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Performance Analysis of a Markovian Model for Two Heterogeneous Servers Accompanied by Retrial, Impatience, Vacation and Additional Server

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Abstract The two heterogeneous servers of the Markovian retrial queue model with an additional server, impatience behavior and vacation are presented in this research paper. An arriving customer who finds accessible servers gets immediate service. Otherwise, if both servers are engaged, an entering customer will join in the orbit to retry and get their service after some random time. If any customers in the orbit discover that the waiting time is longer than expected, they may leave without receiving the service. We consider two servers with different service rates to provide the service based on “First Come, First Served”. When the number of customers in orbit increases occasionally, we will instantly activate an additional server to reduce the queue size. After the orbit becomes null, the server goes for maintenance activity. The practical application is given to justify our model. The proposed model was obtained using the birth-death process and the equations were governed using Chapman-Kolmogorov equations. Finally, we have solved the equations using a recursive approach and the performance indices are derived to improve quality and efficiency.

Keywords Two Heterogeneous Servers, Impatience, Retrial Queue, Vacation, Additional Server

1. Introduction

This paper investigates a model of two-heterogeneous server retrial queue accompanied by additional server, impatience and vacation, which has fascinated many researchers. The domains of computer networks, commerce, manufacturing and service systems heavily rely on this queueing approach. We have all experienced waiting in line to satisfy our demands in various situations, known as the queuing system.

The blocking is done in real-time, for typical strategic behavior of the service provider. During the arrival, if a likely customer expects a long waiting time in the queue, the customer is free to choose whether or not to join the system to receive service. For example, the patients are waiting in the line to get their treatment. If any of the patients, on arriving observes a long queue, may refuse to join in the queue and this situation is termed as Balking.

In a retrial queue, a customer entering the system seeking service, finds all the servers being engaged and proceeds a customer to the waiting area is known as retrial orbit. On where the customers check the availability of the servers until the server is free. Also, if any customer in the queue decides to leave the orbit without receiving the service presuming that their wait will be longer, this is referred to as reneging. For example in telephone systems, the auto-repeat facilities allow the client to automatically redial a busy telephone until a connection is made.

Accordingly, we are considering two heterogeneous servers with different service rates to provide service. Once their expected service is provided, the arriving customers will exit from the system. If the number of customers in the system increases during an occasional period, we will activate an additional server until it bounces back to the normal situation.

During this period, we activate the third server in order to not to lose the probability of any requests to the server. The purpose of the additional server is to avoid the huge queue and reduce their waiting time in the system. During the completion of service epoch, if servers find that there is no customer in the orbit then it takes vacation. At the time of vacation, the server may perform maintenance activity and commit basic prerequisite work to improve service efficiency.

2. Literature Survey

Many researchers have conducted a literature review on queueing theory. One among them is two heterogeneous servers with different rates at the same time of service. The retrial behaviour of arriving customers has been extensively studied in recent decades. Developing a Markovian model for stationary distribution using the Laplace Steiltjes transform by Arivudainambi et al. [1]. Also they have encouraged a feedback system for the unsatisfied customer to retry for their service again.

Arivudainambi and Godhandaraman [2] introduced few compulsory phases of service for customers. As soon as the system becomes null, server leaves for a vacation. The equations for the steady state distributions and the number of customers in the systems are solved. On the other hand, few performance measures are performed on obtaining the results.

The concept of service facilities operating under different service rates is studied effectively. Bouchentouf and Messabihi [3] investigated the model with the two servers of same service with different speed rates. Economic analysis and effectiveness are obtained using steady state measures. Where the reverse balking and retention of customers establishes to improve the efficiency and bring out effectiveness.

This model is all about understanding the effect of impatience and feedbacks from the customer. When the server is unreliable, to improve the service facility, they come out with customer feedback to satisfy their demand. They analyze the model by the Quasi-Birth death model to bring out the rate matrix. We refer to the interested reader in Chang et al. [4] who used Nelder –Mead tool to direct the tasks. They provided analytical results in numerical form as well as their practical application.

The single server arrival of poisson process and the service time is determined by exponential distribution in which retrial queue Geo and Zhang [5] has focused the application in healthcare service centre. The customers who were arriving may enter the group of retrial or may leave. The stationary equations are derived using the generating functions technique and the sojourn times were also calculated. They have also obtained few performance measures of the system which are illustrated numerically.

In this paper, they have adopted the power of diversion when choosing the servers for heterogeneous systems where the service is homogeneous. Gardner et al. [6] introduce the heterogeneity dispatching rules and derived their achievements using stability conditions. The policy they framed is tractable and approaches the server to infinity. Further they also noticed the queue length and their response time to access the system.

The supplementary variable technique is carried out by Jain and Sanga [7] to solve the steady state analysis for the model. Introducing the concept of additional server with finite capacity. The main motto of this additional server is to reduce the queue in the peak hours. The threshold has been fixed and allowed the customers until the finite capacity in the system. Adaptive neuro fuzzy inference system approach is applied in the model. Cost analysis which may help to bring the minimum expected cost.

This model further developed with probability generating functions. Here considered M/M/1 model Sampath and Liu [8] with impatience and vacation. They come out with a new idea of differentiated server vacation with the main role of impatience behaviour to improve the quality. The system which is dependent on time was obtained using continued fractions and hypergeometric function. They mainly focused on the variations of vacations with the customers behaviour. Finally they analysed with the numerical results.

Verdonck et al. [9] established the retrial queueing system in discrete times with an occasional server. The server states were divided among Up and Down periods to provide service for the arriving customers. Their processing times were assumed to be general as well as geometric distributions. Further, the results were obtained using a rational probability generating function. Also the vacation part takes place, when there is no customer in the orbit.

In this paper, Wu and Yang [10] analyzed using matrix geometric method. The main objective of the study is to reduce the cost per unit time with the help of algorithm. They obtained a solution for the cost and customers wait time. The authors appreciated the model with the real-time application. Though they analyzed the performance indices with the sensitivity analysis and cost function.

3. Practical Application

The below Figure 1 model illustrates the practical application of printing T-Shirts scenarios. We consider two servers, server-1 is a fast server and server-2 is a slow server of two different digital printing machines with different speed rates, serving at the same time.

Whenever the T-Shirt order comes, it looks upon the printer whether it is free or allocated. If it is free then the printer takes up T-shirt for printing. If fastest digital printer is busy, then the other printer will take up service as printing T-Shirt and vice versa.

If both printers are busy, then the T-Shirts will be sent to the waiting area called orbit to retry to use the digital printer. From the waiting area, if we look upon the queue whether it takes longer time to wait, by then the client will wind up and cancel the order at anytime.

Therefore the order for T-Shirts printing can be accepted only, if once the digital printer gets free. If the printing orders reach more than the normal level in that situation, they will introduce additional servers in order to reduce the queue and in quality of service.

Meanwhile, the additional server gets inactivated once the order gets back to normal level. After serving all the orders, if the machine finds there are no orders in the orbit then it will go for a period of time called vacation. During vacations, it undergoes activities like maintaining machine, servicing gear chain, applying oil in machine daily, in order to run without breakdown.

Figure 1. T-Shirt Printing Model

4. Model Description

Consider the Markovian model for two servers of M/M/2 retrial queue for the attainment of service, impatient behavior, vacation and additional server. Considering two heterogeneous servers with limitless capacity, where it performs duties on a single customer at a time based on first come first serve service pattern.

The arrival of customers follows Poisson fashion with rate λ. The service time follows exponential distribution with rate μ_1 for faster server-1, μ_2 for slower server-2 and μ_3 for additional server. Upon arrival of customers, if finding any of the servers is null, then the customer gets the service on the available server and leaves the system.

The retrial times are exponentially distributed with rate γ. If both the server is busy then the customer will join the retrial orbit with a probability q and in the case of balking probability $1-q$. After some time, the customers in the orbit will retry one by one to get their turn for service with probability σ and the customer with reneging probability $1-\sigma$. When both servers are busy, seems the queue is long and reaches the limit i.

Here they activates additional server from $(i+1)$ customers until it bounces back to normal queue length i. If the orbit becomes empty after serving all the customers in the queue, then the server activates a vacation for a random duration. Once the vacation gets completed, the server will be activated at a rate θ and waiting for the customer to give the service.

4.1. Service Discipline

The case initiates the discipline, whereas the customers enter into the system are following:

1. When both servers are idle, incoming customer chooses the faster server.
2. When faster server is busy and slower server is idle, then incoming customer will choose the slow server and vice versa.
3. When both servers are busy, then the customer will join the orbit to retry the service from any of the servers, depending on who completes the service first.

5. Analysis of the Model

The mathematical formulation of the constructed model is stated below and shown in figure 2:

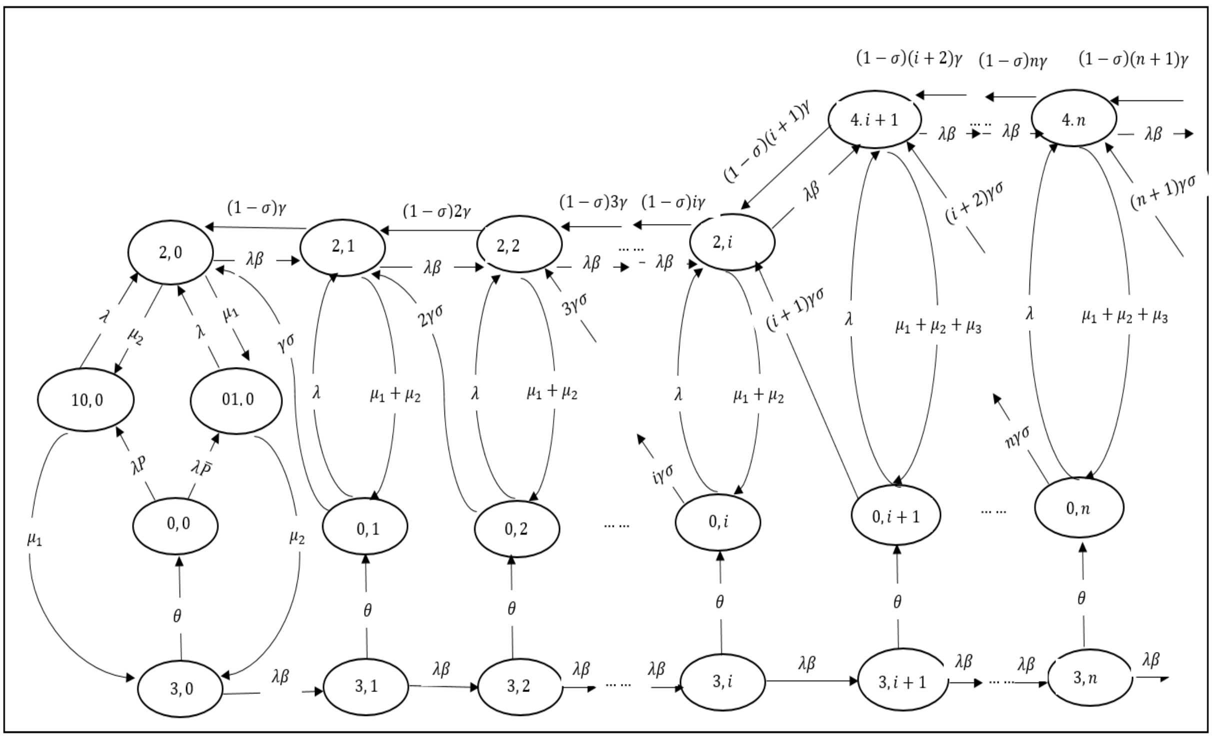

Figure 2. State Transition Diagram

Let $N(t)$ be the total number of customers in the system at any time t and let $C(t)$ specify the server's current state at time t, which is as follows:

$$C(t) = \begin{cases} 0, & \text{when the servers are free} \\ 1, & \text{when either one of the server is busy} \\ 2, & \text{when both the servers are busy} \\ 3, & \text{when both the servers activate a vacation} \\ 4. & \text{when the servers activate an extra server} \end{cases}$$

Then $\{C(t), N(t): t \geq 0\}$ is a Markovian bivariate with discrete state space.

Let $P_{x,n}(t) = Prob\{C(t) = x, N(t) = n\}$ be the probability that there will be n customers in the system. When x takes values 0, 1, 2, 3 and 4.

$P_{x,n}(t) = \lim_{t\to\infty} P_{x,n}(t)$ represents the steady state probability

5.1. Notations

$\lambda =$ Arrival rate
$\mu_1 =$ Service rate for server-1
$\mu_2 =$ Service rate for server-2
$\mu_3 =$ Service rate for server-3
$\beta =$ Customers joining in the system
$\gamma =$ Retrial rate
$\sigma =$ Customers joining in the retrial orbit
$1 - \sigma =$ Reneging probability
$\theta =$ Vacation rate
$N(t) =$ Number of customers in the system
$C(t) =$ Status of the servers

The following probabilities are described in steady state:

- $P_{0,n} =$ The probability when the server is idle and there are n customers in the orbit.
- $P_{1,n} =$ The probability that one server is active at a busy moment when there are n customers in the orbit.
- $P_{2,n} =$ The probability that both servers are active at a busy moment when there are n customers in the orbit.
- $P_{3,n} =$ The probability that servers on a vacation when there are n customers in the orbit.
- $P_{4,n} =$ The probability that additional server is active on an occasional period when there are n customers in the orbit.

6. Stationary Queue Size Distribution

Chapman-Kolmogorov equation has been framed for the steady state space probabilities by following the argument of probability and flow of the Markov chain model balance rule. For the following modes of the server, governing equations of Chapman-Kolmogorov difference equations are considered. The state $x = 0$ when the servers are free, $x = 1$ when either of the server is in busy mode, $x = 2$ when both the servers are in busy mode, $x = 3$ when both the server activates for a vacation mode and $x = 4$ when the servers activates additional server which is shown in figure 2.

$$(\lambda p + \lambda\bar{p})P_{0,0} = \theta P_{3,0}; \quad n = 0 \tag{1}$$

$$(\lambda + n\gamma\sigma)P_{0,n} = (\mu_1 + \mu_2)P_{2,n} + \theta P_{3,n}; 1 \leq n \leq i \tag{2}$$

$$(\lambda + n\gamma\sigma)P_{0,n} = (\mu_1 + \mu_2 + \mu_3)P_{4,n} + \theta P_{3,n}; \quad n \geq (i+1) \tag{3}$$

$$(\lambda + \mu_1)P_{10,0} = \lambda p P_{0,0} + \mu_2 P_{2,0}; \quad n = 0 \tag{4}$$

$$(\lambda + \mu_2)P_{01,0} = \lambda\bar{p}P_{0,0} + \mu_1 P_{2,0}; n = 0 \tag{5}$$

$$(\lambda\beta + \mu_1 + \mu_2)P_{2,0} = \lambda P_{01,0} + \lambda P_{10,0} \quad + (1-\sigma)\gamma P_{2,1} + \gamma\sigma P_{0,1} \tag{6}$$

$$(\lambda\beta + (1-\sigma)n\gamma + \mu_1 + \mu_2)P_{2,n} = (1-\sigma)(n+1)\gamma P_{2,n+1} + \lambda P_{0,n} + \lambda\beta P_{2,n-1} + (n+1)\gamma\sigma P_{0,n+1}; \quad (1 \leq n \leq i) \tag{7}$$

$$(\lambda\beta + \theta)P_{3,0} = (\mu_1 + \mu_2)P_{1,0}; \quad n = 0 \tag{8}$$

$$(\lambda\beta + \theta)P_{3,n} = \lambda\beta P_{3,n-1}; \quad n \geq 1 \tag{9}$$

$$\big(\lambda\beta + (1-\sigma)(i+1)\gamma + (\mu_1 + \mu_2 + \mu_3)\big)P_{4,i+1} = \lambda\beta P_{2,i} + (1-\sigma)(i+2)\gamma P_{4,i+2} + \lambda P_{0,i+1} + (i+2)\gamma\sigma P_{0,i+2} \tag{10}$$

$$\big(\lambda\beta + (1-\sigma)n\gamma + (\mu_1 + \mu_2 + \mu_3)\big)P_{4,n} = \lambda\beta P_{4,n-1} + (n+1)\gamma\sigma P_{0,n+1} + (1-\sigma)(n+1)\gamma P_{4,n+1} + \lambda P_{0,n} \tag{11}$$

To find steady state probabilities explicitly defined in terms of $P_{0,0}$, we employ the recursive method. The normalising condition is also used to locate the $P_{0,0}$ term.

We obtain for the vacation state, when no customer in the orbit

$$P_{3,0} = \frac{\lambda(p+\bar{p})}{\theta} P_{0,0} \tag{12}$$

where $p + \bar{p} = 1$

If the system is in vacation state, when the first customer is arriving for service

$$P_{3,1} = \left[\frac{\lambda^2\beta}{\theta(\lambda\beta+\theta)}\right] P_{0,0} \tag{13}$$

While if the system is in vacation state and second customer is arriving for service

$$P_{3,2} = \left[\frac{\lambda^3\beta^2}{\theta(\lambda\beta+\theta)^2}\right] P_{0,0} \tag{14}$$

Now for the server is in vacation state and the customer is waiting in the queue for the third customer

$$P_{3,3} = \left[\frac{\lambda^4\beta^3}{\theta(\lambda\beta+\theta)^3}\right] \tag{15}$$

For the customer of n-1 waiting during the vacation state

$$P_{3,n-1} = \left[\frac{\lambda^n\beta^{n-1}}{\theta(\lambda\beta+\theta)^{n-1}}\right] P_{0,0} \tag{16}$$

Now finding for n number of customer during the vacation state

$$P_{3,n} = \left[\frac{\lambda^{n+1}\beta^n}{\theta(\lambda\beta+\theta)^n}\right] P_{0,0} \tag{17}$$

When there is no customer in the system and either of the server is busy then we find

$$P_{10,0} = \left[\frac{\mu_2\zeta_0+\lambda p\kappa}{\kappa(\mu_1+\lambda)}\right] P_{0,0} \tag{18}$$

When there is no customer in the system and either of the server is busy then we find

$$P_{01,0} = \left[\frac{\mu_1\zeta_0+\lambda\bar{p}\kappa}{\kappa(\mu_2+\lambda)}\right] P_{0,0} \tag{19}$$

When both the servers are busy and finding no customer in the orbit, then

$$P_{2,0} = \left[\frac{\lambda(\theta+\lambda\beta)(\lambda+\mu_1)(\lambda+\mu_2)-\lambda\theta(\mu_1 p+\mu_2\bar{p})(\lambda+\mu_1)(\lambda+\mu_2)}{\theta(\mu_1+\mu_2)^2}\right] P_{0,0}$$

$$P_{2,0} = \left[\frac{\zeta_0}{\kappa}\right] P_{0,0} \tag{20}$$

where $\zeta_0 = \lambda(\theta+\lambda\beta)(\lambda+\mu_1)(\lambda+\mu_2) -\lambda\theta(\mu_1 p+\mu_2\bar{p})(\lambda+\mu_1)(\lambda+\mu_2)$

and $\kappa = \theta\mu_1\mu_2(2\lambda+\mu_1+\mu_2)$

When first customer is arriving in the orbit, when both the servers are busy

$$P_{2,1} = \left[\frac{\zeta_1}{\kappa^2\xi_1}\right] P_{0,0} \tag{21}$$

where $\zeta_1 = [\zeta_0(\lambda\beta+\mu_1+\mu_2)(\lambda\beta+\theta)(\lambda+\gamma\sigma)^2 (\lambda+\mu_1)(\lambda+\mu_2) - \lambda^2\beta\kappa(\lambda+\mu_1)(\lambda+\mu_2) (\lambda+\gamma\sigma) - \lambda\mu_1\zeta_0(\lambda\beta+\theta)(\lambda+\gamma\sigma)^2(\lambda+\mu_1) -\lambda^2\bar{p}\kappa(\lambda\beta+\theta)(\lambda+\gamma\sigma)^2(\lambda+\mu_1) -\lambda\mu_2\zeta_0(\lambda\beta+\theta)(\lambda+\gamma\sigma)^2(\lambda+\mu_2) -\lambda^2 p\kappa(\lambda\beta+\theta)(\lambda+\gamma\sigma)^2(\lambda+\mu_2)$

and $\kappa = (\lambda+\mu_1)(\lambda+\mu_2)$

$\xi_1 = (\lambda\beta+\theta)(\lambda+\gamma\sigma)[\gamma\sigma(\mu_1+\mu_2) +\gamma(1-\sigma)(\lambda+\gamma\sigma)]$

Obtaining the value for the number of customers in the orbit state

$$P_{0,1} = \left[\frac{(\mu_1+\mu_2)(\lambda\beta+\theta)\zeta_1+\lambda^2\beta\kappa^2\xi_1}{\kappa^2\xi_1(\lambda\beta+\theta)(\lambda+\sigma\gamma)}\right] P_{0,0} \tag{22}$$

When both the servers are busy and serving for the second customer

$$P_{2,2} = \left[\frac{\zeta_2}{\kappa^3\xi_1\xi_2}\right] P_{0,0} \tag{23}$$

Where $\zeta_2 = [\zeta_1(\lambda\beta+\mu_1+\mu_2+(1-\sigma)\gamma)(\lambda\beta+\theta)^2 (\lambda+\gamma\sigma)(\lambda+2\gamma\sigma)^2 - 2\lambda^3\beta^2\gamma\sigma\kappa^2\xi_1 (\lambda+\gamma\sigma)(\lambda+2\gamma\sigma) - \lambda\zeta_1(\mu_1+\mu_2) (\lambda\beta+\theta)^2(\lambda+2\gamma\sigma)^2 - \lambda^3\beta\kappa^2\zeta_1\theta (\lambda\beta+\theta)(\lambda+2\gamma\sigma)^2 - \lambda\zeta_0\beta\kappa\xi_1 (\lambda\beta+\theta)^2(\lambda+\gamma\sigma)(\lambda+2\gamma\sigma)^2$

$\kappa = (\lambda+\gamma\sigma)$

$\xi_2 = (\lambda\beta+\theta)^2(\lambda+2\gamma\sigma)[2\gamma(1-\sigma)(\lambda+2\gamma\sigma) +2\gamma\sigma(\mu_1+\mu_2)]$

When both the servers are busy and serving for the third customer

$$P_{2,3} = \left[\frac{\zeta_3}{\kappa^4\xi_1\xi_2\xi_3}\right] P_{0,0} \tag{24}$$

Where $\zeta_3 = [\zeta_2(\lambda\beta+\mu_1+\mu_2+(1-\sigma)2\gamma)(\lambda\beta+\theta)^3 (\lambda+2\gamma\sigma)(\lambda+3\gamma\sigma)^2 - 3\lambda^4\beta^3\gamma\sigma\kappa^3\xi_1\xi_2 (\lambda+2\gamma\sigma)(\lambda+3\gamma\sigma) - \lambda\zeta_2(\mu_1+\mu_2) (\lambda\beta+\theta)^3(\lambda+3\gamma\sigma)^2 - \lambda^4\beta^2\kappa^3\xi_1\xi_2(\lambda\beta+\theta) (\lambda+3\gamma\sigma)^2 - \lambda\zeta_1\beta\kappa\xi_2(\lambda\beta+\theta)^3 (\lambda+2\gamma\sigma)(\lambda+3\gamma\sigma)^2$

$\kappa = (\lambda+2\gamma\sigma)$

$\xi_3 = (\lambda\beta+\theta)^3(\lambda+3\gamma\sigma)[3\gamma(1-\sigma)(\lambda+3\gamma\sigma) +3\gamma\sigma(\mu_1+\mu_2)]$

For the number of customer in the orbit

$$P_{0,2} = \left[\frac{(\mu_1+\mu_2)(\lambda\beta+\theta)^2\zeta_2+\lambda^3\beta^2\kappa^3\xi_1\xi_2}{\kappa^3\xi_1\xi_2(\lambda\beta+\theta)^2(\lambda+2\sigma\gamma)}\right] P_{0,0} \tag{25}$$

For the third number of customer in the orbit

$$P_{0,3} = \left[\frac{(\mu_1+\mu_2)(\lambda\beta+\theta)^3\zeta_3+\lambda^4\beta^3\kappa^4\xi_1\xi_2\xi_3}{\kappa^4\xi_1\xi_2\xi_3(\lambda\beta+\theta)^3(\lambda+3\sigma\gamma)}\right] P_{0,0} \tag{26}$$

In general, we obtain

$$P_{0,n} = \left[\frac{(\mu_1+\mu_2)(\lambda\beta+\theta)^n\zeta_n+\lambda^{(n+1)}\beta^n\kappa\prod_{(n=1)}^{n}\xi_n}{\kappa^n\prod_{(n=1)}^{i}\xi_n(\lambda+n\gamma\sigma)(\lambda\beta+\theta)^n}\right]; 1 \le n \le I \tag{27}$$

$$P_{0,n} = \left[\frac{\zeta_n(\mu_1+\mu_2+\mu_3)(\lambda\beta+\theta)^n+\lambda^{n+1}\beta^n\kappa^{n+1}\prod_{n=1}^{i}\xi_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n(\lambda\beta+\theta)^n(\lambda+n\gamma\sigma)}\right];$$

$$n \ge i+1 \tag{28}$$

$$P_{2,n} = \zeta_n\prod_{n=2}^{n}\left[\frac{1}{\kappa^{n+1}\prod_{(n=1)}^{n}\xi_n}\right] P_{0,0}; 2 \le n \le i \tag{29}$$

$$P_{3,n} = \left[\frac{\lambda^{n+1}\beta^n}{\theta(\lambda\beta+\theta)^n}\right] P_{0,0};\ n \ge 0 \tag{30}$$

$$P_{4,n} = \psi_n\prod_{n=0}^{(\infty)}\left[\frac{1}{\kappa^{n+1}\xi_n}\right] P_{0,0}; n \ge i+1 \tag{31}$$

To obtain the value of $P_{0,0}$, the normalizing condition is given by

$$\sum_{x=0}^{4}\sum_{n=0}^{\infty} P_{x,n} = 1 \tag{32}$$

Subsequently on operating algebraic, we get

$$P_{0,0} = \frac{1}{1+B+V+W_n+X+Y+Z+\sum_{n=1}^{i}[P]+\sum_{n=2}^{i}[R_n]+\sum_{n=1}^{\infty}[S] + \sum_{n=i+1}^{\infty}[T]+\sum_{n=i+2}^{\infty}[U_n]} \tag{33}$$

where $B = \zeta_n \left[\frac{\zeta_{i+1}}{\kappa^{i+2}\prod_{n=1}^{i+1}\xi_n}\right]$; $V = \left[\frac{\lambda}{\theta}\right]$;

$W_n = \left[\frac{\mu_1\zeta_0+\lambda\bar{p}\kappa}{\kappa(\mu_2+\lambda)}\right]$; $X = \left[\frac{\mu_2\zeta_0+\lambda p\kappa}{\kappa(\mu_1+\lambda)}\right]$;

$Y = \left[\frac{\zeta_0}{\kappa}\right]$; $Z = \left[\frac{\zeta_1}{\kappa^2\xi_1}\right]$;

$R_n = \left[\frac{\zeta_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n}\right]$; $S = \left[\frac{\lambda^{n+1}\beta^n}{\theta(\lambda\beta+\theta)^n}\right]$;

$U_n = \left[\frac{\zeta_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n}\right]$;

$P = \left[\frac{(\mu_1+\mu_2)(\lambda\beta+\theta)^n\zeta_n+\lambda^{(n+1)}\beta^n\kappa\prod_{(n=1)}^{n}\xi_n}{\kappa^n\prod_{(n=1)}^{i}\xi_n(\lambda+n\gamma\sigma)(\lambda\beta+\theta)^n}\right]$

$T = \left[\frac{\zeta_n(\mu_1+\mu_2+\mu_3)(\lambda\beta+\theta)^n+\lambda^{n+1}\beta^n\kappa^{n+1}\prod_{n=1}^{i}\xi_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n(\lambda\beta+\theta)^n(\lambda+n\gamma\sigma)}\right]$

7. Performance Measures

We investigate the system behaviour in this way to achieve the various performance indices. The following provides the long-term probabilities of the servers various states.

For the state when the server is free:

$$P_A = \left[\frac{1}{1+B+V+W_n+X+Y+Z+\sum_{n=1}^{i}[P]+\sum_{n=2}^{i}[R_n] + \sum_{n=1}^{\infty}[S]+\sum_{n=i+1}^{\infty}[T]+\sum_{n=i+2}^{\infty}[U_n]}\right]\left[\frac{1+\sum_{n=1}^{i}\zeta_n(\mu_1+\mu_2)(\lambda\beta+\theta)^n+\lambda^{n+1}\beta^n\kappa^{n+1}\prod_{n=1}^{i}\xi_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n(\lambda\beta+\theta)^n(\lambda+n\gamma\sigma)}\right] \tag{35}$$

For the state when either of the server is busy:

$$P_B = \left[\frac{1}{1+B+V+W_n+X+Y+Z+\sum_{n=1}^{i}[P]+\sum_{n=2}^{i}[R_n] + \sum_{n=1}^{\infty}[S]+\sum_{n=i+1}^{\infty}[T]+\sum_{n=i+2}^{\infty}[U_n]}\right]\left[\frac{\lambda(\lambda\beta+\theta)}{\theta(\mu_1+\mu_2)}\right] \tag{36}$$

For the state when both the server is busy:

$$P_C = \left[\frac{1}{1+B+V+W_n+X+Y+Z+\sum_{n=1}^{i}[P]+\sum_{n=2}^{i}[R_n] + \sum_{n=1}^{\infty}[S]+\sum_{n=i+1}^{\infty}[T]+\sum_{n=i+2}^{\infty}[U_n]}\right]\left[\frac{\zeta_0}{\kappa}+\sum_{n=1}^{i}\zeta_n\prod_{n=0}^{i}\frac{1}{\kappa^{n+1}\xi_n}\right] \tag{37}$$

For the state when the server is in vacation:

$$P_D = 1-\left[\frac{1}{1+B+V+W_n+X+Y+Z+\sum_{n=1}^{i}[P]+\sum_{n=2}^{i}[R_n]+ \sum_{n=1}^{\infty}[S]+\sum_{n=i+1}^{\infty}[T]+\sum_{n=i+2}^{\infty}[U_n]}\right]\left[\frac{\zeta_0}{\kappa}\sum_{n=1}^{i}\zeta_n\prod_{n=0}^{i}\frac{1}{\kappa^{n+1}\xi_n}\right] \tag{38}$$

Expected number of customer in the orbit:

$$E[N_O] = \left[\frac{n}{1+B+V+W_n+X+Y+Z+\sum_{n=1}^{i}[P]+\sum_{n=2}^{i}[R_n] + \sum_{n=1}^{\infty}[S]+\sum_{n=i+1}^{\infty}[T]+\sum_{n=i+2}^{\infty}[U_n]}\right]$$

$$\left[\frac{1+i+\sum_{n=1}^{i}\zeta_n(\mu_1+\mu_2)(\lambda\beta+\theta)^n+\lambda^{n+1}\beta^n\kappa^{n+1}\prod_{n=1}^{i}\xi_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n(\lambda\beta+\theta)^n(\lambda+n\gamma\sigma)}+\sum_{n=i+1}^{\infty}\frac{\psi_n(\mu_1+\mu_2+\mu_3)(\lambda\beta+\theta)^n+\lambda^{n+1}\beta^n\kappa^{n+1}\prod_{n=1}^{i}\xi_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n(\lambda\beta+\theta)^n(\lambda+n\gamma\sigma)}\right] \tag{39}$$

Expected number of customer in the System:

$$E[N_S] = \left[\frac{1}{\begin{matrix}1+B+V+W_n+X+Y+Z+\sum_{n=1}^{i}[P]+\sum_{n=2}^{i}[R_n]\\ +\sum_{n=1}^{\infty}[S]+\sum_{n=i+1}^{\infty}[T]+\sum_{n=i+2}^{\infty}[U_n]\end{matrix}}\right]\left[\frac{\lambda(\lambda\beta+\theta)}{\theta(\mu_1+\mu_2)}+n\left(\frac{\zeta_0}{\kappa}+\sum_{n=1}^{i}\frac{\zeta_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n}\right)\right] \tag{40}$$

The expected waiting time in the orbit:

$$W_O = \frac{n}{\lambda_{eff}\left[\begin{matrix}1+B+V+W_n+X+Y+Z+\sum_{n=1}^{i}[P]+\sum_{n=2}^{i}[R_n]\\ +\sum_{n=1}^{\infty}[S]+\sum_{n=i+1}^{\infty}[T]+\sum_{n=i+2}^{\infty}[U_n]\end{matrix}\right]}$$

$$\left[\frac{1+i+\sum_{n=1}^{i}\zeta_n(\mu_1+\mu_2)(\lambda\beta+\theta)^n+\lambda^{n+1}\beta^n\kappa^{n+1}\prod_{n=1}^{i}\xi_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n(\lambda\beta+\theta)^n(\lambda+n\gamma\sigma)} + \sum_{n=i+1}^{\infty}\frac{\psi_n(\mu_1+\mu_2+\mu_3)(\lambda\beta+\theta)^n+\lambda^{n+1}\beta^n\kappa^{n+1}\prod_{n=1}^{i}\xi_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n(\lambda\beta+\theta)^n(\lambda+n\gamma\sigma)}\right] \tag{41}$$

The expected waiting time in the System:

$$W_S = \left[\frac{1}{\lambda_{eff}\left[\begin{matrix}1+B+V+W_n+X+Y+Z+\sum_{n=1}^{i}[P]+\sum_{n=2}^{i}[R_n]\\ +\sum_{n=1}^{\infty}[S]+\sum_{n=i+1}^{\infty}[T]+\sum_{n=i+2}^{\infty}[U_n]\end{matrix}\right]}\right]\left[\frac{\lambda(\lambda\beta+\theta)}{\theta(\mu_1+\mu_2)}+n\left[\frac{\zeta_0}{\kappa}+\sum_{n=1}^{i}\frac{\zeta_n}{\kappa^{n+1}\prod_{n=1}^{i}\xi_n}\right]\right] \tag{42}$$

The above equations have been obtained by using Little's formula.

8. Conclusions

The study analysis on M/M/2 of Markovian retrial queue model incorporating the impatience behavior of the customer with the retrial queue, vacation and additional server is provided. We can improve system efficiency and significantly lower the number of impatient cases with this model. Recursive methods are utilised to deliver supplies on schedule and to provide insightful updates for future improvements in service quality. To assess the systems performance, a numerical study was conducted. Managing the various parameters for the optimum outcomes, improves performance in the future.

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