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Product Properties for Generalized Pairwise Lindelöf Spaces

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Abstract In topological spaces, although compactness is satisfying the product invariant properties, but for the Lindelöffness, it is not preserved by the product unless one or more factors are assumed to satisfy additional conditions. Similar results yield for the bitopological spaces, that is, the property of pairwise Lindelöf bitopological spaces is not preserved under the product unless one or more factors are assumed to be satisfy additional conditions, for instance, P-spaces. The Cartesian product for arbitrarily many bitopological spaces was defined by Datta in 1972. Since then, many researchers have begun their study for the product bitopological spaces for their reason and direction. In this paper, we shall study finite product of pairwise nearly Lindelöf, pairwise almost Lindelöf and pairwise weakly Lindelöf spaces. We show that, all these generalized pairwise Lindelöf spaces are not preserved under a product by some counterexamples provided. Furthermore, we give some necessary conditions for these three bitopological spaces to be preserved under a finite product. Such condition is that one or more of the spaces has to be P-space or the product have to be pairwise weak P-space. Another interesting result is that the projection of these generalized pairwise Lindelöf spaces product with P-space is a closed map.

Keywords Bitopological Space, (i,j)-Nearly Lindelöf, (i,j)-Almost Lindelöf, (i,j)-Weakly Lindelöf, Product Bitopology

1 Introduction

The Cartesian product of a collection of sets is one of the most important and widely used ideas in mathematics. The theory of product spaces constitutes a very interesting and complex part of set-theoretic topology. The Cartesian product of arbitrarily many topological spaces was defined by Tychonoff in 1930. Then after 42 years, that is, in 1972, Datta [1] defined the Cartesian product of arbitrarily many bitopological spaces. It is known as well, the Tychonoff Product Theorem plays an important role in general product of compact topological spaces.

Kılıçman and Fawakhreh [7, 3] studied the product properties for nearly Lindelöf, almost Lindelöf and weakly Lindelöf spaces. The authors in our papers [13, 8, 10] introduced the notions of pairwise nearly Lindelöf, pairwise almost Lindelöf and pairwise weakly Lindelöf bitopological spaces. Recently the authors [11] studied the product properties for pairwise Lindelöf spaces that was introduced by Fora and Hdeib [4].

Although compactness is satisfying the product invariant properties, but for the Lindelöf spaces are negative that is Lindelöfness is not preserved by the product unless one or more factors are assumed to satisfy additional conditions. Similar results yield for the pairwise Lindelöf bitopological spaces (see [11]). The purpose of the present paper is to study the product properties on pairwise nearly Lindelöf, pairwise almost Lindelöf and pairwise weakly Lindelöf bitopological spaces.

Many of the results on the invariance of covering properties under product are negative, in that covering properties are simply not generally preserved unless one or more of the factors are assumed to satisfy additional conditions. In this paper, we show that pairwise nearly Lindelöf, pairwise almost Lindelöf and pairwise weakly Lindelöf properties do not satisfy product invariant properties. Some conditions in which these properties are preserved under a finite product would be given, for instance, pairwise P-spaces. We provide some counterexamples to show that the product invariant properties are negative. We also give some necessary conditions for these spaces to be preserved under a finite product.

2 Preliminaries

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) always mean topological spaces and bitopological spaces, respectively, unless explicitly stated. If $\mathcal{P}$ is a topological property, we always use the symbol (τ_i, τ_j)-$\mathcal{P}$ to denote τ_i has the property $\mathcal{P}$ with respect to topology τ_j in bitopological spaces, where $i, j \in \{1, 2\}$. Sometimes the prefixes (τ_i, τ_j)- or τ_i- will be replaced by (i, j)- or i- respectively, if there is no chance for confusion. By i-open cover of X, we mean that the cover of X by i-open sets in X; similar for the (i, j)-regular open cover of X, etc.

By $i\text{-int}\,(A)$ and $i\text{-cl}\,(A)$, we shall mean the interior and the closure of a subset A of X with respect to topology τ_i, respectively. We denote by $\text{int}\,(A)$ and $\text{cl}\,(A)$ for the interior and the closure of a subset A of X with respect to topology τ_i for each $i = 1, 2$, respectively. If the bitopological space (X, τ_1, τ_2) is i-Lindelöf, then the topological space (X, τ_i) is Lindelöf.

Definition 1 *Let (X, τ_1, τ_2) be a bitopological space. A subset F of X is said to be*
(i) i-open if F is open with respect to τ_i in X, F is called open in X if it is both 1-open and 2-open in X, or equivalently, $F \in U$ for $U \subseteq (\tau_1 \cap \tau_2)$ in X;
(ii) i-closed if F is closed with respect τ_i in X, F is called closed in X if it is both 1-closed and 2-closed in X, or equivalently, $F \in V$ for $V = X \setminus U$ and $U \subseteq (\tau_1 \cap \tau_2)$ in X.

Definition 2 ([6, 14]) *A subset S of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-regular open (resp. (i, j)-regular closed) if $i\text{-int}\,(j\text{-cl}\,(S)) = S$ (resp. $i\text{-cl}\,(j\text{-int}\,(S)) = S$), where $i, j \in \{1, 2\}$. S is called pairwise regular open (resp. pairwise regular closed) if it is both $(1, 2)$-regular open and $(2, 1)$-regular open (resp. $(1, 2)$-regular closed and $(2, 1)$-regular closed).*

Definition 3 ([5, 6]) *A bitopological space (X, τ_1, τ_2) is said to be (i, j)-regular if for each point $x \in X$ and for each i-open set V containing x, there exists an i-open set U such that $x \in U \subseteq j\text{-cl}\,(U) \subseteq V$. X is said to be pairwise regular if it is both $(1, 2)$-regular and $(2, 1)$-regular.*

Definition 4 *A bitopological space X is said to be*
(i) i-P-space if countable intersection of i-open sets in X is i-open. X is called P-space if it is i-P-space for each $i = 1, 2$;
(ii) (i, j)-weak P-space if for each countable family $\{U_n : n \in \mathbb{N}\}$ of i-open sets in X, we have $j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_n\right) = \bigcup_{n\in\mathbb{N}} j\text{-cl}\,(U_n)$. X is called pairwise weak P-space if it is both $(1, 2)$-weak P-space and $(2, 1)$-weak P-space.

Definition 5 *A bitopological space X is said to be (i, j)-nearly compact if for every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset $\{\alpha_1, \dots, \alpha_n\}$ of Δ such that $X = \bigcup_{k=1}^{n} i\text{-int}\,(j\text{-cl}\,(U_{\alpha_k}))$. X is called pairwise nearly compact if it is both $(1, 2)$-nearly compact and $(2, 1)$-nearly compact.*

Definition 6 ([13, 8, 10]) *A bitopological space X is said to be (i, j)-nearly Lindelöf (resp. (i, j)-almost Lindelöf, (i, j)-weakly Lindelöf) if for every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n\in\mathbb{N}} i\text{-int}\,(j\text{-cl}\,(U_{\alpha_n}))$ $\left(\text{resp. } X = \bigcup_{n\in\mathbb{N}} j\text{-cl}\,(U_{\alpha_n}),\ X = j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} (U_{\alpha_n})\right)\right)$. X is called pairwise nearly Lindelöf (resp. pairwise almost Lindelöf, pairwise weakly Lindelöf) if it is both $(1, 2)$-nearly Lindelöf (resp. $(1, 2)$-almost Lindelöf, $(1, 2)$-weakly Lindelöf) and $(2, 1)$-nearly Lindelöf (resp. $(2, 1)$-almost Lindelöf, $(2, 1)$-weakly Lindelöf).*

Definition 7 (see [1]) *Let $\{(X_\alpha, \tau_\alpha, \sigma_\alpha) : \alpha \in \Delta\}$ be a family of bitopological spaces. On the product set $X = \prod_{\alpha\in\Delta} X_\alpha$, we define a bitopological structure (τ, σ) by taking τ as the product topology generated by the projections which (τ, τ_α)-continuous and σ as the product topology generated by the projections which (σ, σ_α)-continuous where $\alpha \in \Delta$. The product set X with the product bitopology (τ, σ), i.e., (X, τ, σ) is called product bitopological space. The product bitopology (τ, σ) also can be denoted by $\left(\prod_{\alpha\in\Delta} \tau_\alpha, \prod_{\alpha\in\Delta} \sigma_\alpha\right)$.*

3 Product of Pairwise Nearly Lindelöf Spaces

Theorem 1 *Let (X, τ_1, τ_2) be a (τ_i, τ_j)-nearly compact space and (Y, σ_1, σ_2) a (σ_i, σ_j)-nearly compact space. Then $(X \times Y, \rho_1, \rho_2)$ is (ρ_i, ρ_j)-nearly compact where ρ_i is a product topology.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a ρ_i-open cover of $X \times Y$. Then each member of $\mathcal{U}$ is a union of ρ_i-basis elements of the form $V \times W$ with V is τ_i-open set in X and W is σ_i-open set in Y. We may restrict our attention to the cover $\{V_\lambda \times W_\lambda : \lambda \in \Lambda\}$ of $X \times Y$ by these ρ_i-basis elements where each $V_\lambda \times W_\lambda$ is contained in some member of $\mathcal{U}$, since any finite subfamily $\{V_{\lambda_1} \times W_{\lambda_1}, \dots, V_{\lambda_n} \times W_{\lambda_n}\}$ of this basic ρ_i-open cover such that $X \times Y = \bigcup_{k=1}^{n} \rho_i\text{-int}\left(\rho_j\text{-cl}\,(V_{\lambda_k} \times W_{\lambda_k})\right)$ will lead immediately to a finite subfamily chosen from the original cover $\mathcal{U}$ such that $X \times Y = \bigcup_{k=1}^{n} \rho_i\text{-int}\left(\rho_j\text{-cl}\,(U_{\alpha_k})\right)$. For each $x \in X$, let $Y_x = \{x\} \times Y$ which is i-homeomorphic to Y and hence Y_x is (ρ_i, ρ_j)-nearly compact with respect to the inducted bitopology from (ρ_1, ρ_2). So, Y_x is (ρ_i, ρ_j)-nearly compact relative to $X \times Y$ and since $\{V_\lambda \times W_\lambda : \lambda \in \Lambda\}$ also covers Y_x, there must exists a finite subfamily $\left\{V_{x,\lambda_1} \times W_{x,\lambda_1}, \dots, V_{x,\lambda_{n(x)}} \times W_{x,\lambda_{n(x)}}\right\}$ by

ρ_i-open sets which have a nonempty intersection with Y_x such that

$$\begin{aligned} Y_x &\subseteq \bigcup_{k=1}^{n(x)} \rho_i\text{- int}\left(\rho_j\text{- cl}\left(V_{x,\lambda_k} \times W_{x,\lambda_k}\right)\right) \\ &\subseteq \bigcup_{k=1}^{n(c)} \left(\tau_i\text{- int}\left(\tau_j\text{- cl}\left(V_{x,\lambda_k}\right)\right) \times \sigma_i\text{- int}\left(\sigma_j\text{- cl}\left(W_{x,\lambda_k}\right)\right)\right) \\ &\subseteq \left(\bigcup_{k=1}^{n(x)} \tau_i\text{- int}\left(\tau_j\text{- cl}\left(V_{x,\lambda_k}\right)\right)\right) \times \\ &\quad \left(\bigcup_{k=1}^{n(x)} \sigma_i\text{- int}\left(\sigma_j\text{- cl}\left(W_{x,\lambda_k}\right)\right)\right). \end{aligned}$$

Letting $H_x = \bigcap_{k=1}^{n(x)} \left(\tau_i\text{- int}\left(\tau_j\text{- cl}\left(V_{x,\lambda_k}\right)\right)\right)$, we see that H_x is a (τ_i, τ_j)-regular open set of X containing x. The above finite subfamily

$$\left\{V_{x,\lambda_1} \times W_{x,\lambda_1}, \ldots, V_{x,\lambda_{n(x)}} \times W_{x,\lambda_{n(x)}}\right\}$$

actually satisfies the condition $H_x \times Y \subseteq \bigcup_{k=1}^{n(x)} \rho_i\text{-int}\left(\rho_j\text{- cl}\left(V_{x,\lambda_k} \times W_{x,\lambda_k}\right)\right)$. Now $\{H_x : x \in X\}$ is a (τ_i, τ_j)-regular open cover of X. Since X is (τ_i, τ_j)-nearly compact, there exists a finite subcover $\{H_{x_1}, \ldots, H_{x_m}\}$. But then

$$\left\{\left\{V_{x_t,\lambda_k} \times W_{x_t,\lambda_k} : k = 1, \ldots, n\left(x_t\right)\right\} : t = 1, \ldots, m\right\}$$

satisfies the condition

$$\begin{aligned} X \times Y &= \bigcup_{t=1}^{m} \left(\bigcup_{k=1}^{n(x_t)} \rho_i\text{- int}\left(\rho_j\text{- cl}\left(V_{x_t,\lambda_k} \times W_{x_t,\lambda_k}\right)\right)\right) \\ &= \bigcup_{t=1,\ldots,m,k=1,\ldots,n(x_t)} \rho_i\text{- int}\left(\rho_j\text{- cl}\left(V_{x_t,\lambda_k} \times W_{x_t,\lambda_k}\right)\right). \end{aligned}$$

Since $\left\{\left\{V_{x_t,\lambda_k} \times W_{x_t,\lambda_k} : k = 1, \ldots, n\left(x_t\right)\right\} : t = 1, \ldots, m\right\}$ leads to a finite subfamily from $\mathcal{U}$ such that $X \times Y = \bigcup_{k=1}^{n} \rho_i\text{-int}\left(\rho_j\text{- cl}\left(U_{\alpha_k}\right)\right)$, we have that $X \times Y$ is $\left(\rho_i, \rho_j\right)$-nearly compact. ■

Corollary 2 *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be pairwise nearly compact spaces. Then the product $(X \times Y, \rho_1, \rho_2)$ is pairwise nearly compact where ρ_i is a product topology.*

The product still invariants if we take a finite collection of (i, j)-nearly compact spaces as stated in the following corollary. The result will then follow by induction.

Corollary 3 *Let $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots, n\right\}$ be a collection of $\left(\tau_i^k, \tau_j^k\right)$-nearly compact spaces. Then the product $\left(\prod_{k=1}^{n} X_k, \rho_1, \rho_2\right)$ is $\left(\rho_i, \rho_j\right)$-nearly compact.*

Corollary 4 *Let $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots, n\right\}$ be a collection of pairwise nearly compact spaces. Then the product $\left(\prod_{k=1}^{n} X_k, \rho_1, \rho_2\right)$ is pairwise nearly compact.*

Theorem 5 *Let (X, τ_1, τ_2) be a (τ_i, τ_j)-nearly Lindelöf space and (Y, σ_1, σ_2) a (σ_i, σ_j)-nearly compact space. Then the product $(X \times Y, \rho_1, \rho_2)$ is $\left(\rho_i, \rho_j\right)$-nearly Lindelöf.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a ρ_i-open cover of $X \times Y$. Then each member of $\mathcal{U}$ is a union of ρ_i-basis elements of the form $V \times W$ with V is τ_i-open set in X and W is σ_i-open set in Y. We may restrict our attention to the cover $\{V_\lambda \times W_\lambda : \lambda \in \Lambda\}$ of $X \times Y$ by the ρ_i-basis elements where each $V_\lambda \times W_\lambda$ is contained in some member of $\mathcal{U}$, since any countable subfamily $\{V_{\lambda_n} \times W_{\lambda_n} : n \in \mathbb{N}\}$ of this basic ρ_i-open cover such that $X \times Y = \bigcup_{n \in \mathbb{N}} \rho_i\text{-int}\left(\rho_j\text{- cl}\left(V_{\lambda_n} \times W_{\lambda_n}\right)\right)$ will lead immediately to a countable subfamily chosen from the original cover $\mathcal{U}$ such that $X \times Y = \bigcup_{n \in \mathbb{N}} \rho_i\text{-int}\left(\rho_j\text{- cl}\left(U_{\alpha_n}\right)\right)$. For each $x \in X$, let $Y_x = \{x\} \times Y$ which is i-homeomorphic to Y and hence Y_x is $\left(\rho_i, \rho_j\right)$-nearly compact with respect to the inducted bitopology from (ρ_1, ρ_2). So Y_x is $\left(\rho_i, \rho_j\right)$-nearly compact relative to $X \times Y$ and since $\{V_\lambda \times W_\lambda : \lambda \in \Lambda\}$ also covers Y_x, there must exists a finite subfamily $\left\{V_{x,\lambda_1} \times W_{x,\lambda_1}, \ldots, V_{x,\lambda_{n(x)}} \times W_{x,\lambda_{n(x)}}\right\}$ by ρ_i-open sets which have a non-empty intersection with Y_x such that $Y_x \subseteq \bigcup_{k=1}^{n(x)} \rho_i\text{-int}\left(\rho_j\text{- cl}\left(V_{x,\lambda_k} \times W_{x,\lambda_k}\right)\right)$. Letting $H_x = \bigcap_{k=1}^{n(x)} \left(\tau_i\text{- int}\left(\tau_j\text{- cl}\left(V_{x,\lambda_k}\right)\right)\right)$, we see that H_x is a (τ_i, τ_j)-regular open set of X containing x. The above finite subfamily $\left\{V_{x,\lambda_1} \times W_{x,\lambda_1}, \ldots, V_{x,\lambda_{n(x)}} \times W_{x,\lambda_{n(x)}}\right\}$ actually satisfies the condition $H_x \times Y \subseteq \bigcup_{k=1}^{n(x)} \rho_i\text{-int}\left(\rho_j\text{- cl}\left(V_{x,\lambda_k} \times W_{x,\lambda_k}\right)\right)$. Now $\{H_x : x \in X\}$ is a (τ_i, τ_j)-regular open cover of X. Since X is (τ_i, τ_j)-nearly Lindelöf, there exists a countable subcover $\{H_{x_m} : m \in \mathbb{N}\}$. But then $\left\{\left\{V_{x_m,\lambda_k} \times W_{x_m,\lambda_k} : k = 1, \ldots, n\left(x_m\right)\right\} : m \in \mathbb{N}\right\}$ satisfies the condition

$$\begin{aligned} &X \times Y \\ &= \bigcup_{m \in \mathbb{N}} \left(\bigcup_{k=1}^{n(x_m)} \rho_i\text{- int}\left(\rho_j\text{- cl}\left(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k}\right)\right)\right) \\ &= \bigcup_{m \in \mathbb{N}, k=1,\ldots,n(x_m)} \rho_i\text{- int}\left(\rho_j\text{- cl}\left(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k}\right)\right). \end{aligned}$$

Since $\left\{\left\{V_{x_m,\lambda_k} \times W_{x_m,\lambda_k} : k = 1, \ldots, n\left(x_m\right)\right\} : m \in \mathbb{N}\right\}$ leads to a countable subfamily from $\mathcal{U}$ such that $X \times Y = \bigcup_{n \in \mathbb{N}} \rho_i\text{-int}\left(\rho_j\text{- cl}\left(U_{\alpha_n}\right)\right)$, we have that $X \times Y$ is $\left(\rho_i, \rho_j\right)$-nearly Lindelöf. ■

Corollary 6 *Let (X, τ_1, τ_2) be a pairwise nearly Lindelöf space and (Y, σ_1, σ_2) a pairwise nearly compact space. Then the product $(X \times Y, \rho_1, \rho_2)$ is pairwise nearly Lindelöf.*

The above result still holds if we take an (i, j)-nearly Lindelöf space and a finite collection of (i, j)-nearly compact spaces as stated in the following corollary.

Corollary 7 *Let $(X_m, \tau_1^m, \tau_2^m)$ be a $\left(\tau_i^m, \tau_j^m\right)$-nearly Lindelöf space and*

$\{(X_k, \tau_1^k, \tau_2^k) : k = 1, \ldots, n, k \neq m, m \leq n\}$ *a collection of* (τ_i^k, τ_j^k)*-nearly compact spaces. Then the product* $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ *is* (ρ_i, ρ_j)*-nearly Lindelöf.*

Proof. It follows immediately by the fact that the topological product is commutative, associative, the Corollary 3 and Theorem 5. ■

Corollary 8 *Let* $(X_m, \tau_1^m, \tau_2^m)$ *be a pairwise nearly Lindelöf space and* $\{(X_k, \tau_1^k, \tau_2^k) : k = 1, \ldots, n, k \neq m, m \leq n\}$ *a collection of pairwise nearly compact spaces. Then the product* $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ *is pairwise nearly Lindelöf.*

Lemma 9 *Let* (X, τ_1, τ_2) *be a* (τ_i, τ_j)*-regular space and* (Y, σ_1, σ_2) *a* (σ_i, σ_j)*-regular space. Then* $(X \times Y, \rho_1, \rho_2)$ *is* (ρ_i, ρ_j)*-regular where* ρ_i *is a product topology.*

Proof. Let $z = (x, y) \in X \times Y$ and let U be a ρ_i-open set in $X \times Y$ containing z. Then there exist sets $V \in \tau_i, W \in \sigma_i$ with $z \in V \times W \subseteq U$. Since (X, τ_1, τ_2) is (τ_i, τ_j)-regular space, there exists a τ_i-open set C in X such that $x \in C \subseteq \tau_j\text{-cl}(C) \subseteq V$. Again since (Y, σ_1, σ_2) is (σ_i, σ_j)-regular, there exists a σ_i-open set D in Y such that $y \in D \subseteq \sigma_j\text{-cl}(D) \subseteq W$. Put $P = C \times D$, hence $z \in P \subseteq \rho_j\text{-cl}(P) = \tau_j\text{-cl}(C) \times \sigma_j\text{-cl}(D) \subseteq V \times W \subseteq U$. Therefore there exists a ρ_i-open set P such that $z \in P \subseteq \rho_j\text{-cl}(P) \subseteq U$ and completes the proof. ■

Corollary 10 *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be pairwise regular spaces. Then the product* $(X \times Y, \rho_1, \rho_2)$ *is pairwise regular.*

In general the product of any two (i, j)-nearly Lindelöf spaces need not be (i, j)-nearly Lindelöf or the product of any two pairwise nearly Lindelöf spaces need not be pairwise nearly Lindelöf as the following example below shows. The following lemma is needed.

Lemma 11 ([13]) *An* (i, j)*-regular space* X *is* (i, j)*-nearly Lindelöf if and only if it is* i*-Lindelöf.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be an i-open cover of X. Since X is (i, j)-regular, then for each $x \in X$ and for each i-open set $U_x \in \mathcal{U}$ containing x, there is an i-open set V_x such that $x \in V_x \subseteq j\text{-cl}(V_x) \subseteq U_x$. Hence $x \in V_x \subseteq i\text{-int}(j\text{-cl}(V_x)) \subseteq j\text{-cl}(V_x) \subseteq U_x$. So, $X = \bigcup_{x \in X} V_x \subseteq \bigcup_{x \in X} i\text{-int}(j\text{-cl}(V_x))$. Now $\{i\text{-int}(j\text{-cl}(V_x)) : x \in X\}$ forms an (i, j)-regular open cover of X. Since X is (i, j)-nearly Lindelöf, there exists a countable subset of points $x_1, x_2, \ldots, x_n, \ldots$ of X such that $X = \bigcup_{n \in \mathbb{N}} i\text{-int}(j\text{-cl}(V_{x_n})) \subseteq \bigcup_{n \in \mathbb{N}} U_{x_n}$. Thus $\{U_\alpha : \alpha \in \Delta\}$ has a countable subfamily covering X. Therefore X is i-Lindelöf. The converse is obvious because every i-Lindelöf space is an (i, j)-nearly Lindelöf. ■

Example 1 *Let* $\mathcal{B}$ *be a collection of closed-open intervals in the real line* $\mathbb{R}$:

$$\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}, a < b\}.$$

Hence $\mathcal{B}$ *is a base for the lower limit topology (or Sorgenfrey topology)* τ_s *on* $\mathbb{R}$*. Choose usual topology as topology* τ_u *on* $\mathbb{R}$*. Thus* $(\mathbb{R}, \tau_s, \tau_u)$ *is a Lindelöf bitopological space (see [15]). Therefore it is* (τ_s, τ_u)*-nearly Lindelöf. Note that, sets of the form* $(-\infty, a), [a, b)$ *or* $[a, \infty)$ *are both* τ_s*-open and* τ_s*-closed in* $\mathbb{R}$ *and sets of the form* (a, b) *and* (a, ∞) *are* τ_s*-open in* $\mathbb{R}$ *(see [15, p. 75]). It is easy to check that* $(\mathbb{R}, \tau_s, \tau_u)$ *is* (τ_s, τ_u)*-regular since for each* $x \in \mathbb{R}$ *and for each* τ_s*-open set of the form* $[a, b)$ *in* $\mathbb{R}$ *containing* x*, there exists a* τ_s*-open set* $[a, b - \epsilon)$ *with* $\epsilon > 0$ *such that* $x \in [a, b - \epsilon) \subseteq \tau_u\text{-cl}([a, b - \epsilon)) = [a, b - \epsilon] \subseteq [a, b)$*. We leave to the reader to check for other forms of* τ_s*-open sets in* $\mathbb{R}$*. So,* $(\mathbb{R} \times \mathbb{R}, \tau_s \times \tau_s, \tau_u \times \tau_u)$ *is* $(\tau_s \times \tau_s, \tau_u \times \tau_u)$*-regular by Lemma 9. It is known that* $\mathbb{R} \times \mathbb{R}$ *is not* $(\tau_s \times \tau_s)$*-Lindelöf since the* τ_s*-closed subspace* $L = \{(x, y) : y = -x\}$ *is not* $(\tau_s \times \tau_s)$*-Lindelöf for it is a discrete subspace. Since* $\mathbb{R} \times \mathbb{R}$ *is* $(\tau_s \times \tau_s, \tau_u \times \tau_u)$*-regular but not* $(\tau_s \times \tau_s)$*-Lindelöf, then it is not* $(\tau_s \times \tau_s, \tau_u \times \tau_u)$*-nearly Lindelöf by Lemma 11.*

Recall that a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called i-closed if $f(U)$ is σ_i-closed set in Y for every τ_i-closed set U in X, f is said closed if it is i-closed for each $i = 1, 2$. From elementary general topology, if X is a topological space and suppose a neighbourhood base has been fixed at each $x \in X$, then $F \subseteq X$ is closed if and only if each point $x \notin F$ has a basic neighbourhood disjoint from F (see [16]). So we can prove the following proposition.

Proposition 12 *Let* (X, τ_1, τ_2) *be a* (τ_i, τ_j)*-nearly Lindelöf space and* (Y, σ_1, σ_2) *a* σ_i*-P-space. Then the projection* $\pi_Y : (X \times Y, \rho_1, \rho_2) \to (Y, \sigma_1, \sigma_2)$ *is* i*-closed where* ρ_i *is a product topology.*

Proof. Let U be a ρ_i-closed set in $X \times Y$ and let $y_0 \notin \pi_Y(U)$. Clearly $(X \times \{y_0\}) \cap U = \emptyset$ and so the point $(x, y_0) \notin U$ has a ρ_i-basic neighbourhood $V_x \times W_{x,y_0}$ disjoint from U where V_x is τ_i-open set in X containing x and W_{x,y_0} is σ_i-open set in Y containing y_0. Now $\{V_x \times W_{x,y_0} : x \in X\}$ form a cover of $X \times \{y_0\}$ by ρ_i-basis elements of $X \times Y$. Since $X \times \{y_0\}$ is i-homeomorphic to X, then $X \times \{y_0\}$ is (ρ_i, ρ_j)-nearly Lindelöf with respect to the inducted bitopology from (ρ_1, ρ_2). So $X \times \{y_0\}$ is (ρ_i, ρ_j)-nearly Lindelöf relative to $X \times Y$ and hence there exists a countable subfamily $\{V_{x_n} \times W_{x_n,y_0} : n \in \mathbb{N}\}$ such that

$$\begin{aligned} X \times \{y_0\} &\subseteq \bigcup_{n \in \mathbb{N}} \rho_i\text{-int}\left(\rho_j\text{-cl}\left(V_{x_n} \times W_{x_n,y_0}\right)\right) \\ &\subseteq \bigcup_{n \in \mathbb{N}} \left(\tau_i\text{-int}\left(\tau_j\text{-cl}\left(V_{x_n}\right)\right) \times \sigma_i\text{-int}\left(\sigma_j\text{-cl}\left(W_{x_n,y_0}\right)\right)\right) \\ &\subseteq \left(\bigcup_{n \in \mathbb{N}} \tau_i\text{-int}\left(\tau_j\text{-cl}\left(V_{x_n}\right)\right)\right) \times \\ &\quad \left(\bigcup_{n \in \mathbb{N}} \sigma_i\text{-int}\left(\sigma_j\text{-cl}\left(W_{x_n,y_0}\right)\right)\right). \end{aligned}$$

Set $W = \bigcap_{n \in \mathbb{N}} \sigma_i\text{-int}(\sigma_j\text{-cl}(W_{x_n,y_0}))$ and since Y is a σ_i-P-space, W is a σ_i-open neighbourhood of y_0 such that $W \cap \pi_Y(U) = \emptyset$. Thus $\pi_Y(U)$ is σ_i-closed set in Y. This implies that π_Y is i-closed. ■

Corollary 13 *Let (X, τ_1, τ_2) be a pairwise nearly Lindelöf space and (Y, σ_1, σ_2) a P-space. Then the projection $\pi_Y : (X \times Y, \rho_1, \rho_2) \to (Y, \sigma_1, \sigma_2)$ is closed where ρ_i is a product topology.*

4 Product of Pairwise Almost Lindelöf Spaces

Theorem 14 *Let (X, τ_1, τ_2) be a (τ_i, τ_j)-almost Lindelöf space and (Y, σ_1, σ_2) a (σ_i, σ_j)-nearly compact space. Then $(X \times Y, \rho_1, \rho_2)$ is (ρ_i, ρ_j)-almost Lindelöf.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a ρ_i-open cover of $X \times Y$. Then as in the proof of Theorem 5, we may restrict our attention to the cover $\{V_\lambda \times W_\lambda : \lambda \in \Lambda\}$ of $X \times Y$ by the ρ_i-basis elements where each $V_\lambda \times W_\lambda$ is contained in some member of $\mathcal{U}$, since any countable subfamily $\{V_{\lambda_n} \times W_{\lambda_n} : n \in \mathbb{N}\}$ of this basic ρ_i-open cover such that $X \times Y = \bigcup_{n\in\mathbb{N}} \rho_j\text{-}\mathrm{cl}\,(V_{\lambda_n} \times W_{\lambda_n})$ will lead immediately to a countable subfamily chosen from the original cover $\mathcal{U}$ such that $X \times Y = \bigcup_{n\in\mathbb{N}} \rho_j\text{-}\mathrm{cl}\,(U_{\alpha_n})$. For each $x \in X$, let $Y_x = \{x\} \times Y$ which is i-homeomorphic to Y and hence Y_x is (ρ_i, ρ_j)-nearly compact with respect to the inducted bitopology from (ρ_1, ρ_2). So Y_x is (ρ_i, ρ_j)-nearly compact relative to $X \times Y$ and since $\{V_\lambda \times W_\lambda : \lambda \in \Lambda\}$ also covers Y_x, there must exists a finite subfamily $\{V_{x,\lambda_1} \times W_{x,\lambda_1}, \dots, V_{x,\lambda_{n(x)}} \times W_{x,\lambda_{n(x)}}\}$ by ρ_i-open sets which have a non-empty intersection with Y_x such that $Y_x \subseteq \bigcup_{k=1}^{n(x)} \rho_i\text{-int}\left(\rho_j\text{-}\mathrm{cl}\,(V_{x,\lambda_k} \times W_{x,\lambda_k})\right)$. Letting $H_x = \bigcap_{k=1}^{n(x)} V_{x,\lambda_k}$, we see that H_x is a τ_i-open set of X containing x. The above finite subfamily $\{V_{x,\lambda_1} \times W_{x,\lambda_1}, \dots, V_{x,\lambda_{n(x)}} \times W_{x,\lambda_{n(x)}}\}$ actually satisfies the condition $H_x \times Y \subseteq \bigcup_{k=1}^{n(x)} \rho_i\text{-int}\left(\rho_j\text{-}\mathrm{cl}\,(V_{x,\lambda_k} \times W_{x,\lambda_k})\right)$. Now $\{H_x : x \in X\}$ is a τ_i-open cover of X. Since X is (τ_i, τ_j)-almost Lindelöf, there exists a countable subfamily $\{H_{x_m} : m \in \mathbb{N}\}$ such that $X = \bigcup_{m\in\mathbb{N}} \tau_j\text{-}\mathrm{cl}\,(H_{x_m})$. But then $\{\{V_{x_m,\lambda_k} \times W_{x_m,\lambda_k} : k = 1, \dots, n(x_m)\} : m \in \mathbb{N}\}$ satisfying the condition

$$\begin{aligned}
& X \times Y \\
= & \bigcup_{m\in\mathbb{N}} \rho_j\text{-}\mathrm{cl}\left(\bigcup_{k=1}^{n(x_m)} \rho_i\text{-}\mathrm{int}\left(\rho_j\text{-}\mathrm{cl}\,(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k})\right)\right) \\
\subseteq & \bigcup_{m\in\mathbb{N}} \rho_j\text{-}\mathrm{cl}\left(\bigcup_{k=1}^{n(x_m)} \rho_j\text{-}\mathrm{cl}\,(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k})\right) \\
= & \bigcup_{m\in\mathbb{N}} \rho_j\text{-}\mathrm{cl}\left(\rho_j\text{-}\mathrm{cl}\left(\bigcup_{k=1}^{n(x_m)} (V_{x_m,\lambda_k} \times W_{x_m,\lambda_k})\right)\right) \\
= & \bigcup_{m\in\mathbb{N}} \rho_j\text{-}\mathrm{cl}\left(\bigcup_{k=1}^{n(x_m)} (V_{x_m,\lambda_k} \times W_{x_m,\lambda_k})\right) \\
= & \bigcup_{m\in\mathbb{N}} \left(\bigcup_{k=1}^{n(x_m)} \rho_j\text{-}\mathrm{cl}\,(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k})\right) \\
= & \bigcup_{m\in\mathbb{N},k=1,\dots,n(x_m)} \rho_j\text{-}\mathrm{cl}\,(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k}).
\end{aligned}$$

Since $\{\{V_{x_m,\lambda_k} \times W_{x_m,\lambda_k} : k = 1, \dots, n(x_m)\} : m \in \mathbb{N}\}$ leads to a countable subfamily from $\mathcal{U}$ such that $X \times Y = \bigcup_{n\in\mathbb{N}} \rho_j\text{-}\mathrm{cl}\,(U_{\alpha_n})$, we have that $X \times Y$ is (ρ_i, ρ_j)-almost Lindelöf. ■

Corollary 15 *Let (X, τ_1, τ_2) be a pairwise almost Lindelöf space and (Y, σ_1, σ_2) a pairwise nearly compact space. Then $(X \times Y, \rho_1, \rho_2)$ is pairwise almost Lindelöf.*

The above result is still holds if we take an (i, j)-almost Lindelöf space and a finite collection of (i, j)-nearly compact spaces as stated in the following corollary.

Corollary 16 *Let $(X_m, \tau_1^m, \tau_2^m)$ be a (τ_i^m, τ_j^m)-almost Lindelöf space and $\{(X_k, \tau_1^k, \tau_2^k) : k = 1, \dots, n, k \neq m, m \leq n\}$ a collection of (τ_i^k, τ_j^k)-nearly compact spaces. Then the product $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ is (ρ_i, ρ_j)-almost Lindelöf.*

Proof. It follows immediately by the fact that the topological product is commutative, associative, the Corollary 3 and Theorem 14. ■

Corollary 17 *Let $(X_m, \tau_1^m, \tau_2^m)$ be a pairwise almost Lindelöf space and $\{(X_k, \tau_1^k, \tau_2^k) : k = 1, \dots, n, k \neq m, m \leq n\}$ a collection of pairwise nearly compact spaces. Then the product $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ is pairwise almost Lindelöf.*

Lemma 18 *[8] An (i, j)-regular space is (i, j)-almost Lindelöf if and only if it is i-Lindelöf.*

The space in Example 1 shows that the product of any two (i, j)-almost Lindelöf spaces need not be (i, j)-almost Lindelöf or the product of any two pairwise almost Lindelöf spaces need not be pairwise almost Lindelöf by applying the Lemma 18. But if we add one extra condition such as τ_i-P-space to the (i, j)-almost Lindelöf space and another space is (i, j)-nearly Lindelöf, and we assume that the product space is $(\tau_i \times \sigma_i, \tau_j \times \sigma_j)$-weak P-space, we will obtain that the product space is $(\tau_i \times \sigma_i, \tau_j \times \sigma_j)$-almost Lindelöf.

Proposition 19 *Let (X,τ_1,τ_2) be a (τ_i,τ_j)-almost Lindelöf τ_i-P-space and (Y,σ_1,σ_2) a (σ_i,σ_j)-nearly Lindelöf space. If the product $(X\times Y,\rho_1,\rho_2)$ is (ρ_i,ρ_j)-weak P-space, then $(X\times Y,\rho_1,\rho_2)$ is (ρ_i,ρ_j)-almost Lindelöf.*

Proof. Let $\mathcal{U}=\{U_\alpha:\alpha\in\Delta\}$ be a ρ_i-open cover of $X\times Y$. Then as in the proof of Theorem 5, we may restrict our attention to the cover $\{V_\alpha\times W_\alpha:\alpha\in\Delta\}$ of $X\times Y$ by the ρ_i-basis elements where each $V_\alpha\times W_\alpha$ is contained in some member of $\mathcal{U}$, since any countable subfamily $\{V_{\alpha_n}\times W_{\alpha_n}:n\in\mathbb{N}\}$ of this basic ρ_i-open cover will lead immediately to a countable subfamily chosen from the original cover $\mathcal{U}$ such that $X\times Y=\bigcup_{n\in\mathbb{N}}\rho_j\text{-cl}\,(V_{\alpha_n}\times W_{\alpha_n})$. For each $x\in X$, let $Y_x=\{x\}\times Y$ which is i-homeomorphic to Y and hence Y_x is (ρ_i,ρ_j)-nearly Lindelöf with respect to the inducted bitopology from (ρ_1,ρ_2). So Y_x is (ρ_i,ρ_j)-nearly Lindelöf relative to $X\times Y$ and since $\{V_\alpha\times W_\alpha:\alpha\in\Delta\}$ also covers Y_x, there must exists a countable subfamily $\{V_{x,\alpha_k}\times W_{x,\alpha_k}:k\in\mathbb{N}\}$ by ρ_i-open sets which have a non-empty intersection with Y_x such that $Y_x\subseteq\bigcup_{k\in\mathbb{N}}\rho_i\text{-int}\left(\rho_j\text{-cl}\,(V_{x,\alpha_k}\times W_{x,\alpha_k})\right)$. Letting $H_x=\bigcap_{k\in\mathbb{N}}V_{x,\alpha_k}$, we see that H_x is a τ_i-open set of X containing x since X is τ_i-P-space. The above countable subfamily $\{V_{x,\alpha_k}\times W_{x,\alpha_k}:k\in\mathbb{N}\}$ actually satisfying the condition $H_x\times Y\subseteq\bigcup_{k\in\mathbb{N}}\rho_i\text{-int}\left(\rho_j\text{-cl}\,(V_{x,\alpha_k}\times W_{x,\alpha_k})\right)$. Now $\{H_x:x\in X\}$ is a τ_i-open cover of X. Since X is (τ_i,τ_j)-almost Lindelöf, there exists a countable subfamily $\{H_{x_n}:n\in\mathbb{N}\}$ such that $X=\bigcup_{n\in\mathbb{N}}\tau_j\text{-cl}\,(H_{x_n})$. But since $(X\times Y,\rho_1,\rho_2)$ is (ρ_i,ρ_j)-weak P-space, then $\{\{V_{x_n,\alpha_k}\times W_{x_n,\alpha_k}:k\in\mathbb{N}\}:n\in\mathbb{N}\}$ satisfying the condition

$$\begin{aligned} & X\times Y \\ = \;& \bigcup_{n\in\mathbb{N}}\rho_j\text{-cl}\left(\bigcup_{k\in\mathbb{N}}\rho_i\text{-int}\left(\rho_j\text{-cl}\,(V_{x_n,\alpha_k}\times W_{x_n,\alpha_k})\right)\right) \\ = \;& \bigcup_{n\in\mathbb{N}}\left(\bigcup_{k\in\mathbb{N}}\rho_j\text{-cl}\left(\rho_i\text{-int}\left(\rho_j\text{-cl}\,(V_{x_n,\alpha_k}\times W_{x_n,\alpha_k})\right)\right)\right) \\ \subseteq \;& \bigcup_{n\in\mathbb{N}}\left(\bigcup_{k\in\mathbb{N}}\rho_j\text{-cl}\left(\rho_j\text{-cl}\,(V_{x_n,\alpha_k}\times W_{x_n,\alpha_k})\right)\right) \\ = \;& \bigcup_{n\in\mathbb{N},k\in\mathbb{N}}\rho_j\text{-cl}\,(V_{x_n,\alpha_k}\times W_{x_n,\alpha_k}). \end{aligned}$$

Since $\{\{V_{x_n,\alpha_k}\times W_{x_n,\alpha_k}:k\in\mathbb{N}\}:n\in\mathbb{N}\}$ is a countable subfamily, we have that $X\times Y$ is (ρ_i,ρ_j)-almost Lindelöf. ■

Corollary 20 *Let (X,τ_1,τ_2) be a pairwise almost Lindelöf P-space and (Y,σ_1,σ_2) a pairwise nearly Lindelöf space. If the product $(X\times Y,\rho_1,\rho_2)$ is pairwise weak P-space, then $(X\times Y,\rho_1,\rho_2)$ is pairwise almost Lindelöf.*

Corollary 21 *Let (X,τ_1,τ_2) be a (τ_i,τ_j)-nearly Lindelöf τ_i-P-space and (Y,σ_1,σ_2) a (σ_i,σ_j)-nearly Lindelöf space. If the product $(X\times Y,\rho_1,\rho_2)$ is (ρ_i,ρ_j)-weak P-space, then $(X\times Y,\rho_1,\rho_2)$ is (ρ_i,ρ_j)-almost Lindelöf.*

Proof. It follows immediately from the fact that every (i,j)-nearly Lindelöf space is (i,j)-almost Lindelöf and Proposition 19. ■

Corollary 22 *Let (X,τ_1,τ_2) be a pairwise nearly Lindelöf P-space and (Y,σ_1,σ_2) a pairwise nearly Lindelöf space. If the product $(X\times Y,\rho_1,\rho_2)$ is pairwise weak P-space, then $(X\times Y,\rho_1,\rho_2)$ is pairwise almost Lindelöf.*

Lemma 23 *(see [11]). Let $\left\{\left(X_k,\tau_1^k,\tau_2^k\right):k=1,\ldots,n\right\}$ be a collection of τ_i^k-P-spaces. Then $(\prod_{k=1}^n X_k,\rho_1,\rho_2)$ is ρ_i-P-space where ρ_i is a product topology.*

The result of Proposition 19 can be extended to an (i,j)-almost Lindelöf i-P-space, a collection of finite (i,j)-nearly Lindelöf i-P-spaces and an (i,j)-nearly Lindelöf space as stated in the following corollary.

Corollary 24 *Let (X_m,τ_1^m,τ_2^m) be a $\left(\tau_i^m,\tau_j^m\right)$-almost Lindelöf τ_i^m-P-space,*

$$\{\left(X_k,\tau_1^k,\tau_2^k\right):k=1,\ldots,n,k\neq m\leq n,k\neq p\leq n,\ \text{and } m\neq p\}$$

a collection of $\left(\tau_i^k,\tau_j^k\right)$-nearly Lindelöf τ_i^k-P-spaces and (X_p,τ_1^p,τ_2^p) a $\left(\tau_i^p,\tau_j^p\right)$-nearly Lindelöf. If the product $(\prod_{k=1}^n X_k,\rho_1,\rho_2)$ is (ρ_i,ρ_j)-weak-P-space, then $(\prod_{k=1}^n X_k,\rho_1,\rho_2)$ is (ρ_i,ρ_j)-almost Lindelöf.

Proof. It follows by induction of k, the fact that topological product is associative, the Lemma 23 and Proposition 19. ■

Corollary 25 *Let (X_m,τ_1^m,τ_2^m) be a pairwise almost Lindelöf P-space,*

$$\{\left(X_k,\tau_1^k,\tau_2^k\right):k=1,\ldots,n,k\neq m\leq n,k\neq p\leq n,\ \text{and } m\neq p\}$$

a collection of pairwise nearly Lindelöf P-spaces and (X_p,τ_1^p,τ_2^p) a pairwise nearly Lindelöf. If the product $(\prod_{k=1}^n X_k,\rho_1,\rho_2)$ is pairwise weak-P-space, then $(\prod_{k=1}^n X_k,\rho_1,\rho_2)$ is pairwise almost Lindelöf.

Proposition 26 *Let (X,τ_1,τ_2) be a (τ_i,τ_j)-almost Lindelöf space and (Y,σ_1,σ_2) a σ_i-P-space. Then the projection $\pi_Y:(X\times Y,\rho_1,\rho_2)\to(Y,\sigma_1,\sigma_2)$ is i-closed where ρ_i is a product topology.*

Proof. Let U be a ρ_i-closed set in $X\times Y$ and let $y_0\notin\pi_Y(U)$. Clearly $(X\times\{y_0\})\cap U=\emptyset$ and so the point $(x,y_0)\notin U$ has a ρ_i-basic neighbourhood $V_x\times W_{x,y_0}$ disjoint from U where V_x is τ_i-open set in X containing x and W_{x,y_0} is σ_i-open set in Y containing y_0. Now $\{V_x\times W_{x,y_0}:x\in X\}$ form a ρ_i-basic open cover of $X\times\{y_0\}$ by ρ_i-basis elements of $X\times Y$. Since $X\times\{y_0\}$ is i-homeomorphic to X, then $X\times\{y_0\}$ is (ρ_i,ρ_j)-almost Lindelöf with respect to the inducted bitopology from (ρ_1,ρ_2). So $X\times\{y_0\}$ is (ρ_i,ρ_j)-almost Lindelöf relative to $X \times Y$ and hence there exists a countable subfamily $\{V_{x_n} \times W_{x_n,y_0} : n \in \mathbb{N}\}$ such that

$$\begin{aligned} X \times \{y_0\} &\subseteq \bigcup_{n\in\mathbb{N}} \rho_j\text{- cl}\left(V_{x_n} \times W_{x_n,y_0}\right) \\ &= \bigcup_{n\in\mathbb{N}} \left(\tau_j\text{- cl}\left(V_{x_n}\right) \times \sigma_j\text{- cl}\left(W_{x_n,y_0}\right)\right) \\ &\subseteq \left(\bigcup_{n\in\mathbb{N}} \tau_j\text{- cl}\left(V_{x_n}\right)\right) \times \left(\bigcup_{n\in\mathbb{N}} \sigma_j\text{- cl}\left(W_{x_n,y_0}\right)\right). \end{aligned}$$

Set $W = \bigcap_{n\in\mathbb{N}} W_{x_n,y_0}$ and since Y is a σ_i-P-space, W is a σ_i-open neighbourhood of y_0 such that $W \cap \pi_Y(U) = \emptyset$. Thus $\pi_Y(U)$ is σ_i-closed set in Y. This implies that π_Y is i-closed and completes the proof. ■

Corollary 27 *Let (X, τ_1, τ_2) be a pairwise almost Lindelöf space and (Y, σ_1, σ_2) a P-space. Then the projection $\pi_Y : (X \times Y, \rho_1, \rho_2) \to (Y, \sigma_1, \sigma_2)$ is closed where ρ_i is a product topology.*

5 Product of Pairwise Weakly Lindelöf Spaces

Theorem 28 *Let (X, τ_1, τ_2) be a (τ_i, τ_j)-weakly Lindelöf space and (Y, σ_1, σ_2) a (σ_i, σ_j)-nearly compact space. Then the product $(X \times Y, \rho_1, \rho_2)$ is (ρ_i, ρ_j)-weakly Lindelöf.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a ρ_i-open cover of $X \times Y$. Then as in the proof of Theorem 5, we may restrict our attention to the cover $\{V_\lambda \times W_\lambda : \lambda \in \Lambda\}$ of $X \times Y$ by the ρ_i-basis elements where each $V_\lambda \times W_\lambda$ is contained in some member of $\mathcal{U}$, since any countable subfamily $\{V_{\lambda_n} \times W_{\lambda_n} : n \in \mathbb{N}\}$ of this basic ρ_i-open cover such that $X \times Y = \rho_j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} \left(V_{\lambda_n} \times W_{\lambda_n}\right)\right)$ will lead immediately to a countable subfamily chosen from the original cover $\mathcal{U}$ such that $X \times Y = \rho_j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}\right)$. For each $x \in X$, let $Y_x = \{x\} \times Y$ which is i-homeomorphic to Y and hence Y_x is (ρ_i, ρ_j)-nearly compact with respect to the inducted bitopology from (ρ_1, ρ_2). So Y_x is (ρ_i, ρ_j)-nearly compact relative to $X \times Y$ and since $\{V_\lambda \times W_\lambda : \lambda \in \Lambda\}$ also covers Y_x, there must exists a finite subfamily $\left\{V_{x,\lambda_1} \times W_{x,\lambda_1}, \ldots, V_{x,\lambda_{n(x)}} \times W_{x,\lambda_{n(x)}}\right\}$ by ρ_i-open sets which have a non-empty intersection with Y_x such that $Y_x \subseteq \bigcup_{k=1}^{n(x)} \rho_i\text{-int}\left(\rho_j\text{- cl}\left(V_{x,\lambda_k} \times W_{x,\lambda_k}\right)\right)$. Letting $H_x = \bigcap_{k=1}^{n(x)} V_{x,\lambda_k}$, we see that H_x is a τ_i-open set in X containing x. The above finite subfamily $\left\{V_{x,\lambda_1} \times W_{x,\lambda_1}, \ldots, V_{x,\lambda_{n(x)}} \times W_{x,\lambda_{n(x)}}\right\}$ actually satisfies the condition $H_x \times Y \subseteq \bigcup_{k=1}^{n(x)} \rho_i\text{-int}\left(\rho_j\text{- cl}\left(V_{x,\lambda_k} \times W_{x,\lambda_k}\right)\right)$. Now $\{H_x : x \in X\}$ is a τ_i-open cover of X. Since X is (τ_i, τ_j)-weakly Lindelöf, there exists a countable subfamily $\{H_{x_m} : m \in \mathbb{N}\}$ such that $X = \tau_j\text{-cl}\left(\bigcup_{m\in\mathbb{N}} H_{x_m}\right)$. But then $\{\{V_{x_m,\lambda_k} \times W_{x_m,\lambda_k} : k = 1, \ldots, n(x_m)\} : m \in \mathbb{N}\}$ satisfies the condition

$$\begin{aligned} & X \times Y = \\ & \rho_j\text{- cl}\left(\bigcup_{m\in\mathbb{N}} \left(\bigcup_{k=1}^{n(x_m)} \rho_i\text{- int}\left(\rho_j\text{- cl}\left(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k}\right)\right)\right)\right) \\ & \subseteq \rho_j\text{- cl}\left(\bigcup_{m\in\mathbb{N}} \left(\bigcup_{k=1}^{n(x_m)} \rho_j\text{- cl}\left(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k}\right)\right)\right) \\ & = \rho_j\text{- cl}\left(\bigcup_{m\in\mathbb{N}} \rho_j\text{- cl}\left(\bigcup_{k=1}^{n(x_m)} \left(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k}\right)\right)\right) \\ & \subseteq \rho_j\text{- cl}\left(\rho_j\text{- cl}\left(\bigcup_{m\in\mathbb{N}} \left(\bigcup_{k=1}^{n(x_m)} \left(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k}\right)\right)\right)\right) \\ & = \rho_j\text{- cl}\left(\bigcup_{m\in\mathbb{N},k=1,\ldots,n(x_m)} \left(V_{x_m,\lambda_k} \times W_{x_m,\lambda_k}\right)\right). \end{aligned}$$

Since $\{\{V_{x_m,\lambda_k} \times W_{x_m,\lambda_k} : k = 1, \ldots, n(x_m)\} : m \in \mathbb{N}\}$ leads to a countable subfamily from $\mathcal{U}$ such that $X \times Y = \rho_j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}\right)$, we have that $X \times Y$ is (ρ_i, ρ_j)-weakly Lindelöf. ■

Corollary 29 *Let (X, τ_1, τ_2) be a pairwise weakly Lindelöf space and (Y, σ_1, σ_2) a pairwise nearly compact space. Then the product $(X \times Y, \rho_1, \rho_2)$ is pairwise weakly Lindelöf.*

The above result is still hold if we take an (i, j)-weakly Lindelöf space and a collection of finite (i, j)-nearly compact spaces as stated in the following corollary.

Corollary 30 *Let $(X_m, \tau_1^m, \tau_2^m)$ be a (τ_i^m, τ_j^m)-weakly Lindelöf space and $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots, n, k \neq m, m \leq n\right\}$ a collection of $\left(\tau_i^k, \tau_j^k\right)$-nearly compact spaces. Then the product $\left(\prod_{k=1}^n X_k, \rho_1, \rho_2\right)$ is (ρ_i, ρ_j)-weakly Lindelöf.*

Proof. It follows immediately by the fact that the topological product is associative, the Corollary 3 and Theorem 28. ■

Corollary 31 *Let $(X_m, \tau_1^m, \tau_2^m)$ be a pairwise weakly Lindelöf space and $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots, n, k \neq m, m \leq n\right\}$ a collection of pairwise nearly compact spaces. Then the product $\left(\prod_{k=1}^n X_k, \rho_1, \rho_2\right)$ is pairwise weakly Lindelöf.*

Lemma 32 *(see [10]) An (i, j)-weakly Lindelöf, (i, j)-regular and (i, j)-weak P-space is i-Lindelöf.*

In general the product of any two (i, j)-weakly Lindelöf spaces need not be (i, j)-weakly Lindelöf or the product of any two pairwise weakly Lindelöf spaces need not be pairwise weakly Lindelöf as the following example below shows.

Example 2 *Let $(\mathbb{R}, \tau_s, \tau_u)$ be a Lindelöf bitopological space from Example 1 above. Therefore it is (τ_s, τ_u)-weakly Lindelöf. It is easy to check that $(\mathbb{R}, \tau_s, \tau_u)$ is (τ_s, τ_u)-regular and (τ_s, τ_u)-weak P-space. So $(\mathbb{R} \times \mathbb{R}, \tau_s \times \tau_s, \tau_u \times \tau_u)$*

is $(\tau_s \times \tau_s, \tau_u \times \tau_u)$-regular (see Example 1). It is clear that $\mathbb{R} \times \mathbb{R}$ is $(\tau_s \times \tau_s, \tau_u \times \tau_u)$-weak P-space. It is known that $\mathbb{R} \times \mathbb{R}$ is not $(\tau_s \times \tau_s)$-Lindelöf. Since $\mathbb{R} \times \mathbb{R}$ is $(\tau_s \times \tau_s, \tau_u \times \tau_u)$-regular and $(\tau_s \times \tau_s, \tau_u \times \tau_u)$-weak P-space but not $(\tau_s \times \tau_s)$-Lindelöf, then it is not $(\tau_s \times \tau_s, \tau_u \times \tau_u)$-weakly Lindelöf by Lemma 32.

Although the product of (τ_i, τ_j)-weakly Lindelöf and (σ_i, σ_j)-weakly Lindelöf spaces need not be $(\tau_i \times \sigma_i, \tau_j \times \sigma_j)$-weakly Lindelöf, but if we add one extra condition such as τ_i-P-space to one of the space, we will obtain that the product is $(\tau_i \times \sigma_i, \tau_j \times \sigma_j)$-weakly Lindelöf.

Proposition 33 *Let (X, τ_1, τ_2) be a (τ_i, τ_j)-weakly Lindelöf τ_i-P-space and (Y, σ_1, σ_2) a (σ_i, σ_j)-weakly Lindelöf space. Then the product $(X \times Y, \rho_1, \rho_2)$ is $\left(\rho_i, \rho_j\right)$-weakly Lindelöf.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a ρ_i-open cover of $X \times Y$. Then as in the proof of Theorem 5, we may restrict our attention to the cover $\{V_\lambda \times W_\lambda : \lambda \in \Lambda\}$ of $X \times Y$ by the ρ_i-basis elements where each $V_\lambda \times W_\lambda$ is contained in some member of $\mathcal{U}$, since any countable subfamily $\{V_{\lambda_n} \times W_{\lambda_n} : n \in \mathbb{N}\}$ of this basic ρ_i-open cover such that $X \times Y = \rho_j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} (V_{\lambda_n} \times W_{\lambda_n})\right)$ will lead immediately to a countable subfamily chosen from the original cover $\mathcal{U}$ such that $X \times Y = \rho_j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}\right)$. For each $x \in X$, let $Y_x = \{x\} \times Y$ which is i-homeomorphic to Y and hence Y_x is $\left(\rho_i, \rho_j\right)$-weakly Lindelöf with respect to the inducted bitopology from (ρ_1, ρ_2). So Y_x is $\left(\rho_i, \rho_j\right)$-weakly Lindelöf relative to $X \times Y$ and since $\{V_\lambda \times W_\lambda : \lambda \in \Lambda\}$ also covers Y_x, there must exists a countable subfamily $\{V_{x,\lambda_k} \times W_{x,\lambda_k} : k \in \mathbb{N}\}$ by ρ_i-open sets which have a non-empty intersection with Y_x such that $Y_x \subseteq \rho_j\text{-cl}\left(\bigcup_{k\in\mathbb{N}} (V_{x,\lambda_k} \times W_{x,\lambda_k})\right)$. Letting $H_x = \bigcap_{k\in\mathbb{N}} V_{x,\lambda_k}$, we see that H_x is a τ_i-open set of X containing x since X is τ_i-P-space. The above countable subfamily $\{V_{x,\lambda_k} \times W_{x,\lambda_k} : k \in \mathbb{N}\}$ actually satisfies the condition $H_x \times Y \subseteq \rho_j\text{-cl}\left(\bigcup_{k\in\mathbb{N}} (V_{x,\lambda_k} \times W_{x,\lambda_k})\right)$. Now $\{H_x : x \in X\}$ is a τ_i-open cover of X. Since X is (τ_i, τ_j)-weakly Lindelöf, there exists a countable subfamily $\{H_{x_n} : n \in \mathbb{N}\}$ such that $X = \tau_j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} (H_{x_n})\right)$. But then $\{\{V_{x_n,\lambda_k} \times W_{x_n,\lambda_k} : k \in \mathbb{N}\} : n \in \mathbb{N}\}$ satisfies the condition

$$\begin{aligned} & X \times Y \\ = \; & \rho_j\text{- cl}\left(\bigcup_{n\in\mathbb{N}} \rho_j\text{- cl}\left(\bigcup_{k\in\mathbb{N}} (V_{x_n,\lambda_k} \times W_{x_n,\lambda_k})\right)\right) \\ \subseteq \; & \rho_j\text{- cl}\left(\rho_j\text{- cl}\left(\bigcup_{n\in\mathbb{N}}\left(\bigcup_{k\in\mathbb{N}} (V_{x_n,\lambda_k} \times W_{x_n,\lambda_k})\right)\right)\right) \\ = \; & \rho_j\text{- cl}\left(\bigcup_{n\in\mathbb{N}, k\in\mathbb{N}} (V_{x_n,\lambda_k} \times W_{x_n,\lambda_k})\right). \end{aligned}$$

Since $\{\{V_{x_n,\lambda_k} \times W_{x_n,\lambda_k} : k \in \mathbb{N}\} : n \in \mathbb{N}\}$ leads to a countable subfamily from $\mathcal{U}$ such that $X \times Y = \rho_j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_n}\right)$, we have that $X \times Y$ is $\left(\rho_i, \rho_j\right)$-weakly Lindelöf. ■

Corollary 34 *Let (X, τ_1, τ_2) be a pairwise weakly Lindelöf P-space and (Y, σ_1, σ_2) a pairwise weakly Lindelöf space. Then the product $(X \times Y, \rho_1, \rho_2)$ is pairwise weakly Lindelöf.*

The result of Proposition 33 can be extended to a finite collection (i, j)-weakly Lindelöf i-P-spaces and an (i, j)-weakly Lindelöf space as stated in the following corollary.

Corollary 35 *Let*

$$\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \dots, n, k \neq m, m \leq n\right\}$$

be a collection of $\left(\tau_i^k, \tau_j^k\right)$-weakly Lindelöf τ_i^k-P-spaces and $(X_m, \tau_1^m, \tau_2^m)$ a $\left(\tau_i^m, \tau_j^m\right)$-weakly Lindelöf space. Then the product $\left(\prod_{k=1}^n X_k, \rho_1, \rho_2\right)$ is $\left(\rho_i, \rho_j\right)$-weakly Lindelöf.

Proof. It follows by induction of k, the Lemma 23 and the Proposition 33. ■

Corollary 36 *Let*

$$\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \dots, n, k \neq m, m \leq n\right\}$$

be a collection of pairwise weakly Lindelöf P-spaces and $(X_m, \tau_1^m, \tau_2^m)$ a pairwise weakly Lindelöf space. Then $\left(\prod_{k=1}^n X_k, \rho_1, \rho_2\right)$ is pairwise weakly Lindelöf.

Proposition 37 *Let (X, τ_1, τ_2) be a (τ_i, τ_j)-weakly Lindelöf space and (Y, σ_1, σ_2) a σ_i-P-space. Then the projection $\pi_Y : (X \times Y, \rho_1, \rho_2) \to (Y, \sigma_1, \sigma_2)$ is i-closed where ρ_i is a product topology.*

Proof. Let U be a ρ_i-closed set in $X \times Y$ and let $y_0 \notin \pi_Y(U)$. Clearly $(X \times \{y_0\}) \cap U = \emptyset$ and so the point $(x, y_0) \notin U$ has a ρ_i-basic neighbourhood $V_x \times W_{x,y_0}$ disjoint from U where V_x is τ_i-open set in X containing x and W_{x,y_0} is σ_i-open set in Y containing y_0. Now $\{V_x \times W_{x,y_0} : x \in X\}$ form a ρ_i-basic open cover of $X \times \{y_0\}$ by ρ_i-basis elements of $X \times Y$. Since $X \times \{y_0\}$ is i-homeomorphic to X, then $X \times \{y_0\}$ is $\left(\rho_i, \rho_j\right)$-weakly Lindelöf with respect to the inducted bitopology from (ρ_1, ρ_2). So $X \times \{y_0\}$ is $\left(\rho_i, \rho_j\right)$-weakly Lindelöf relative to $X \times Y$ and hence there exists a countable subfamily $\{V_{x_n} \times W_{x_n,y_0} : n \in \mathbb{N}\}$ such that

$$\begin{aligned} & X \times \{y_0\} \\ \subseteq \; & \rho_j\text{- cl}\left(\bigcup_{n\in\mathbb{N}} (V_{x_n} \times W_{x_n,y_0})\right) \\ \subseteq \; & \rho_j\text{- cl}\left(\left(\bigcup_{n\in\mathbb{N}} V_{x_n}\right) \times \left(\bigcup_{n\in\mathbb{N}} W_{x_n,y_0}\right)\right) \\ = \; & \left(\tau_j\text{- cl}\left(\bigcup_{n\in\mathbb{N}} V_{x_n}\right)\right) \times \left(\sigma_j\text{- cl}\left(\bigcup_{n\in\mathbb{N}} W_{x_n,y_0}\right)\right). \end{aligned}$$

Set $W = \bigcap_{n\in\mathbb{N}} W_{x_n,y_0}$ and since Y is a σ_i-P-space, W is a σ_i-open neighbourhood of y_0 such that $W \cap \pi_Y(U) = \emptyset$. Thus $\pi_Y(U)$ is σ_i-closed set in Y. This implies that π_Y is i-closed and completes the proof. ■

Corollary 38 *Let (X, τ_1, τ_2) be a pairwise weakly Lindelöf space and (Y, σ_1, σ_2) a P-space. Then the projection $\pi_Y : (X \times Y, \rho_1, \rho_2) \to (Y, \sigma_1, \sigma_2)$ is closed where ρ_i is a product topology.*

Theorem 39 *Suppose that $\{(X_\alpha, \tau_1^\alpha, \tau_2^\alpha) : \alpha \in \Delta\}$ be a collection of nonempty bitopological spaces. If $\left(\prod_{\alpha\in\Delta} X_\alpha, \rho_1, \rho_2\right)$ is $\left(\rho_i, \rho_j\right)$-nearly Lindelöf $($resp. pairwise nearly Lindelöf, $\left(\rho_i, \rho_j\right)$-almost Lindelöf, pairwise almost Lindelöf, $\left(\rho_i, \rho_j\right)$-weakly Lindelöf, pairwise weakly Lindelöf, $\left(\rho_i, \rho_j\right)$-nearly compact, pairwise nearly compact$)$, then each X_α is $\left(\tau_i^\alpha, \tau_j^\alpha\right)$-nearly Lindelöf $($resp. pairwise nearly Lindelöf, $\left(\tau_i^\alpha, \tau_j^\alpha\right)$-almost Lindelöf, pairwise almost Lindelöf, $\left(\tau_i^\alpha, \tau_j^\alpha\right)$-weakly Lindelöf, pairwise weakly Lindelöf, $\left(\tau_i^\alpha, \tau_j^\alpha\right)$-nearly compact, pairwise nearly compact$)$ where ρ_i is a product topology.*

Proof. Since each projection map $\pi_\alpha : \prod_{\alpha\in\Delta} X_\alpha \to X_\alpha$ is continuous open surjection, the theorem is clearly proved. ■

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