\mu}(z)*g_{\mu}(z)\big)', \tag{1.3}$$

and

$$\mathcal{H}_{\lambda,g}^{m}F_{\mu}(z)=\mathcal{H}_{\lambda,g}^{1}\left(\mathcal{H}_{\lambda,g}^{m-1}F_{\mu}(z)\right). \tag{1.4}$$

If $F_{\mu}\in\Pi_{\mu}$, then from (1.3) and (1.4) we may easily deduce that

$$\mathcal{H}_{\lambda,g}^{m}F_{\mu}(z)=z+\sum_{n=2}^{\infty}[1+\lambda(\mathrm{n}\mu-1)]^{\mathrm{m}}\ \Gamma_{n}^{m}a_{n}z^{n\mu},(m\in\ N_{0}=N\cup\{0\}\ and\ \lambda\geq 0). \tag{1.5}$$

If we let $\lambda=1$ and $\Gamma_{n}=1$ in (1.5), then $\mathcal{H}_{\lambda,g}^{m}F_{\mu}$ reduces to the operator defined by Abdulnaby et al. [2].

Let $\mathcal{P}$ denote the well-known class of functions with a positive real part. Kanas and Wiśniowska in [10, 11] (also see [13,14]) defined the function $\hat{p}_{k,\sigma}(z)$ which plays the role of extremal functions those related to the conic domain and is given by

$$\hat{p}_{k,\sigma}(z)=\begin{cases}\frac{1+(1-2\sigma)z}{1-z}, & if\ \ k=0\\ 1+\frac{2(1-\sigma)}{\pi^{2}}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{2}, & if\ \ k=1\\ 1+\frac{2(1-\sigma)}{1-k^{2}}\sinh^{2}\left[\left(\frac{2}{\pi}\arccos k\right)\operatorname{arctanh}\sqrt{z}\ \right], & if\ \ 0<k<1\\ 1+\frac{2(1-\sigma)}{1-k^{2}}\sin\left(\frac{\pi}{2R(t)}\int_{0}^{\frac{u(z)}{t}}\frac{1}{\sqrt{1-x^{2}}\ \sqrt{1-(tx)^{2}}}dx\right)+\frac{1}{k^{2}-1}, & if\ \ k>1\end{cases} \tag{1.6}$$

where $u(z)=\frac{z-\sqrt{t}}{1-\sqrt{tz}}, t\in(0,1)$ and t is chosen such that $k=\cosh\left(\frac{\pi R'(t)}{4\ R(t)}\right)$, with $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. Clearly, $\hat{p}_{k,\sigma}(z)$ is in $\mathcal{P}$ with the expansion of the form

$$\hat{p}_{k,\sigma}(z)=1+\delta_{1}z+\delta_{2}z^{2}+\cdots,\big(\delta_{j}=p_{j}(k,\sigma),j=1,2,3,\ldots\big), \tag{1.7}$$

we get

$$\delta_{1}=\begin{cases}\frac{8(1-\sigma)(\arccos k)^{2}}{\pi^{2}(1-k^{2})},\ if\ \ 0\leq k<1,\\ \frac{8(1-\sigma)}{\pi^{2}}, if\ \ k=1\\ \frac{\pi^{2}(1-\sigma)}{4\sqrt{t}\ \ (k^{2}-1)R^{2}(t)(1+t)}, if\ \ k>1\end{cases} \tag{1.8}$$

Breaz et al. [5], (also see [8]) introduced and studied the following function

$$\aleph_{p}[X,Y;\beta;\psi(z)]=\frac{[(1+X)p+\beta(Y-X)]\psi(z)+[(1-X)p-\beta(Y-X)]}{[(Y+1)\psi(z)+(1-Y)]} \tag{1.9}$$

The function $\aleph_{p}[X,Y;\beta;\psi(z)]$ was motivated by [3, 6], and is a generalization of the well-known Janowski function. Motivated by Breaz *et al. [5]* and [4, 7, 8, 10,12], we now define the following:

Definition 1.1. For $-1\leq Y<X\leq 1, 0\leq\alpha,\lambda\leq 1, 0\leq\beta,\sigma<1$ and $\hat{p}_{k,\sigma}(z)$ be defined as in (1.6), a function $F_{\mu}(z)\in\Pi_{\mu}$ is said to be in class $k-\mathcal{E}_{g}(X,Y;\alpha,\lambda;m)$ if it satisfies the subordination condition

$$\frac{\mathcal{H}_{\lambda,g}^{m}F_{\mu}(z)}{(1-\alpha)z+\alpha z\ \mathcal{H}_{\lambda,g}^{m}F_{\mu}{'}(z)}\prec\aleph_{1}[X,Y;\beta;\psi(z)], \tag{1.10}$$

where $\prec$ denotes *subordination*.

Remark 1.1. Equivalently, $h(z)\prec\aleph_{1}\big[X,Y;\beta;\hat{p}_{k,\sigma}(z)\big]$ implies

$$Re\ \ \left(\frac{(Y-1)h(z)+[(1-X)-\beta(Y-X)]}{(Y+1)h(z)-[(1+X)+\beta(Y-X)]}\right)>.$$
$$k\left|\frac{(Y-1)h(z)+[(1-X)-\beta(Y-X)]}{(Y+1)h(z)-[(1+X)+\beta(Y-X)]}-1\right|+\sigma\ \ .$$

Remark 1.2. Now we highlight some special cases.

Setting $m=\beta=0,\alpha=1,X=1\ \ and\ \ Y=-1\ \ in\ \ Definition\ \ 1.1, then$ $k-\mathcal{E}_{g}(X,Y;\alpha,\lambda;m)$ reduces to

$$k-S(\sigma)=\left\{\ F_{\mu}\in\Pi_{\mu};Re\left(\frac{F_{\mu}(z)}{zF_{\mu}{'}(z)}\right)>k\left|\frac{F_{\mu}(z)}{zF_{\mu}{'}(z)}-1\right|+\sigma\right\}.$$

Setting $m=\alpha=\beta=\sigma=0, X=1\ \ and\ \ Y=-1\ \ in\ \ Definition\ \ 1.1, then$ $k-\mathcal{E}_g(X,Y;\alpha,\lambda;m)$ reduces to

$$k-S(\sigma)=\left\{\ F_\mu\in\Pi_\mu; Re\left(\frac{F_\mu(z)}{z}\right)>k\left|\frac{F_\mu(z)}{z}-1\right|\right\}.$$

Further we define $k-T\mathcal{E}_g(X,Y;\alpha,\lambda;m)=k-\mathcal{E}_g(X,Y;\alpha,\lambda;m)\cap T,$ where T is a class consisting of analytic functions of the form

$$F_\mu(z)=z-\sum_{n=2}^{\infty}a_n z^{n\mu}, a_n\geq 0, \forall n\geq 2. \tag{1.11}$$

2. Main Results

In this section, we will obtain several properties of the class $k-T\mathcal{E}_g(X,Y;\alpha,\lambda;m)$. We will begin with the following:
Theorem: 2.1. Let the function $F_\mu(z)$ be defined by (1.2) and let

$$\sum_{n=2}^{\infty}\left[\frac{2(k+1)}{1-\sigma}|\alpha n\mu-1|+|(Y+1)-[(1+X)+\beta(Y-X)]\alpha n\mu|\right]$$
$$[1+\lambda(n\mu-1)]^m|\Gamma_n a_n|\leq(X-Y)(1-\beta) \tag{2.1}$$

hold, then $F_\mu(z)$ belongs to $k-\mathcal{E}_g(X,Y;\alpha,\lambda;m)$.
Proof. Suppose that the inequality holds. Then we have for $z\in\mathcal{U}$

$$k\left|\frac{(Y-1)\dfrac{\mathcal{H}_{\lambda,g}^m F_\mu(z)}{(1-\alpha)z+\alpha z\ \ \mathcal{H}_{\lambda,g}^m F_\mu{}'(z)}+[(1-X)-\beta(Y-X)]}{(Y+1)\dfrac{\mathcal{H}_{\lambda,g}^m F_\mu(z)}{(1-\alpha)z+\alpha z\ \ \mathcal{H}_{\lambda,g}^m F_\mu{}'(z)}-[(1+X)+\beta(Y-X)]}-1\right|$$

$$=k\left|\frac{(Y-1)\mathcal{H}_{\lambda,g}^m F_\mu(z)+[(1-X)-\beta(Y-X)]\left[(1-\alpha)z+\alpha z\ \ \mathcal{H}_{\lambda,g}^m F_\mu{}'(z)\right]}{(Y+1)\mathcal{H}_{\lambda,g}^m F_\mu(z)-[(1+X)+\beta(Y-X)]\left[(1-\alpha)z+\alpha z\ \ \mathcal{H}_{\lambda,g}^m F_\mu{}'(z)\right]}-1\right|$$

$$=2k\left|\frac{\sum_{n=2}^{\infty}[1+\lambda(n\mu-1)]^m(\alpha n\mu-1)\Gamma_n a_n z^{n\mu}}{(Y-X)(1-\beta)z+\sum_{n=2}^{\infty}[1+\lambda(n\mu-1)]^m\{(Y+1)-[(1+X)+\beta(Y-X)]\alpha n\mu\}\Gamma_n a_n z^{n\mu}}\right|$$

$$\leq\frac{2k\sum_{n=2}^{\infty}[1+\lambda(n\mu-1)]^m|\alpha n\mu-1||\Gamma_n a_n||z|^{n\mu}}{(X-Y)(1-\beta)|z|-\sum_{n=2}^{\infty}[1+\lambda(n\mu-1)]^m|(Y+1)-[(1+X)+\beta(Y-X)]\alpha n\mu||\Gamma_n a_n||z|^{n\mu}} \tag{2.2}$$

From (2.1), we have

$$(X-Y)(1-\beta)|z|-\sum_{n=2}^{\infty}[1+\lambda(n\mu-1)]^m|(Y+1)-[(1+X)+\beta(Y-X)]\alpha n\mu||\Gamma_n a_n||z|^{n\mu}>0.$$

Now if $F_\mu(z)\in k-\mathcal{E}_g(X,Y;\alpha,\lambda;m)$, we have to establish

$$k\left|\frac{(Y-1)\dfrac{\mathcal{H}_{\lambda,g}^m F_\mu(z)}{(1-\alpha)z+\alpha z\ \ \mathcal{H}_{\lambda,g}^m F_\mu{}'(z)}+[(1-X)-\beta(Y-X)]}{(Y+1)\dfrac{\mathcal{H}_{\lambda,g}^m F_\mu(z)}{(1-\alpha)z+\alpha z\ \ \mathcal{H}_{\lambda,g}^m F_\mu{}'(z)}-[(1+X)+\beta(Y-X)]}-1\right|-$$

$$Re\left(\frac{(Y-1)\dfrac{\mathcal{H}_{\lambda,g}^m F_\mu(z)}{(1-\alpha)z+\alpha z\ \ \mathcal{H}_{\lambda,g}^m F_\mu{}'(z)}+[(1-X)-\beta(Y-X)]}{(Y+1)\dfrac{\mathcal{H}_{\lambda,g}^m F_\mu(z)}{(1-\alpha)z+\alpha z\ \ \mathcal{H}_{\lambda,g}^m F_\mu{}'(z)}-[(1+X)+\beta(Y-X)]}-1\right)<1-\sigma.$$

From (2.2), we have

$$k\left|\frac{(Y-1)\dfrac{\mathcal{H}_{\lambda,g}^{m}F_{\mu}(z)}{(1-\alpha)z+\alpha z\ \mathcal{H}_{\lambda,g}^{m}F_{\mu}{}'(z)}+[(1-X)-\beta(Y-X)]}{(Y+1)\dfrac{\mathcal{H}_{\lambda,g}^{m}F_{\mu}(z)}{(1-\alpha)z+\alpha z\ \mathcal{H}_{\lambda,g}^{m}F_{\mu}{}'(z)}-[(1+X)+\beta(Y-X)]}-1\right|-$$

$$Re\ \left(\frac{(Y-1)\dfrac{\mathcal{H}_{\lambda,g}^{m}F_{\mu}(z)}{(1-\alpha)z+\alpha z\ \mathcal{H}_{\lambda,g}^{m}F_{\mu}{}'(z)}+[(1-X)-\beta(Y-X)]}{(Y+1)\dfrac{\mathcal{H}_{\lambda,g}^{m}F_{\mu}(z)}{(1-\alpha)z+\alpha z\ \mathcal{H}_{\lambda,g}^{m}F_{\mu}{}'(z)}-[(1+X)+\beta(Y-X)]}-1\right)$$

$$\leq (k+1)\left|\frac{(Y-1)\dfrac{\mathcal{H}_{\lambda,g}^{m}F_{\mu}(z)}{(1-\alpha)z+\alpha z\ \mathcal{H}_{\lambda,g}^{m}F_{\mu}{}'(z)}+[(1-X)-\beta(Y-X)]}{(Y+1)\dfrac{\mathcal{H}_{\lambda,g}^{m}F_{\mu}(z)}{(1-\alpha)z+\alpha z\ \mathcal{H}_{\lambda,g}^{m}F_{\mu}{}'(z)}-[(1+X)+\beta(Y-X)]}-1\right|$$

$$\leq \frac{2(k+1)\sum_{n=2}^{\infty}[1+\lambda(n\mu-1)]^{m}|\alpha n\mu-1||\Gamma_n a_n||z|^{n\mu}}{(X-Y)(1-\beta)|z|-\sum_{n=2}^{\infty}[1+\lambda(n\mu-1)]^{m}|(Y+1)-[(1+X)+\beta(Y-X)]\alpha n\mu||\Gamma_n a_n||z|^{n\mu}}.$$

The above inequality would be bounded by $1-\sigma$, if

$$2(k+1)\sum_{n=2}^{\infty}[1+\lambda(n\mu-1)]^{m}|\alpha n\mu-1||\Gamma_n a_n| \leq (1-\sigma)\Bigg((X-Y)(1-\beta)|z| - \sum_{n=2}^{\infty}[1+\lambda(n\mu-1)]^{m}|(Y+1)-[(1+X)+\beta(Y-X)]\alpha n\mu||\Gamma_n a_n|\Bigg).$$

On simplifying, we get

$$\sum_{n=2}^{\infty}\left[\frac{2(k+1)}{1-\sigma}|\alpha n\mu-1|+|(Y+1)-[(1+X)+\beta(Y-X)\alpha n\mu|\right][1+\lambda(n\mu-1)]^{m}|\Gamma_n a_n| \leq (X-Y)(1-\beta)$$

which shows that $F_{\mu}(z)$ belongs to $k-\mathcal{E}_g(X,Y;\alpha,\lambda;m)$.

Corollary 2.1. Let the function $F_{\mu}(z)$ defined by (1.11) belong to $k-T\mathcal{E}_g(X,Y;\alpha,\lambda;m)$. Then

$$a_n \leq \frac{(X-Y)(1-\beta)}{\left[\frac{2(k+1)}{1-\sigma}|\alpha n\mu-1|+|(Y+1)-[(1+X)+\beta(Y-X)\alpha n\mu|\right]\Theta_n}, (n\geq 2), \tag{2.3}$$

with $\Theta_n=[1+\lambda(n\mu-1)^{m}]|\Gamma_n|$.

We will denote

$$\Upsilon_{\mathrm{n}}(\alpha,\beta,\mu;k,\sigma)=\left[\frac{2(k+1)}{1-\sigma}|\alpha n\mu-1|+|(Y+1)-[(1+X)+\beta(Y-X)\alpha n\mu|\right]\Theta_n. \tag{2.4}$$

Theorem 2.2. Let $F_{\mu}\in k-T\mathcal{E}_g(X,Y;\alpha,\lambda;m)$ and let $\alpha\mu>\frac{1}{2}$. If $\{\Upsilon_{\mathrm{n}}(\alpha,\beta,\mu;k,\sigma)\}_{n=2}^{\infty}$ is a non-decreasing sequence, then, for $|z|=r<1$

$$|z|-\frac{(X-Y)(1-\beta)}{\Upsilon_2(\alpha,\beta,\mu;k,\sigma)}|z|^{2\mu}\leq\left|F_{\mu}(z)\right|\leq|z|+\frac{(X-Y)(1-\beta)}{\Upsilon_2(\alpha,\beta,\mu;k,\sigma)}|z|^{2\mu} \tag{2.5}$$

and if $\{\Upsilon_{\mathrm{n}}(\alpha,\beta,\mu;k,\sigma)/n\}_{n=2}^{\infty}$ is a non-decreasing sequence, then for $|z|=r<1$,

$$1-\frac{2(X-Y)(1-\beta)}{\Upsilon_2(\alpha,\beta,\mu;k,\sigma)}|z|^{2\mu-1}\leq\left|F_{\mu}{}'(z)\right|\leq 1+\frac{2(X-Y)(1-\beta)}{\Upsilon_2(\alpha,\beta,\mu;k,\sigma)}|z|^{2\mu-1} \tag{2.6}$$

The results (2.5) and (2.6) are sharp for the function $f(z)$ given by

$$f(z)=z-\frac{(X-Y)(1-\beta)}{\Upsilon_2(\alpha,\beta,\mu;k,\sigma)}z^{2\mu}. \tag{2.7}$$

Proof. Since $f\in k-T\mathcal{E}_g(X,Y;\alpha,\lambda;m)$, by Theorem (2.1),

$$\sum_{n=2}^{\infty}\left[\frac{2(k+1)}{1-\sigma}|\alpha n\mu-1|+|(Y+1)-[(1+X)+\beta(Y-X)\alpha n\mu|\right]$$

$$[1+\lambda(n\mu-1)]^m|\Gamma_n a_n| \le (X-Y)(1-\beta).$$

Now

$$\Upsilon_2(\alpha,\beta,\mu;k,\sigma)\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \Upsilon_2(\alpha,\beta,\mu;k,\sigma)a_n$$
$$\le \sum_{n=2}^{\infty} \Upsilon_{\mathrm{n}}(\alpha,\beta,\mu;k,\sigma)a_n$$
$$\le (X-Y)(1-\beta)$$

and therefore

$$\sum_{n=2}^{\infty} a_n \le \frac{(X-Y)(1-\beta)}{\Upsilon_2(\alpha,\beta,\mu;k,\sigma)}. \tag{2.8}$$

Since $F_\mu(z) = z - \sum_{n=2}^{\infty} a_n z^{n\mu}$,

$$|F_\mu(z)| \le |z| + |z|^{2\mu}\sum_{n=2}^{\infty} a_n \le |z| + \frac{(X-Y)(1-\beta)}{\Upsilon_2(\alpha,\beta,\mu;k,\sigma)}|z|^{2\mu}$$

which is the right hand side of (2.5). The left hand side of (2.5) can be proved using similar arguments.

Also from Theorem (2.1), we have

$$\frac{\Upsilon_2(\alpha,\beta,\mu;k,\sigma)}{2\mu}\sum_{n=2}^{\infty} n\mu a_n \le \sum_{n=2}^{\infty} \Upsilon_{\mathrm{n}}(\alpha,\beta,\mu;k,\sigma)a_n \le (X-Y)(1-\beta).$$

Thus,

$$|F_\mu{}'(z)| \le 1 + \sum_{n=2}^{\infty} n\mu \ a_n|z|^{n\mu-1} \le 1 + |z|^{2\mu-1}\sum_{n=2}^{\infty} n\mu a_n \le 1 + \frac{2(X-Y)(1-\beta)}{\Upsilon_2(\alpha,\beta,\mu;k,\sigma)}|z|^{2\mu-1}$$

On the other hand,

$$|F_\mu{}'(z)| \ge 1 - \sum_{n=2}^{\infty} n\mu \ a_n|z|^{n\mu-1} \ge 1 - |z|^{2\mu-1}\sum_{n=2}^{\infty} n\mu a_n \ge 1 - \frac{2(X-Y)(1-\beta)}{\Upsilon_2(\alpha,\beta,\mu;k,\sigma)}|z|^{2\mu-1}$$

This completes the proof.

Theorem 2.3.

If $F_\mu \in k - T\mathcal{E}_g(X,Y;\alpha,\lambda;m), and \ \ \alpha\mu > \frac{1}{2}$, then F_μ is convex of order ρ in $|z| < R$, where

$$R = \inf_{n\ge 2}\left[\frac{(1-\rho)\Upsilon_{\mathrm{n}}(\alpha,\beta,\mu;k,\sigma)}{n\mu(n\mu-\rho)(X-Y)(1-\beta)}\right]^{\frac{1}{n\mu-1}}. \tag{2.9}$$

Proof. By a computation, we have

$$\left|\frac{zF_\mu''(z)}{F_\mu'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n\mu(n\mu-1)a_n z^{n\mu-1}}{1-\sum_{n=2}^{\infty} n\mu a_n z^{n\mu-1}}\right| \le \frac{\sum_{n=2}^{\infty} n\mu|n\mu-1||a_n||z|^{n\mu-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n\mu-1}}.$$

Thus F_μ is convex of order ρ if

$$\sum_{n=2}^{\infty}\frac{n\mu|n\mu-\rho|}{1-\rho} \ \ a_n|z|^{n\mu-1} \le 1. \tag{2.10}$$

Since $f \in k - T\mathcal{E}_g(X,Y;\alpha,\lambda;m)$, we have

$$\sum_{n=2}^{\infty}\frac{\Upsilon_{\mathrm{n}}(\alpha,\beta,\mu;k,\sigma)}{(X-Y)(1-\beta)} \ \ a_n \le 1. \tag{2.11}$$

Now, (2.11) holds if

$$\frac{n\mu|n\mu-\rho|}{1-\rho}|z|^{n\mu-1} \le \frac{\Upsilon_{\mathrm{n}}(\alpha,\beta,\mu;k,\sigma)}{(X-Y)(1-\beta)},.$$

that is, if

$$|z| \le \left[\frac{(1-\rho)\Upsilon_{\mathrm{n}}(\alpha,\beta,\mu;k,\sigma)}{n\mu|n\mu-\rho|(X-Y)(1-\beta)}\right]^{\frac{1}{n\mu-1}} \tag{2.12}$$

which yields the desired result.

Remark 2.1. Though the class $k - \mathcal{E}_g(X,Y;\alpha,\lambda;m)$ is closely related to various subclasses of analytic functions, unfortunately we could not provide any special cases of our results.

3. Conclusions

In this paper, we have studied a new family of analytic functions of fractional powers with negative coefficients. It was challenging as it involved a function whose expansion had fractional powers. Further, we have discussed some geometrical and analytic properties of the function class in detail. Necessary and sufficient conditions, coefficient estimates, growth and distortion bounds have been obtained for the defined function class.

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