$$f(x) = \frac{\alpha\gamma}{1+e^{-x}}\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma} \quad , x,\ \alpha,\ \gamma > 0, \tag{3}$$

$$F(x) = 1 - \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma} \tag{4}$$

In this paper, we propose a new four-parameters extension of the generalized half-logistic distribution (GHLD) using the compounding technique to add two parameters from the Kumaraswamy distribution to the two-parameter GHLD. The new generalized distribution is named by Kumaraswamy-generalized half logistic (KW-GHL). First of all, we drive and prove the mathematical statistical properties of the KW-GHLD as moments, survival function, hazard function, incomplete moments, stochastic ordering and others. We implement such a distribution to fit some real data set. The primary goal of this paper is to propose a simpler and more flexible model, as well as to investigate new mathematical expressions to capitalize on some mathematical ideas such as algorithms, and computational techniques. Also we need some basic concepts like the Renyi entropy formula and the binomial expansion, and the negative series expansion, which are given by the following respectively.

$$h_{\vartheta} = \frac{1}{1-\vartheta}\log\int_0^{\infty} g^{\vartheta}(x) \quad dx, \text{ where } \vartheta > 0 \text{ and } \vartheta \neq 0 \tag{5}$$

$$(1-z)^n = \sum_{j=0}^{\infty}(-1)^j \binom{n}{j} z^j, \quad |z| < 1, n > 0, \tag{6}$$

$$(1+z)^{-n} = \sum_{j=0}^{\infty}(-1)^j \binom{n+j-1}{j} z^j \tag{7}$$

The paper is structured as follows: Section 2 provides an interpretation of the pdf and cdf for an observed distribution, including its hazard and survival rates, and derives other properties such as the moment, moment generating function, incomplete moments, Renyi entropy, stochastic ordering, probability weighted moments and explains some properties of Kw-GHL distribution such as quantile function, median survival time, and order statistics. In section 3, the greatest likelihood method is utilized to make estimates for the KW-GHL distribution's unknown parameters, and the application and flexibility of the model as compared to other models. The whole study is concluded in section 4.

2. Derivations and Properties

The cdf and pdf of (KW-GHLD) respectively of the random variable X can be obtained by substituting (4) into (1) then this yields a new cdf, that is.

$$G(x) = 1 - \left[1 - \left(1 - \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta} \quad 0 \leq x \leq \infty \quad , \quad \alpha, \gamma, \omega, \theta > 0 \tag{8}$$

By differentiating $G(x)$ with respect to x, we obtain the pdf of KW-GHLD as given in (9). Also, we can show other statistical properties like survival function, hazard rate function, and cumulative hazard rate function respectively.

$$g(x) = \frac{\alpha\gamma\theta\omega}{1+e^{-\alpha x}} \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma} \left(1 - \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega-1} \left[1 - \left(1 - \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta-1} \tag{9}$$

$$S(x) = \left[1 - \left(1 - \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta} \tag{10}$$

$$h(x) = \frac{\frac{\alpha\gamma\theta\omega}{1+e^{-\alpha x}} \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma} \left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega-1}}{1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right) \omega} \tag{11}$$

$$H(x) = -\theta \log\left[\ 1 - \left(1 - \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right] \tag{12}$$

The graphs for the KW-GHLD can be seen in Figure 1 and Figure 2. The figures below show that the graphs of KW-GHLD are approximately corresponding to the graphs of the original distribution GHLD.

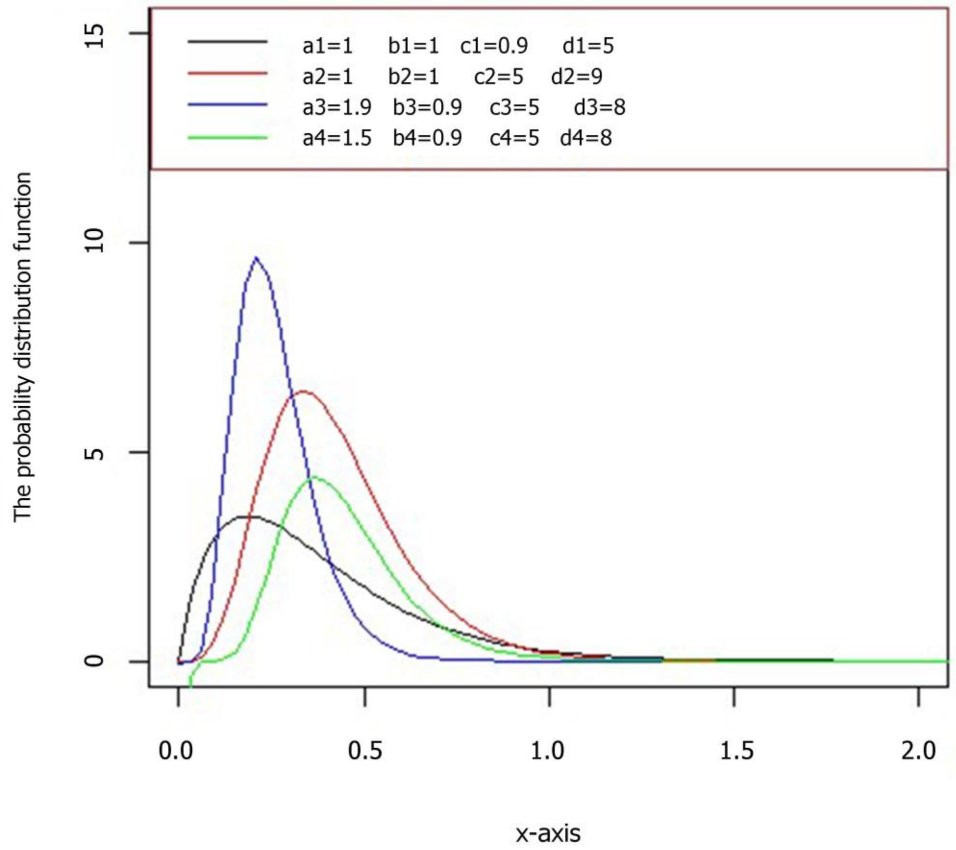

Figure 1. The pdf of KW-GHLD

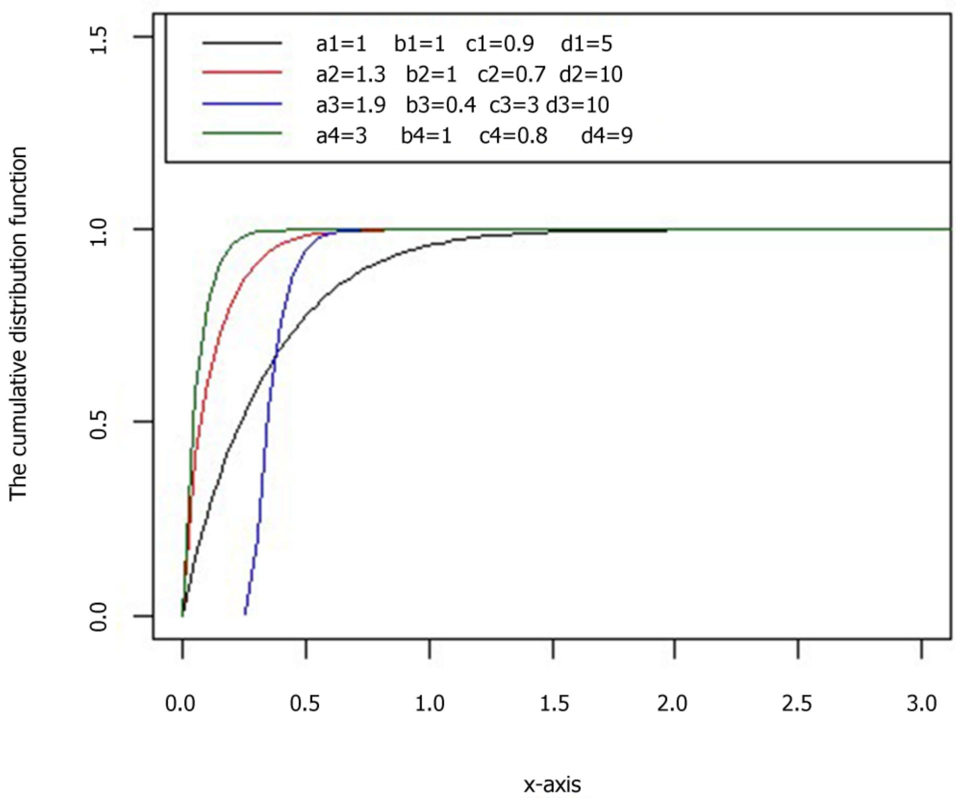

Figure 2. The cdf of KW-GHLD

2.1. Moments

Any probability distribution can be studied more effectively by using the moments. In this theorem, we derive the expression of the r^{th} moment for the KW-GHLD.

Theorem. 2.1 A random variable $\boldsymbol{X}$ from Kw-GHLD has the $\boldsymbol{r^{th}}$ moment as given in (13)

$$E(X^r) = \alpha\gamma\theta\omega \sum_{i,j,k=0}^{\infty} y_{i,j,k} \left(\frac{1}{\alpha(\gamma i+\gamma+k)}\right)^{r+1} \Gamma(r+1) \tag{13}$$

where

$$y_{i,j,k} = (-1)^{i+j}\, 2^{\gamma i+\gamma} \binom{\theta-1}{j}\binom{\omega j+\omega-1}{i}\binom{(\gamma i+\gamma+k)}{k} \tag{14}$$

Proof. the, $\boldsymbol{r^{th}}$ moment of the distribution function is **defined by**

$$E(X^r) = \int_0^{\infty} x^r \; g(x)\, dx$$

Thus

$$E(X^r) = \int_0^{\infty} \frac{\frac{x^r\gamma\alpha\theta\omega}{1+e^{-\alpha x}}\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}}{\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega-1}\left[1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta-1}} dx$$

By using the binomial series expansion in (6) and (7), we can express $E(X^r)$ as follows

$$E(X^r) = \alpha\gamma\theta\omega \int_0^{\infty} x^r \sum_{i,j,k=0}^{\infty} y_{i,j,k}\, e^{-\alpha(\gamma i+\gamma+k)x}\, dx$$

Let

$$\boldsymbol{p = x\alpha(\gamma i+\gamma+k)}$$

$$\boldsymbol{x = \frac{p}{\alpha(\gamma i+\gamma+k)}} \qquad \textbf{implies that} \qquad \boldsymbol{dx = \frac{1}{\alpha(\gamma i+\gamma+k)}\, dp}$$

$$\boldsymbol{E(X^r) = \alpha\gamma\theta\omega \int_0^{\infty} \sum_{i,j,k=0}^{\infty} y_{i,j,k} \left(\frac{p}{\alpha(\gamma i+\gamma+k)}\right)^{r} e^{-p} \frac{1}{\alpha(\gamma i+\gamma+k)} dp}$$

By solving the above integral, we obtain that

$$E(X^r) = \alpha\gamma\theta\omega \sum_{i,j,k=0}^{\infty} y_{i,j,k} \left(\frac{1}{\alpha(\gamma i+\gamma+k)}\right)^{r+1} \Gamma(r+1)$$

This completes the proof.

2.2. Moment Generating Function

The moment-generating function (mgf) is a helpful tool to derive some other statistical properties like moments. In the next theorem, we find the mgf of KW-GHLD.

Theorem. 2.2. If the random variable X follows KW-GHLD, then the moment generating function (mgf) of X is

$$M_x(t) = E(e^{tx}) = \sum_{i,j,k=0}^{\infty} y_{i,j,k} \frac{\alpha\theta\gamma\omega}{(\alpha(\gamma i+\gamma+k)-t)} e^{-(\alpha(\gamma i+\gamma+k)-t)}$$

Proof. A definition of (mgf) for X is

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \; g(x)\, dx$$

$$E(e^{tx}) = \int_0^{\infty} e^{tx} \frac{\alpha\gamma\theta\omega}{1+e^{-\alpha x}} \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega-1}\left[1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta-1} dx$$

By using "(14)", we get

$$E(e^{tx}) = \alpha\theta\gamma\omega \int_0^\infty e^{xt} \sum_{i,j,k=0}^{\infty} y_{i,j,k}\, e^{-(\alpha(\gamma i+\gamma+k)-t)}\, dx$$

$$E(e^{tx}) = \sum_{i,j,k=0}^{\infty} y_{i,j,k} \frac{\alpha\theta\gamma\omega}{(\alpha(\gamma i+\gamma+k)-t)} e^{-(\alpha(\gamma i+\gamma+k)-t)}$$

2.3. Incomplete Moments

Let 1_A be the indicator function of over some event A, where $1_A = 1$ if A is fulfill and $1_A = 0$ otherwise. Then, we have a definition for the $\boldsymbol{r^{th}}$ incomplete moment of X.

$$\boldsymbol{E\left(X^r 1_{\{X\le t\}}\right) = \int_0^t x^r \;\; g(x)\, dx}$$

From equation (9) and (14), we obtain

$$E\left(X^r 1_{\{X\le t\}}\right) = \alpha\gamma\theta\omega \int_0^t x^r \;\; \sum_{i,j,k=0}^{\infty} y_{i,j,k}\, e^{-\alpha(\gamma i+\gamma+k)x}\, dx$$

$$E\left(X^r 1_{\{X\le t\}}\right) = \alpha\gamma\theta\omega \sum_{i,j,k=0}^{\infty} y_{i,j,k}\, e^{-\alpha(\gamma i+\gamma+k)t} \frac{r!}{(\alpha(\gamma i+\gamma+k))^{r+1}}\left[1 - \sum_{m=0}^{r} \frac{(\alpha(\gamma i+\gamma+k))^m}{m!} t^m\right]$$

2.4. Renyi Entropy

The Renyi entropy is a measure of the variance of uncertainty. It is important in ecology and statistics as indicators of diversity.

Theorem. 2.4 A random variable X has the cdf and the pdf of KW-GHLD. Then the Renyi entropy of X can be defined as

$$h_\vartheta = \frac{1}{1-\vartheta} \log \sum_{i,j,k=0}^{\infty} z_{i,j,k} \frac{2^{\gamma j+\gamma\vartheta}(\gamma\alpha\theta)^\vartheta}{\alpha(k+\gamma j+\gamma\vartheta)} \;\; e^{-\alpha x \;\; (k+\gamma j+\gamma\vartheta)}$$

Proof. By recalling the Rényi entropy formula in equation (5), we obtain

$$h_\vartheta = \frac{1}{1-\vartheta} \;\; log \int_0^\infty \left(\frac{\alpha\gamma\theta\omega}{1+e^{-\alpha x}}\right)^\vartheta \;\; \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma\vartheta} \left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^\gamma\right)^{\vartheta(\omega-1)} \left[1 - \left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^\gamma\right)^\omega\right]^{\vartheta(\theta-1)} dx$$

By simplifying the expression $(1+e^{-\alpha x})^{-\vartheta(1+\gamma)}$, $\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^\gamma\right)^{\vartheta(\omega-1)}$ and $\left[1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^\gamma\right)^\omega\right]^{\vartheta(\theta-1)}$, with the binomial series expansion in (5) and (6) we obtain that

$$h_\vartheta = \frac{1}{1-\vartheta} log \int_0^\infty \sum_{i,j,k=0}^{\infty} z_{i,j,k}\, 2^{\gamma j+\gamma\vartheta}(\gamma\alpha\theta)^\vartheta \;\; e^{-\alpha x \;\; (k+\gamma j+\gamma\vartheta)} \;\; dx,$$

where, $z_{i,j,k} = (-1)^{i+j}\binom{\vartheta\;\;(\theta-1)}{j}\binom{wj+\vartheta(w-1)}{i}\binom{\gamma i+\vartheta(\gamma+1)+k-1}{k}$

$$h_\vartheta = \frac{1}{1-\vartheta} \log \sum_{i,j,k=0}^{\infty} z_{i,j,k} \frac{2^{\gamma j+\gamma\vartheta}(\gamma\alpha\theta)^\vartheta}{\alpha(k+\gamma j+\gamma\vartheta)} \;\; e^{-\alpha x \;\; (k+\gamma j+\gamma\vartheta)}$$

2.5. Quantile and Median

The quantile of a random variable X from KW-GHLD can be found by recalling equation (8). Then

$$1 - \left[1 - \left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^\gamma\right)^\omega\right]^\theta = q$$

$$\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}=1-[1-q]^{\frac{1}{\theta}}$$

$$x=\frac{1}{\alpha}log\left[2-\left[1-\left[1-[1-q]^{\frac{1}{\theta}}\right]^{\frac{1}{\omega}}\right]^{\frac{1}{\gamma}}\right]+\frac{1}{\gamma\alpha}\log\left[1-\left[1-[1-q]^{\frac{1}{\theta}}\right]^{\frac{1}{\omega}}\right] \tag{15}$$

By substituting $q=\frac{1}{2}$, we obtain

$$x_{0.5}=\frac{1}{\alpha}\log\left[2-\left(1-\left(1-2^{\frac{-1}{\theta}}\right)^{\frac{1}{\omega}}\right)^{\frac{1}{\gamma}}\right]-\frac{1}{\gamma\alpha}\log\left[1-\left(1-2^{\frac{-1}{\theta}}\right)^{\frac{1}{\omega}}\right] \tag{16}$$

2.6. Stochastic Ordering

Below is a theorem on the stochastic ordering involving the KW-GHLD with fixed parameters α,γ,ω and θ.

Theorem. 2.6 Let X and Y be random variable, with X having the pdf $g_1(x)$ provided by equation (9) with parameters (α,γ,ω and θ_1) and Y having the pdf $g_2(x)$ given by equation (9) with parameters (α,γ,ω and θ_2). If $\theta_1\leq\ \theta_2$ so, we have $X\ \leq_{lr} Y$.

Proof. We have

$$\frac{g_1(x)}{g_2(x)}=\frac{\theta_1}{\theta_2}\left[1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta_1-\theta_2}$$

Since $\theta_1\leq\theta_2$, we have

$$\frac{g_1(x)}{g_2(x)}\leq 1$$

Therefore, the ratio function $g_1(x)\ /g_2(x)$ is decreasing, implying that $\leq_{lr} Y$. This ends the proof of Theorem 2.4.

2.7. Probability Weighted Moments

Assuming the existence of the ordinary moments of a random variable, probability-weighted moments (PWM) are expectations of the functions of that variable. In addition to being useful as a starting point for maximum likelihood estimation (MLE), it is also useful in situations where MLE fails or is computationally challenging, such as when estimating the parameters of a distribution whose inverse cannot be represented clearly. The estimation based on PWM is frequently seen as being superior to the estimation using the standard moment. The PWM approach was first taken into consideration by [12]. PWM applications and specifics are covered in [13,14]. For a random variable, X the probability-weighted moments are defined by $\overline{\ \delta}_{r,s}=E[X^rG^s(X)]$, where $G(x)$ and $g(x)$ are the cdf and pdf of X. Now, we obtain the PWMs of the Kw-GHLD as follows.

$$\overline{\ \delta}_{r,s}=E[X^rG^s(X)],$$

$$\overline{\ \delta}_{r,s}=\int_0^{\infty}x^rG^s(x)g(x)dx$$

By using the binomial series expansion in (5) and (6), we get

$$\overline{\ \delta}_{r,s}=\alpha\gamma\theta\omega\int_0^{\infty}x^r\ \sum_{i,j,m,k=0}^{\infty}b_{i,j,s,k}\,e^{-\alpha(\gamma i+\gamma+k)x}\,dx$$

By solving the above integral we obtain

$$\overline{\ \delta}_{r,s}=\alpha\gamma\theta\omega\sum_{i,j,m,k=0}^{\infty}b_{i,j,k}\ \left(\frac{1}{\alpha(\gamma i+\gamma+k)}\right)^{r+1}\ \Gamma(r+1)$$

where

$$b_{i,j,\mathrm{m},k}=\ (-1)^{i+j+m}\ 2^{\gamma i+\gamma}\binom{s}{m}\binom{\theta(m+1)-1}{j}\binom{\omega j+\omega-1}{i}\binom{(\gamma i+\gamma+k)}{k}$$

2.8. Order Statistic of KW-GHLD

In statistic theory and application, order statistics make it important in many areas like statistical inference, and estimates of some unknown parameters, and it can be found as follows.

If $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denotes the order statistic of the random sample $X_1, X_2, \ldots, X_n$ from a continuous population with general forms is given as.

$$g_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}(G_X(x))^{k-1}(1-G_X(x))^{n-k} g_X(x),$$

for $k = 1,2, \ldots, n$ the pdf of the k^{th} order statistic for the KW-GHLD is given by

$$g_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!} \quad \frac{\alpha\gamma\theta\omega}{1+e^{-x}} \quad \left(\frac{2e^{-\alpha x_n}}{1+e^{-\alpha x_n}}\right)^{\gamma} \left[1-\left[1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta}\right]^{k-1}$$
$$\left(1-\left(\frac{2e^{-\alpha x_n}}{1+e^{-\alpha x_n}}\right)^{\gamma}\right)^{\omega-1} \quad \left[1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta(n-k+1)-1}$$

Therefore, the pdf of the smallest-order statistic $X_{(1)}$ is

$$g_{X_{(1)}}(x) = \quad \frac{n\alpha\gamma\theta\omega}{1+e^{-x}} \quad \left(\frac{2e^{-\alpha x_n}}{1+e^{-\alpha x_n}}\right)^{\gamma} \left(1-\left(\frac{2e^{-\alpha x_n}}{1+e^{-\alpha x_n}}\right)^{\gamma}\right)^{\omega-1} \quad \left[1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta n-1}$$

and the pdf of the largest-order statistic $X_{(n)}$ is

$$g_{X_{(n)}}(x) = \frac{n\alpha\gamma\theta\omega}{1+e^{-x}} \quad \left(\frac{2e^{-\alpha x_n}}{1+e^{-\alpha x_n}}\right)^{\gamma} \left[1-\left[1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta}\right]^{n-1} \left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega-1} \quad \left[1\right.$$
$$\left.-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\right]^{\theta-1}$$

It is clear that the pdf of the smallest order statistic or largest statistic can lead to many other statistical properties.

3. Results and Discussion

3.1. Estimation

In this part, we investigate the MLE's estimation of the parameters model of the generated distribution KW-GHLD. This method is preferred over other methods because it almost produces unbiased estimators. If $X_1, X_2, \ldots, X_n$ be a random sample from a KW- GHLD of size n, then the likelihood function of that KW- GHLD distribution is

$$L = \prod_{i=1}^{n} g(x; \alpha, \gamma, \theta, \omega) \tag{17}$$

By substituting the pdf (9) into "(16)", we obtain the likelihood function, that is

$$L \ = \frac{\alpha^n \gamma^n \theta^n \omega^n}{\prod_{i=1}^{n}(1+e^{-\alpha x_i})} \prod_{i=1}^{n}\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma} \prod_{i=1}^{n} \left(1-\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}\right)^{\omega-1} \quad \prod_{i=1}^{n} \quad \left[1\right.$$
$$\left.-\left(1-\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}\right)^{\omega}\right]^{\theta-1}$$

Then, the log-likelihood function is

$$\ell = \log L(x; \alpha, \gamma, \theta, \omega)$$

$$\ell = n(\log\alpha + log\,\gamma + log\,\theta + log\,\omega) - \sum_{i=1}^{n} log(1+e^{-\alpha x_i}) + \gamma\sum_{i=1}^{n}\log\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right) + \quad (\omega-1)\sum_{i=1}^{n} log\left(1-\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}\right) + (\theta-1)\sum_{i=1}^{n} log\left(1-\left(1-\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}\right)^{\omega}\right) \tag{18}$$

The estimation equations are obtained by differentiating (18) with respect to the desire parameters (α, γ, θ and ω) and it equals the obtained equations to zero, we have.

$$\frac{\partial \ell}{\partial \alpha}=\frac{n}{\alpha}+(1+\gamma)\sum_{i=1}^{n}\left(\frac{x_i e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)-\gamma\sum_{i=1}^{n}x_i+\alpha\gamma(\omega-1)\sum_{i=1}^{n}\frac{\frac{1}{1+e^{-\alpha x_i}}\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}}{1-\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}}-(\theta-1)\omega\alpha\gamma\sum_{i=1}^{n}\frac{\frac{1}{1+e^{-\alpha x_i}}\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega-1}}{1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}}=0 \quad (19)$$

$$\frac{\partial \ell}{\partial \gamma}=\frac{n}{\gamma}+\log 2-\sum_{i=1}^{n}\alpha x_i-\sum_{i=1}^{n}\log(1+e^{-\alpha x_i})-(\omega-1)\sum_{i=1}^{n}\frac{\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}\log\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)}{1-\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}}+(\theta-1)\omega\sum_{i=1}^{n}\frac{log\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega-1}}{1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}}=0 \quad (20)$$

$$\frac{\partial \ell}{\partial \theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log 1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}=0 \quad (21)$$

$$\frac{\partial \boldsymbol{\ell}}{\partial \boldsymbol{\omega}}=\frac{n}{\omega}+\sum_{i=1}^{n}\log\left(1-\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}\right)-(\theta-1)\sum_{i=1}^{n}\frac{1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}\log\left(1-\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\gamma}\right)}{1-\left(1-\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}\right)^{\omega}}=0 \quad (22)$$

The maximum likelihood estimator for the unknown parameters cannot be solved in equations (19), (20), (21), and (22) because the derived equations are of complex forms. To maximize the aforementioned log- likelihood function, it is practical to use the nonlinear Newton-Raphson algorithm for solving the above equations, and obtain the estimated values of the parameters $(\alpha,\gamma,\theta \text{ and } \omega)$.

3.2. Applications

Here, we test the following data

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1.00, 1.00, 1.02, 1.05, 1.07, 0.7, 0.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.20, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.60, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.30, 2.31, 2.40, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

The data show how long 72 Guinea pigs survived after being contaminated with a dangerous bacterial tubercle. The data set was collected by Usman, Hag, and Talib (2017). We use the above data to substantiate the proposed model by comparison with GHLD. The model parameters of the MLEs are shown in Table 1. The programming language R was used to implement the Newton-Raphson algorithm, which solved the MLE problem. Table 2 shows that KW-GHLD is the best-fit model for data collection using the test statistic: AIC, BIC, and CAIC, demonstrating that the information criteria show how well the model is at explaining the relationship between variables. When compared to the other model, the statistic produces the lowest values.

Table 1. MLEs of the model parameters.

Model	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\omega}$	$\hat{\theta}$
GHLD	1.3687	0.5398		
KW-GHLD	2.0008	0.1825	1.7732	3.8358
Std. Error	1.3382	0.2691	0.3486	5.6132

Note that the standard error in Table (1) represent the standard error value of the parameters of KW-GHLD

Table 2. The AIC, BIC, and CAIC values for the set data.

Model	$\ell\hat{}$	AIC	BIC	CAIC	HQIC
GHLD	-104.7106	213.4211	217.9744	213.595	215.2338
KW-GHLD	-96.5062	201.0124	210.1191	201.6094	204.6378

4. Conclusions

The study introduces a new expansion of generalized half-logistic distribution named Kumaraswamy generalized half logistic distribution KW_GHLD and expresses its mathematical statistical properties. Also, we estimated the parameters (α, γ, θ and ω) of the desired distribution by using the maximum likelihood method. Furthermore, compare the KW-GHLD with the original distribution (GHLD). Note that the KW-GHLD is the best fit. Finally, real applications complete data used to achieve the desired goal.

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