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Isomorphism Criteria for A Subclass of Filiform Leibniz Algebras

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Abstract In the paper, we propose three isomorphism criteria for a subclass of finite-dimensional Leibniz algebras. Isomorphism Criterion 1 has been given earlier (see [5]). We introduce notations for new structure constants. Using the new notation, we state the isomorphism criterion 2. To formulate Isomorphism Criterion 3, we introduce "semi-invariant functions" needed. We prove that these three Isomorphism Criteria are equivalent. The isomorphism criterion 3 is convenient to find the invariant functions to represent isomorphism classes. The proof of the isomorphism criteria in the general case is computational and is based on hypothetic convolution identities given in [11]. Therefore, we give details in the ten-dimensional case.

Keywords Leibniz Algebra, Isomorphism Criterion, Adapted Basis

1 Introduction

The classification problem of algebras is the most important problem of algebra. In a finite-dimensional case it can be broken up into two separate tasks: the description of semisimple and solvable algebras. A semisimple algebra is written in form of a direct sum of simple one and the simple algebras are completely described. Therefore, the first task can be successfully solved, whereas the second task is one of the hardest problems of algebra. This approach was applied to classify two classes of algebras cases: associative algebras and Lie algebras. J.L.Loday [6] introduced a generalization of Lie algebra called Leibniz algebra. D.Barnes [3] proved that a finite-dimensional complex Leibniz algebra can be written as a simidirect sum of a semisimple Lie and a solvable Leibniz algebra. Thus, the classification problem of complex Leibniz algebras is reduced to the classification of the solvable Leibniz algebras. For more information and latest results on Leibniz algebras we refer the reader to [2]. In this paper, we will be dealing with a subclass of solvable Leibniz algebras in dimension n which is denoted by Lb_n, called filiform Leibniz algebras. The notion of filiform Leibniz algebra was introduced in [1]. This class arises from the naturally graded filiform Leibniz algebras. The set Lb_n is represented as a union of three subclasses $Lb_n = FLb_n \cup SLb_n \cup TLb_n$ which are stable under the action of the linear group $GL(V)$ ("automorphisms of V"). Therefore, it suffices to classify each of them separately. The classes of FLb_n and SLb_n were classified up to dimension $n \leq 9$. While the class TLb_n was classified up to dimension $n \leq 8$.

In this paper we will be treating the class FLb_n. The isomorphism criteria in low-dimensional cases, up to dimension nine, have been constructed in many previous papers. The cases from five to seven were given in [10]. The classification in dimension eight can be found in PhD Thesis [13]. The isomorphism criteria and complete lists of isomorphism classes of this class in dimension nine can be found in [8, 9, 14]. The general criterion was given in [7]. In solving the classification problem in these papers, it was noticed that the usage of Criterion 1 is very computational. Therefore, it was decided to suggest a few more isomorphism criteria. The present paper is devoted to create three isomorphism criteria (called Criterion 1, Criterion 2 and Criterion 3) for FLb_n and each one is an improvement on the previous one. The need of such an improvement is motivated by the discussion made in the section Preliminaries. We prove the equivalence of the criteria. The details of the computations are given in dimension ten.

2 Preliminaries

Let G be a group and X be a G-set. An invariant function is a function $f : X \to K$ such that $f(g \cdot x) = f(x)$ for any $g \in G$ and $x \in X$.

Let K be an algebraically closed field and G be an algebraic group acting on an irreducible algebraic variety X. This action generates an action of G on the field of rational function $K(X)$

of X :

$$(g \cdot f)(x) = f(g^{-1} \cdot x), \text{ for all } f \in K(X).$$

The set of all invariant rational functions $K(X)^G$ forms a subfield of $K(X)$. Being a subfield of a finitely generated field $K(X)$ the field $K(X)^G$ possess finitely many generators.

It is said that an invariant function f separates orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ if it is defined on the both orbits and takes on them different values. A set of invariant functions S separates orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ if S contains an element separating these two orbits. We say that a set of invariants S separates generic orbits if there exists a non-empty open subset X_0 of X such that the set S separates orbits in X_0. The existence such a set of invariants S for the case where an algebraic group G acts on an irreducible algebraic variety X has been given by M. Rosenlicht (see [12] and [4] for a modern version). However, the description and the number of its elements depend on the representation considered. The following result holds true (see [15]).

Lemma 1 *If a finite set $S \subset K(X)^G$ separates orbits of points in general position then S generates $K(X)^G$.*

M. Rosenlicht has also shown that orbits in general position in X can be separated by rational invariants.

It is said that $f \in K(X)$ has invariant property over an orbit $\mathcal{O}$ if it is defined at any element of $\mathcal{O}$. Also $f|_{\mathcal{O}}$ is constant or it does not vanish at any element of $\mathcal{O}$. That is $f|_{\mathcal{O}}$ does not vanish at all. Let us denote by $In(\mathcal{O})$ the set of all such functions. We say that $f \in K(X)$ separates orbits $\mathcal{O}_1, \mathcal{O}_2$ if $f \in In(\mathcal{O}_1) \cap In(\mathcal{O}_2)$ and $f|_{\mathcal{O}_1}, f|_{\mathcal{O}_2}$ are constant functions with different values or one of them is identically zero and the second one does not vanish at all.

We say that a finite set S of elements of $K(X)$ separates a given set of orbits if for any two orbits, there exists an element of S which separates these two orbits. If we look for a finite set of functions separating all and not only generic orbits we need to construct a finite set of invariant functions. In this sense, one should expect that in most, if not in all cases, there exists a finite set of functions separating all orbits.

An algebra structure $L = (V, \lambda)$ on an n-dimensional vector space V is element $\lambda(L)$ of $Alg_n(K) \cong K^{n^3}$, where $\lambda\colon V \otimes V \to V$ is a bilinear mapping as a binary algebraic operation on L. Let $\{e_1, e_2, ..., e_n\}$ be a basis of the algebra L, then the table of multiplication of L is given by a point (γ_{ij}^k) of K^{n^3} as follows:

$$\lambda(e_i, e_j) = \sum_{k=1}^{n} \gamma_{ij}^k e_k \text{ where } i, j = 1, 2, ..., n.$$

The entries γ_{ij}^k of the cubic matrix $\left[\gamma_{ij}^k\right]_{i,j,k=1,2,...,n}$ are called structure constants of L. We consider the action $(g \cdot \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y)))$ of the automorphism group $GL(V)$ on $Alg_n(K)$.

Definition 1 *Two algebra structures (V, λ_1) and (V, λ_2) are said to be isomorphic if there exists $g \in GL(V)$ such that*

$$\lambda_2(x, y) = (g \cdot \lambda_1)(x, y) = g^{-1}(\lambda_1(g(x), g(y)))$$

for all $x, y \in V$.

Each orbit consists of isomorphic to each other algebras from $Alg_n(K)$. The set of all laws isomorphic to λ is denoted by $O(\lambda)$ (the orbit of λ).

Throughout the work we assume that the algebras considered are defined over the field of complex numbers C and denote the law $\lambda(\cdot, \cdot)$ by the bracket notation $[\cdot, \cdot]$.

Definition 2 *An algebra L is called a Leibniz algebra, if $[\cdot, \cdot]$ satisfies the following Leibniz identity:*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

for all $x, y, z \in L$.

Let LB_n be the set of all Leibniz algebra laws on V. Note that LB_n is subvariety of $Alg_n(K)$ specified by the Leibniz identity above.

We define the lower central series as follows:

$$L^1 = L, \; L^{k+1} = [L^k, L], \; k \in N. \tag{1}$$

Definition 3 *A Leibniz algebra L is called nilpotent if the series* (1) *terminates, i.e., there exists an integer $s \in N$ such that*

$$L^1 \supset L^2 \supset \ldots \supset L^s = 0.$$

If the termination occurs by the rule $dim\ L^i = n - i$, where $2 \le i \le n$ then the n-dimensional Leibniz algebra L is said to be filiform.

There is a so-called adapted basis $\{e_0, e_1, .., e_n\}$ (see [5]) of the underlying vector space V of FLb_{n+1} with respect to that the table of multiplication of algebras from FLb_{n+1} is given as follows:

$$FLb_{n+1} = \begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, i = 1, 2, ..., n-1 \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + ... + \alpha_{n-1} e_{n-1} + \theta e_n, \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + ... + \alpha_{n+1-j} e_n, \\ j = 1, 2, ..., n-2 \text{ and } \alpha_3, \alpha_4, ..., \alpha_n, \theta \in \mathrm{C}. \end{cases}$$

Elements of FLb_{n+1} we denote by $L(\alpha_3, \alpha_4, ..., \alpha_n, \theta)$. Let L be a filiform Leibniz algebra with the adapted basis $\{e_0, e_1, e_2, ..., e_n\}$.

Definition 4 *The basis transformation $\sigma \in GL(V)$ is said to be adapted, if the set of vectors $\{\sigma(e_0), \sigma(e_1), ..., \sigma(e_n)\}$ is adapted.*

A subgroup of $GL(V)$ consisting of all adapted transformations of FLb_{n+1} is denoted by G_{ad}. In G_{ad}, we consider the following transformations of FLb_{n+1} called elementary.

$$\text{First type } \tau_{A,B}(k) = \begin{cases} \sigma(e_0) = e_0 + Ae_k, \\ \sigma(e_1) = e_1 + Be_k, \\ \text{where } A, B \in \mathrm{C} \\ \text{and } k = 2, 3, ..., n, \\ \sigma(e_{i+1}) = [\sigma(e_i), \sigma(e_0)], \\ i = 1, 2, ..., n-1, \\ \sigma(e_2) = [\sigma(e_0), \sigma(e_0)]. \end{cases}$$

$$\text{Second type } \nu_{A,B} = \begin{cases} \sigma(e_0) = Ae_0 + Be_1, \\ \sigma(e_1) = (A+B)e_1 \\ \quad +b(\theta - \alpha_n)e_{n-1}, \\ \text{where } A, B, \theta, \alpha_n \in \mathrm{C} \\ \text{and } A(A+B) \neq 0 \\ \sigma(e_{i+1}) = [\sigma(e_i), \sigma(e_0)], \\ i = 1, 2, ..., n-1, \\ \sigma(e_2) = [\sigma(e_0), \sigma(e_0)]. \end{cases}$$

The proof of the following proposition is straightforward.

Proposition 1

1. *Let σ be an adapted transformation of FLb_{n+1}, then* $\sigma = \tau_{a_n,b_n}(n) \circ \tau_{a_{n-1},a_{n-1}}(n-1) \circ ... \circ \tau_{a_2,a_2}(2) \circ \nu_{a_0,a_1}$.
2. *The structure constants of algebras from FLb_{n+1} are stable under the transformation*

$$\tau_{a_n,b_n}(n) \circ \tau_{a_{n-1},a_{n-1}}(n-1) \circ ... \circ \tau_{a_2,a_2}(2).$$

Due to the proposition above the action of G_{ad} on algebras from FLb_{n+1} is reduced to that of $\nu_{a,b}$.

3 Isomorphism Criteria

In this section we give the isomorphism criteria for elements of FLb_{n+1} in terms of the structure constants.

Theorem 1 (Creterion 1) *The algebras $L(\alpha)$ and $L(\alpha')$ from FLb_{n+1}, with $\alpha = (\alpha_3, \alpha_4, ..., \alpha_n, \theta)$ and $\alpha' = (\alpha'_3, \alpha'_4, ..., \alpha'_n, \theta')$ are isomorphic, if and only if there exist A and B ($A, B \in$ C), such that $A(A+B) \neq 0$ and the following conditions hold true*

$$\alpha'_t = \frac{1}{A^{t-2}} \varphi_t \left(\frac{B}{A}; \alpha\right), \quad 3 \le t \le n, \tag{2}$$

$$\theta' = \frac{1}{A^{n-2}} \varphi_{n+1} \left(\frac{B}{A}; \alpha\right), \tag{3}$$

where

$$\begin{aligned}
&\varphi_t(y; \mathbf{u}) \\
&= \varphi_t(y; u_3, u_4, ..., u_n, u_{n+1}) \\
&= (1+y)u_t - \sum_{k=3}^{t-1} \Bigg(\tbinom{k-1}{k-2} y u_{t+2-k} + \tbinom{k-1}{k-3} y^2 \sum_{i_1=k+2}^{t} u_{t+3-i_1} \\
&\quad \cdot u_{i_1+1-k} + \tbinom{k-1}{k-4} y^3 \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{i_2} u_{t+3-i_2} \\
&\quad \cdot u_{i_2+3-i_1} \cdot u_{i_1-k} + ... + \tbinom{k-1}{1} y^{k-2} \sum_{i_{k-3}=2k-2}^{t} \\
&\quad \cdot \sum_{i_{k-4}=2k-2}^{i_{k-3}} ... \sum_{i_1=2k-2}^{i_2} u_{t+3-i_{k-3}} \cdot u_{i_{k-3}+3-i_{k-4}} \cdot ... \cdot u_{i_2+3-i_1} \\
&\quad \cdot u_{i_1+5-2k} + y^{k-1} \sum_{i_{k-2}=2k-1}^{t} \sum_{i_{k-3}=2k-1}^{i_{k-2}} ... \sum_{i_1=2k-1}^{i_2} u_{t+3-i_{k-2}} \\
&\quad \cdot u_{i_{k-2}+3-i_{k-3}} \cdot ... \cdot u_{i_2+3-i_1} \cdot u_{i_1+4-2k} \Bigg) \cdot \varphi_k(y; u), \\
&\qquad \text{for } 3 \le t \le n, \\
&\varphi_{n+1}(y; \mathbf{u}) \\
&= \varphi_{n+1}(y; u_3, u_4, ..., u_n, u_{n+1}) \\
&= u_{n+1} - u_n + \varphi_n(y; \mathbf{u}).
\end{aligned}$$

Proof. The proof was given in [5].

In order to create the next isomorphism criteria we introduce the following notations:

$$\begin{aligned}
\Delta_3 &= \alpha_3, \\
\Theta &= \theta - \alpha_s \\
\Delta_s &= \alpha_s + (-1)^s C_{s-2} \alpha_3^{s-2}
\end{aligned} \tag{4}$$

$$\begin{aligned}
\Delta'_3 &= \alpha'_3, \\
\Theta' &= \theta' - \alpha'_s \\
\Delta'_s &= \alpha'_s + (-1)^s C_{s-2} \alpha_3'^{s-2},
\end{aligned} \tag{5}$$

where $s = 4, 5, \ldots$ and $C_k = \frac{(2k!)}{(k+1)!k!}$ are Catalan numbers for $k = 0, 1, \ldots$

Remark. An adapted basis change from the basis $\{e_0, e_1, \ldots, e_n\}$ to $\{e'_0, e'_1, \ldots, e'_n\}$ gives rise the change of the structure constants $\alpha_3, \alpha_4, \ldots, \alpha_n, \theta$ to $\Delta_3, \Delta_4, \Delta_5, \ldots, \Delta_n, \Theta$. We keep use the notation $L(\mathbf{\Delta}) = L(\Delta_3, \Delta_4, \Delta_5, \ldots, \Delta_n, \Theta)$ for algebras from FLb_{n+1}.

We introduce the notations $\Delta_3 = \alpha_3$, $\Delta'_3 = \alpha'_3$ and the functions $\Psi_t(y; \mathbf{u}), t = 3, 4, \ldots, n+1$ as follows:

$$\begin{aligned}
&\Psi_t(y; \mathbf{u}) \\
&= \Psi_t(y; u_3, u_4, \ldots, u_n, u_{n+1}) \\
&= u_t - \sum_{k=3}^{t-1} \Bigg(\begin{pmatrix} k-1 \\ k-2 \end{pmatrix} y (u_{t+2-k} + (-1)^{t+1-k} C_{t-k} u_3^{t-k}) \\
&+ \begin{pmatrix} k-1 \\ k-3 \end{pmatrix} y^2 \sum_{i_1=k+3}^{t} (u_{t+3-i_1} + (-1)^{t-i_1} C_{t+1-i_1} u_3^{t+1-i_1}) \\
&(u_{i_1+1-k} + (-1)^{i_1-k} C_{i_1-1-k} u_3^{i_1-1-k}) + \begin{pmatrix} k-1 \\ k-4 \end{pmatrix} y^3 \sum_{i_2=k+3}^{t} \\
&\sum_{i_1=k+3}^{i_2} (u_{t+3-i_2} + (-1)^{t-i_2} C_{t+1-i_2} u_3^{t+1-i_2}).(u_{i_2+3-i_1} \\
&+(-1)^{i_2-i_1} C_{i_2+1-i_1} u_3^{i_2+1-i_1})(u_{i_1-k} + (-1)^{i_1-k-1} C_{i_1-k-2} \\
&\cdot u_{i_2+1-i_1})(u_{i_1-k} + (-1)^{i_1-k-1} C_{i_1-k-2} u_3^{i_1-k-2}) + \ldots \\
&+ \begin{pmatrix} k-1 \\ 1 \end{pmatrix} y^{k-2} \sum_{i_{k-3}=2k-2}^{t} \sum_{i_{k-4}=2k-2}^{i_{k-3}} \cdots \sum_{i_1=2k-2}^{i_2} (u_{t+3-i_{k-3}} \\
&+(-1)^{t-i_{k-3}} C_{t+1-i_{k-3}} u_3^{t+1-i_{k-3}})(u_{i_{k-3}+3-i_{k-4}} \\
&+(-1)^{i_{k-3}-i_{k-4}} C_{i_{k-3}-i_{k-4}} u_3^{i_{k-3}+1-i_{k-4}}) \ldots (u_{i_2+3-i_1} \\
&+(-1)^{i_2-i_1} C_{i_2+1-i_1} u_3^{i_2+1-i_1})(u_{i_1+5-2k} + (-1)^{i_2-i_1} C_{i_2+3-i_1} \\
&\cdot u_3^{i_2+3-i_1}) + y^{k-1} \sum_{i_{k-2}=2k-1}^{t} \sum_{i_{k-3}-2k-1}^{i_{k-2}} \cdots \sum_{i_1=2k-1}^{i_2} (u_{t+3-i_{k-2}} \\
&\quad +(-1)^{i_{k-2}-i_{k-3}} C_{i_{k-2}+1-i_{k-3}} u_3^{i_{k-2}+1-i_{k-3}}) \ldots (u_{i_2+3-i_1} \\
&\qquad +(-1)^{t-i_{k-2}} C_{t+1-i_{k-2}} u_3^{t+1-i_{k-2}})(u_{i_{k-2}+3-i_{k-3}} \\
&\quad +(-1)^{i_2-i_1} C_{i_2+1-i_1} u_3^{i_2+1-i_1}).\ (u_{i_1+4-2k} + (-1)^{i_1+1-2k} \\
&\qquad C_{i_1+2-2k} u_3^{i_1+2-2k}) \Bigg) \Psi_k(y; \mathbf{u}) + (-1)^t (1+y)^{t-3} u_3^{t-3} \\
&\qquad\qquad - (-1)^{t-2} C_{t-2} u_3^{t-2}.
\end{aligned}$$

The following theorem holds true.

Theorem 2 (Criterion 2) *The algebras $L(\mathbf{\Delta})$ and $L(\mathbf{\Delta}')$ from FLb_{n+1}, with*

$$\mathbf{\Delta} = (\Delta_3, \Delta_4, \ldots, \Delta_n, \Theta),$$

and

$$\mathbf{\Delta}' = (\Delta'_3, \Delta'_4, \ldots, \Delta'_n, \Theta'),$$

are isomorphic if and only if there exist A and B ($A, B \in$ C), such that $A(A+B) \neq 0$ and the following conditions hold true

$$\begin{aligned}
\Delta'_t &= \tfrac{1}{A^{t-2}} \left(1 + \tfrac{B}{A}\right) \Psi_t(\tfrac{B}{A}; \mathbf{\Delta}), \\
&\qquad 3 \le t \le n, \\
\Theta' &= \tfrac{1}{A^{n-2}} \Theta
\end{aligned}$$

In order to state Criterion 3 we consider functions defined as follows:

$$
\begin{aligned}
& f_t(\mathbf{u}) \\
& = f_t(u_3,u_4,\dots,u_n,u_{n+1}) = \Psi_t(-1;\mathbf{u}) \\
& = \Psi_t(-1;u_3,u_4,\dots,u_n,u_{n+1}) \\
& = u_t + \sum_{k=3}^{t-1}\Bigg(\left(\begin{array}{c} k-1 \\ k-2 \end{array}\right)(u_{t+2-k}+(-1)^{t+1-k}C_{t-k}u_3^{t-k}) \\
& -\left(\begin{array}{c} k-1 \\ k-3 \end{array}\right)\sum_{i_1=k+2}^{t}(u_{t+3-i_1}+(-1)^{t-i_1}C_{t+1-i_1}u_3^{t+1-i_1}) \\
& \left(u_{i_1+1-k}+(-1)^{i_1-k}C_{i_1-1-k}u_3^{i_1-1-k}\right)+\left(\begin{array}{c} k-1 \\ k-4 \end{array}\right) \\
& \cdot\sum_{i_2=k+3}^{t}\sum_{i_1=k+3}^{i_2}(u_{t+3-i_2}+(-1)^{t-i_2}C_{t+1-i_2}u_3^{t+1-i_2}) \\
& \cdot(u_{t_2+2-i_1}+(-1)^{i_2-i_1}C_{i_2+1-i_1}u_3^{i_2+1-i_1}) \\
& \cdot(u_{i_1-k}+(-1)^{i_1-k-1}C_{i_1-k-2}u_3^{i_1-k-2})+\dots \\
& +(-1)^{k-2}\left(\begin{array}{c} k-1 \\ 1 \end{array}\right)\sum_{i_{k-3}=2k-2}^{t}\sum_{i_{k-4}=2k-2}^{i_{k-3}}\cdots \\
& \cdot\sum_{i_1=2k-2}^{i_2}(u_{t+3-i_{k-3}}+(-1)^{t-i_{k-3}}C_{t+1-i_{k-3}}u_3^{t+1-i_{k-3}}) \\
& \cdot(u_{i_{k-3}+3-i_{k-4}}+(-1)^{i_{k-3}-i_{k-4}}C_{i_{k-3}+1-i_{k-4}}u_3^{i_{k-3}+1-i_{k-4}}) \\
& \dots(u_{i_2+3-i_1}+(-1)^{i_2-i_1}C_{i_2+1-i_1}u_3^{i_2+1-i_1})\cdot(u_{i_1+5-2k}+ \\
& (-1)^{i_2-i_1}C_{i_2+3-i_1}u_3^{i_2+3-i_1})+(-1)^{k-1}\sum_{i_{k-2}=2k-1}^{t}\sum_{i_{k-3}=2k-1}^{i_{k-2}} \\
& \dots\sum_{i_1=2k-1}^{i_2}(u_{t+3-i_{k-2}}+(-1)^{t-i_{k-2}}C_{t+1-i_{k-2}}u_3^{t+1-i_{k-2}}) \\
& (u_{i_{k-2}+3-i_{k-3}}+(-1)^{i_{k-2}-i_{k-3}}C_{i_{k-2}+1-i_{k-3}}u_3^{i_{k-2}+1-i_{k-3}}) \\
& \dots(u_{i_2+3-i_1}+(-1)^{i_2-i_1}C_{i_2+1-i_1}u_3^{i_2+1-i_1}) \\
& (u_{i_1+4-2k}+(-1)^{i_1+1-2k}C_{i_1+2-2k}u_3^{i_1+2-2k})\Bigg)f_k(\mathbf{u}) \\
& +(-1)^{t-1}C_{t-2}u_3^{t-2}, \quad \text{for } 3\le t\le n.
\end{aligned}
$$

The Criterion 3 is now stated as follows.

Theorem 3 (Criterion 3) *The algebras $L(\mathbf{\Delta})$ and $L(\mathbf{\Delta}')$ from FLb_{n+1}, with*

$$\mathbf{\Delta} = (\Delta_3,\Delta_4,\dots,\Delta_n,\Theta),$$

and

$\mathbf{\Delta}' = (\Delta'_3,\Delta'_4,\dots,\Delta'_n,\Theta')$, *are isomorphic if and only if there exist $A,B\in$ C, such that $A(A+B)\neq 0$ and the following conditions hold true:*

$$f_t(\mathbf{\Delta}') = \tfrac{1}{A^{t-2}}\left(1+\tfrac{B}{A}\right)f_t(\mathbf{\Delta}), \quad 3\le t\le n.$$

and

$$\Theta' = \tfrac{1}{A^{n-2}}\Theta.$$

Theorem 4 *For Leibniz algebras from FLb_n the ctiteria* **Criterion 1**, **Criterion 2** *and* **Criterion 3** *are equivalent.*

Proof. The proof is based on the convolution identities below from [11].

Let $F(n) = t-2-\sum_{i=1}^{n}s_i$, where n and t ($t\ge 3$) are natural numbers.

Theorem 5 *For any positive integers t ($t\ge 3$) and k ($1\le k\le t-3$), the following convolution identities hold true*

$$
\begin{aligned}
A(k)= & \sum_{s_1=1}^{t-3}\tbinom{s_1+1}{s_1}\tbinom{s_1-1}{k-1}C_{s_1}C_{F(1)} \\
& -\sum_{s_1=1}^{t-3}\tbinom{s_1+1}{s_1-1}\tbinom{s_1-1}{k-2}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}C_{F(2)} \\
& +\sum_{s_1=2}^{t-3}\tbinom{s_1+1}{s_1-2}\tbinom{s_1-1}{k-3}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\sum_{s_3=1}^{F(2)-1}C_{s_3}C_{F(3)} \\
& -\sum_{s_1=3}^{t-3}\tbinom{s_1+1}{s_1-3}\tbinom{s_1-1}{k-4}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\sum_{s_3=1}^{F(2)-1}C_{s_3}\sum_{s_4=1}^{F(4)-1}C_{s_4}C_{F(4)} \\
& +\ldots+(-1)^r\sum_{s_1=r}^{t-3}\tbinom{s_1+1}{s_1-r}\tbinom{s_1-1}{k-(r+1)}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\dots \\
& \sum_{s_{r+1}=1}^{F(r)-1}C_{s_{r+1}}C_{F(r+1)}+\ldots+(-1)^{k-2}\sum_{s_1=k-2}^{t-3}\tbinom{s_1+1}{s_1-(k-2)} \\
& \cdot\tbinom{s_1-1}{1}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\cdots\sum_{s_{k-1}=1}^{F(k-2)-1}C_{s_{k-1}}C_{F(k-1)} \\
& +(-1)^{k-1}\sum_{s_1=k-1}^{t-3}\tbinom{s_1+1}{s_1-(k-1)}C_{s_1}\sum_{s_2=1}^{F(1)-1}C_{s_2}\cdots\sum_{s_{k-1}=1}^{F(k-2)-1} \\
& C_{k-1}\sum_{s_k=1}^{F(k-1)-1}C_{s_k}C_{F(k)} = \tbinom{t-3}{k}C_{t-2}.
\end{aligned}
$$

Indeed, to get **Criterion 2** from **Criterion 1** we write formulas for Δ'_n and Θ', from (4) and express them via Δ_i and Θ, where $i=3,\dots n-1$. To simplify we make use the identities of Theorem 5.

To get **Criterion 3** from **Criterion 2** we plug in Δ'_i and Θ', where $i=3,\dots n-1$ in the functions f_j, where $j=3,4,\dots n$.

To get **Criterion 1** from **Criterion 3** we solve the equations $f_t(\mathbf{\Delta}') = \frac{1}{A^{t-2}}\left(1+\frac{B}{A}\right)f_t(\mathbf{\Delta})$, $3\le t\le n$ and $\Theta' = \frac{1}{A^2}\Theta$ for α'_i, where $i=3,4,\dots,n$ expressing Δ'_i, Θ' and Δ_i where $i=3,\dots n$ via α_i.

The details for 10-dimensional case are given in the next section.

4 DETAILS FOR FLb_{10}

In this section, we prove the equivalence of the criteria for FLb_{10} by using explicit formulas.

Proposition 2 (Criterion 1) *Algebras $L(\alpha_3,\alpha_4,\dots,\alpha_9,\theta)$ and $L(\alpha'_3,\alpha'_4,\dots,\alpha'_9,\theta')$ are isomorphic if and only if there exist $A,B\in$ C such that $A(A+B)\neq 0$ and the following conditions hold true:*

$$
\begin{aligned}
\alpha'_3 = & \tfrac{1}{A}\left(1+\tfrac{B}{A}\right)\alpha_3, \\
\alpha'_4 = & \tfrac{1}{A^2}\left(1+\tfrac{B}{A}\right)\left[\alpha_4-2\tfrac{B}{A}\alpha_3^2\right], \\
\alpha'_5 = & \tfrac{1}{A^3}\left(1+\tfrac{B}{A}\right)\left[\alpha_5-5\tfrac{B}{A}\alpha_3\alpha_4+5\left(\tfrac{B}{A}\right)^2\alpha_3^3\right], \\
\alpha'_6 = & \tfrac{1}{A^4}\left(1+\tfrac{B}{A}\right)\Big[\alpha_6-6\tfrac{B}{A}\alpha_3\alpha_5+21\left(\tfrac{B}{A}\right)^2\alpha_3^2\alpha_4 \\
& -3\tfrac{B}{A}\alpha_4^2-14\left(\tfrac{B}{A}\right)^3\alpha_3^4\Big], \\
\alpha'_7 = & \tfrac{1}{A^5}\left(1+\tfrac{B}{A}\right)\Big[\alpha_7-7\tfrac{B}{A}\alpha_3\alpha_6+28\left(\tfrac{B}{A}\right)^2\alpha_3\alpha_4^2 \\
& +28\left(\tfrac{B}{A}\right)^2\alpha_3^2\alpha_5-7\tfrac{B}{A}\alpha_4\alpha_5-84\left(\tfrac{B}{A}\right)^3\alpha_3^3\alpha_4 \\
& +42\left(\tfrac{B}{A}\right)^4\alpha_3^5\Big], \\
\alpha'_8 = & \tfrac{1}{A^6}\left(1+\tfrac{B}{A}\right)\Big[\alpha_8-8\tfrac{B}{A}\alpha_3\alpha_7+36\left(\tfrac{B}{A}\right)^2\alpha_3^2\alpha_6 \\
& +72\left(\tfrac{B}{A}\right)^2\alpha_3\alpha_4\alpha_5-8\tfrac{B}{A}\alpha_4\alpha_6+12\left(\tfrac{B}{A}\right)^2\alpha_4^3 \\
& -180\left(\tfrac{B}{A}\right)^3\alpha_3^2\alpha_4^2-120\left(\tfrac{B}{A}\right)^3\alpha_3^3\alpha_5 \\
& +330\left(\tfrac{B}{A}\right)^4\alpha_3^4\alpha_4-4\tfrac{B}{A}\alpha_5^2-132\left(\tfrac{B}{A}\right)^5\alpha_6^3\Big], \\
\alpha'_9 = & \tfrac{1}{A^7}\left(1+\tfrac{B}{A}\right)\Big[\alpha_9-9\tfrac{B}{A}\alpha_3\alpha_8+45\left(\tfrac{B}{A}\right)^2\alpha_3^2\alpha_7 \\
& -165\left(\tfrac{B}{A}\right)^3\alpha_3^3\alpha_6+495\left(\tfrac{B}{A}\right)^4\alpha_3^4\alpha_5 \\
& -9\tfrac{B}{A}\alpha_4\alpha_7+45\left(\tfrac{B}{A}\right)^2\alpha_4^2\alpha_5+90\left(\tfrac{B}{A}\right)^2\alpha_3\alpha_4\alpha_6
\end{aligned}
$$

$$-495\left(\frac{B}{A}\right)^3\alpha_3^2\alpha_4\alpha_5+990\left(\frac{B}{A}\right)^4\alpha_3^3\alpha_4^2$$
$$-9\frac{B}{A}\alpha_5\alpha_6+45\left(\frac{B}{A}\right)^2\alpha_3\alpha_5^2-165\left(\frac{B}{A}\right)^3\alpha_3\alpha_4^3$$
$$\left.-1287\left(\frac{B}{A}\right)^5\alpha_3^5\alpha_4+429\left(\frac{B}{A}\right)^6\alpha_3^7\right].$$

According to the notations (4) we have:

$$\begin{array}{ll}
\Delta_3=\alpha_3, & \Delta_3'=\alpha_3',\\
\Delta_4=\alpha_4+2\alpha_3^2, & \Delta_4'=\alpha_4'+2\alpha_3'^2,\\
\Delta_5=\alpha_5-5\alpha_3^3, & \Delta_5'=\alpha_5'-5\alpha_3'^3,\\
\Delta_6=\alpha_6+14\alpha_3^4, & \Delta_6'=\alpha_6'+14\alpha_3'^4,\\
\Delta_7=\alpha_7-42\alpha_3^5, & \Delta_7'=\alpha_7'-42\alpha_3'^5,\\
\Delta_8=\alpha_8+132\alpha_3^6, & \Delta_8'=\alpha_8'+132\alpha_3'^6,\\
\Delta_9=\alpha_9-429\alpha_3^7, & \Delta_9'=\alpha_9'-429\alpha_3'^7,\\
\Theta=\theta-\alpha_9, & \Theta'=\theta'-\alpha_9'.
\end{array}$$

Note that the equivalence of the criteria for FLb_n up to $n=8$ has been done earlier, we will be dealing with $n=9$.

Proposition 3 (Criterion 2) *The algebras*

$$L(\Delta_3,\Delta_4,\Delta_5,\Delta_6,\Delta_7,\Delta_8,\Delta_9,\Theta)$$

and

$$L(\Delta_3',\Delta_4',\Delta_5',\Delta_6',\Delta_7',\Delta_8',\Delta_9',\Theta')$$

from FLb_{10} are isomorphic if and only if

$$\begin{array}{rl}
\Delta_3'= & \frac{1}{A}\left(1+\frac{B}{A}\right)\Delta_3\\
\Delta_4'= & \frac{1}{A^2}\left(1+\frac{B}{A}\right)\Delta_4\\
\Delta_5'= & \frac{1}{A^3}\left(1+\frac{B}{A}\right)\left(\Delta_5-5\frac{B}{A}\Delta_3\Delta_4\right)\\
\Delta_6'= & \frac{1}{A^4}\left(1+\frac{B}{A}\right)\Big(\Delta_6-6\frac{B}{A}\Delta_3\Delta_5+21\left(\frac{B}{A}\right)^2\Delta_3^2\Delta_4\\
& -3\frac{B}{A}\Delta_4^2+12\frac{B}{A}\Delta_4\Delta_3^2\Big)\\
\Delta_7'= & \frac{1}{A^5}\left(1+\frac{B}{A}\right)\big(\Delta_7-7\frac{B}{A}\big(\Delta_3\Delta_6+\Delta_4\Delta_5+5\Delta_3^3\Delta_4\\
& -2\Delta_3^2\Delta_5\big)+28\Delta_3\left(\frac{B}{A}\right)^2\big(\Delta_4^2+\Delta_3\Delta_5-4\Delta_3^2\\
& -3\frac{B}{A}\Delta_3^2\Delta_4\big)\big)\\
\Delta_8'= & \frac{1}{A^6}\left(1+\frac{B}{A}\right)\big(\Delta_8-4\frac{B}{A}\big(2\Delta_3\Delta_7-4\Delta_3^2\Delta_6+\Delta_5^2\\
& -28\Delta_4\Delta_3^4+10\Delta_3^3\Delta_5\big)+12\left(\frac{B}{A}\right)^2\big(3\Delta_3^2\Delta_6\\
& -12\Delta_3^3\Delta_5-6\Delta_3^2\Delta_4^2+42\Delta_3^4\Delta_4+\Delta_4^3\big)\\
& -60\left(\frac{B}{A}\right)^3\big(2\Delta_3^3\Delta_5+3\Delta_3^2\Delta_4^2-12\Delta_3^4\Delta_4\big)\\
& +330\left(\frac{B}{A}\right)^4\Delta_3^4\Delta_4\Big)
\end{array}$$

Introduce the following functions:

$$\begin{array}{rcl}
\Psi_5(-1;\mathbf{u}) & = & f_5(u_3,u_4,u_5)\\
& = & u_5+5u_3u_4;\\
\Psi_6(-1;\mathbf{u}) & = & f_6(u_3,u_4,u_5,u_6)\\
& = & u_6+6u_3u_5+9u_3^2+u_4+3u_4^2;\\
\Psi_7(-1;\mathbf{u}) & = & f_7(u_3,u_4,u_5,u_6,u_7)\\
& = & u_7+7u_3u_6+7u_4u_5+7u_3^3,u_4\\
& & +14u_3^2u_5+28u_3u_4^2;\\
\Psi_8(-1;\mathbf{u}) & = & f_8(u_3,u_4,u_5,u_6,u_7,u_8)\\
& = & u_8+8u_3u_7+8u_4u_6+20u_3^2u_6+4u_5^2+2u_3^4\\
& & +16u_3^3u_5+72u_3u_4u_5+108u_3^2u_4^2+12u_4^3;\\
\Psi_9(-1;\mathbf{u}) & = & f_9(u_3,u_4,u_5,u_6,u_7,u_8,u_9)\\
& = & u_9+9u_3u_8+9u_4u_7+9u_5u_6+45u_3^2u_7\\
& & +90u_3u_4u_6+45u_3u_5^2+45u_4^2u_5+165u_3^3u_6\\
& & +495u_3^2u_4u_5+165u_3u_4^3+495u_3^4u_5\\
& & +990u_3^3u_4^2+1287u_3^5u_4,
\end{array}$$

where $\mathbf{u}=(u_3,u_4,\dots,u_i)$, $i=5,...,9$.

Proposition 4 (Criterion 3) *For algebras*

$$L(\Delta_3,\Delta_4,\Delta_5,\Delta_6,\Delta_7,\Delta_8,\Delta_9,\Theta)$$

and

$$L(\Delta_3',\Delta_4',\Delta_5',\Delta_6',\Delta_7',\Delta_8',\Delta_9',\Theta')$$

from FLb_{10} to be isomorphic, the validity of the following equalities are necessary and sufficient

$$\Delta_3'=\frac{1}{A}\left(1+\frac{B}{A}\right)\Delta_3$$

$$\Delta_4'=\frac{1}{A}\left(1+\frac{B}{A^2}\right)\Delta_4$$

$$f_5(\Delta_3',\Delta_4',\Delta_5')=\frac{1}{A^3}\left(1+\frac{B}{A}\right)f_5(\Delta_3,\Delta_4\Delta_5)$$

$$f_6(\Delta_3',\Delta_4',\Delta_5',\Delta_6')=\frac{1}{A^4}\left(1+\frac{B}{A}\right)f_6(\Delta_3,\Delta_4,\Delta_5,\Delta_6)$$

$$f_7(\Delta_3',\Delta_4',\Delta_5',\Delta_6',\Delta_7')$$

$$=\frac{1}{A^5}\left(1+\frac{B}{A}\right)f_7(\Delta_3,\Delta_4,\Delta_5,\Delta_6,\Delta_7)$$

$$f_8(\Delta_3',\Delta_4',\Delta_5',\Delta_6',\Delta_7',\Delta_8')$$

$$=\frac{1}{A^6}\left(1+\frac{B}{A}\right)f_8(\Delta_3,\Delta_4,\Delta_5,\Delta_6,\Delta_7,\Delta_8)$$

$$f_9(\Delta_3',\Delta_4,\Delta_5',\Delta_6',\Delta_7',\Delta_8',\Delta_9')$$

$$=\frac{1}{A^7}\left(1+\frac{B}{A}\right)f_9(\Delta_3,\Delta_4,\Delta_5,\Delta_6,\Delta_7,\Delta_8,\Delta_9).$$

Criterion 1 $\Longrightarrow$ Criterion 2:

$$\Delta_9' = \frac{1}{A_7}\left(1+\frac{B}{A}\right)\Big[\alpha_9 - 9\frac{B}{A}\alpha_3\alpha_8 + 45\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_7$$
$$-165\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_6 + 495\left(\frac{B}{A}\right)^4\alpha_3^4\alpha_5 - 9\frac{B}{A}\alpha_4\alpha_7$$
$$+45\left(\frac{B}{A}\right)^2\alpha_4^2\alpha_5 + 90\left(\frac{B}{A}\right)^2\alpha_3\alpha_4\alpha_6$$
$$-495\left(\frac{B}{A}\right)^3\alpha_3^2\alpha_4\alpha_5 + 990\left(\frac{B}{A}\right)^4\alpha_3^3\alpha_4^2 - 9\frac{B}{A}\alpha_5\alpha_6$$
$$+45\left(\frac{B}{A}\right)^2\alpha_3\alpha_5^2 - 165\left(\frac{B}{A}\right)^3\alpha_3\alpha_4^3 - 1287\left(\frac{B}{A}\right)^5\alpha_3^5\alpha_4$$
$$+429\left(\frac{B}{A}\right)^6\alpha_3^7\Big] - 429\left(1+\frac{B}{A}\right)^7\alpha_3^7$$
$$= \frac{1}{A_7}\left(1+\frac{B}{A}\right)\Big[\alpha_9 - 9\frac{B}{A}\alpha_3\alpha_8 + 45\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_7$$
$$-165\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_6 + 495\left(\frac{B}{A}\right)^4\alpha_3^4\alpha_5$$
$$-9\frac{B}{A}\alpha_4\alpha_7 + 45\left(\frac{B}{A}\right)^2\alpha_4^2\alpha_5 + 90\left(\frac{B}{A}\right)^2\alpha_3\alpha_4\alpha_6$$
$$-495\left(\frac{B}{A}\right)^3\alpha_3^2\alpha_4\alpha_5 + 990\left(\frac{B}{A}\right)^4\alpha_3^3\alpha_4^2$$
$$-9\frac{B}{A}\alpha_5\alpha_6 + 45\left(\frac{B}{A}\right)^2\alpha_3\alpha_5^2 - 165\left(\frac{B}{A}\right)^3\alpha_3\alpha_4^3$$
$$-1287\left(\frac{B}{A}\right)^5\alpha_3^5\alpha_4 + 429\left(\frac{B}{A}\right)^6\alpha_3^7 - 429\left(1+\frac{B}{A}\right)^6\alpha_3^7\Big]$$
$$= \frac{1}{A_7}\left(1+\frac{B}{A}\right)\Big[\alpha_9 - 9\frac{B}{A}\alpha_3\alpha_8 + 45\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_7$$
$$-165\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_6 + 495\left(\frac{B}{A}\right)^4\alpha_3^4\alpha_5 - 9\frac{B}{A}\alpha_4\alpha_7$$
$$+45\left(\frac{B}{A}\right)^2\alpha_4^2\alpha_5 + 90\left(\frac{B}{A}\right)^2\alpha_3\alpha_4\alpha_6 - 495\left(\frac{B}{A}\right)^3\alpha_3^2\alpha_4\alpha_5$$
$$+990\left(\frac{B}{A}\right)^4\alpha_3^3\alpha_4^2 - 9\frac{B}{A}\alpha_5\alpha_6 + 45\left(\frac{B}{A}\right)^2\alpha_3\alpha_5^2$$
$$-165\left(\frac{B}{A}\right)^3\alpha_3\alpha_4^3 - 1287\left(\frac{B}{A}\right)^5\alpha_3^5\alpha_4 + 429\left(\frac{B}{A}\right)^6\alpha_3^7$$
$$-429\alpha_3^7 - 2574\frac{B}{A}\alpha_3^7 - 6435\left(\frac{B}{A}\right)^2\alpha_3^7 - 8580\left(\frac{B}{A}\right)^3\alpha_3^7$$
$$-6435\left(\frac{B}{A}\right)^4\alpha_3^7 - 2574\left(\frac{B}{A}\right)^5\alpha_3^7 - 429\left(\frac{B}{A}\right)^6\alpha_3^7\Big]$$
$$= \frac{1}{A_7}\left(1+\frac{B}{A}\right)\Big[\left(\alpha_9 - 429\alpha_3^7\right) - 9\frac{B}{A}\alpha_3\left(\Delta_8 - 132\alpha_3^6\right)$$
$$-165\left(\frac{B}{A}\right)^3\alpha_3^3\left(\Delta_6 - 14\alpha_3^4\right) + 45\left(\frac{B}{A}\right)^2\alpha_3^2\left(\Delta_7 + 42\alpha_3^5\right)$$
$$+495\left(\frac{B}{A}\right)^4\alpha_3^4\left(\Delta_5 + 5\alpha_3^3\right) - 9\frac{B}{A}\left(\Delta_4 - 2\alpha_3^2\right)$$
$$\cdot\left(\Delta_7 + 42\alpha_3^5\right) + 45\left(\frac{B}{A}\right)^2\left(\Delta_4 - 2\alpha_3^2\right)^2\left(\Delta_5 + 5\alpha_3^3\right)$$
$$+90\left(\frac{B}{A}\right)^2\alpha_3\left(\Delta_4 - 2\alpha_3^2\right)\left(\Delta_6 - 14\alpha_3^4\right) - 495\left(\frac{B}{A}\right)^3$$
$$\cdot\alpha_3^2\left(\Delta_4 - 2\alpha_3^2\right)\left(\Delta_5 + 5\alpha_3^3\right) + 990\left(\frac{B}{A}\right)^4\alpha_3^3$$
$$\cdot\left(\Delta_4 - 2\alpha_3^2\right)^2 - 9\frac{B}{A}\left(\Delta_5 + 5\alpha_3^3\right)\left(\Delta_6 - 14\alpha_3^4\right)$$
$$+45\left(\frac{B}{A}\right)^2\alpha_3\left(\Delta_5 + 5\alpha_3^3\right) - 165\left(\frac{B}{A}\right)^3\alpha_3\left(\Delta_4 - 2\alpha_3^2\right)^3$$
$$-1287\left(\frac{B}{A}\right)^5\alpha_3^5\left(\Delta_4 - 2\alpha_3^2\right) + 429\left(\frac{B}{A}\right)^6\alpha_3^7$$
$$-2574\frac{B}{A}\alpha_3^7 - 6435\left(\frac{B}{A}\right)^2\alpha_3^7 - 8580\left(\frac{B}{A}\right)^3\alpha_3^7$$
$$-6435\left(\frac{B}{A}\right)^4\alpha_3^7 - 2574\left(\frac{B}{A}\right)^5\alpha_3^7 - 429\left(\frac{B}{A}\right)^6\alpha_3^7\Big]$$
$$\cdot\alpha_3^2\Delta_7 + 1890\left(\frac{B}{A}\right)^2\alpha_3^7 - 165\left(\frac{B}{A}\right)^3\alpha_3^3\Delta_6$$
$$+2310\left(\frac{B}{A}\right)^3\alpha_3^7 + 495\left(\frac{B}{A}\right)^4\alpha_3^4\Delta_5 + 2475\left(\frac{B}{A}\right)^4\alpha_3^7$$
$$-9\frac{B}{A}\Delta_4\Delta_7 - 378\frac{B}{A}\alpha_3^5\Delta_4 + 18\frac{B}{A}\alpha_3^2\Delta_7 + 756\frac{B}{A}\alpha_3^7$$
$$+45\left(\frac{B}{A}\right)^2\Delta_4^2\Delta_5 + 225\left(\frac{B}{A}\right)^2\alpha_3^3\Delta_4^2 - 180\left(\frac{B}{A}\right)^2\alpha_3^2$$
$$\cdot\Delta_4\Delta_5 - 900\left(\frac{B}{A}\right)^2\alpha_3^5\Delta_4 + 180\left(\frac{B}{A}\right)^2\alpha_3^4\Delta_5$$
$$+900\left(\frac{B}{A}\right)^2\alpha_3^7 + 90\left(\frac{B}{A}\right)^2\alpha_3\Delta_4\Delta_6 - 1260\left(\frac{B}{A}\right)^2$$
$$\cdot\alpha_3^5\Delta_4 - 180\left(\frac{B}{A}\right)^2\alpha_3^3\Delta_6 + 2520\left(\frac{B}{A}\right)^2\alpha_3^7$$
$$-495\left(\frac{B}{A}\right)^3\alpha_3^2\Delta_4\Delta_5 - 2475\left(\frac{B}{A}\right)^3\alpha_3^5\Delta_4$$
$$+990\left(\frac{B}{A}\right)^3\alpha_3^4\Delta_5 + 4950\left(\frac{B}{A}\right)^3\alpha_3^7 + 990\left(\frac{B}{A}\right)^4\alpha_3^3\Delta_4^2$$
$$-3960\left(\frac{B}{A}\right)^4\alpha_3^5\Delta_4 + 3960\left(\frac{B}{A}\right)^4\alpha_3^7 - 9\frac{B}{A}\Delta_5\Delta_6$$
$$+126\frac{B}{A}\Delta_5\alpha_3^4 - 45\frac{B}{A}\alpha_3^3\Delta_6 + 630\frac{B}{A}\alpha_3^7 + 45\left(\frac{B}{A}\right)^2$$
$$\cdot\alpha_3\Delta_5^2 + 450\left(\frac{B}{A}\right)^2\alpha_3^4\Delta_5 + 1125\left(\frac{B}{A}\right)^2\alpha_3^7$$
$$-165\left(\frac{B}{A}\right)^3\alpha_3\Delta_4^3 + 990\left(\frac{B}{A}\right)^3\alpha_3^3\Delta_4^2 - 1980\left(\frac{B}{A}\right)^3$$
$$\cdot\alpha_3^5\Delta_4 + 1320\left(\frac{B}{A}\right)^3\alpha_3^7 - 1287\left(\frac{B}{A}\right)^5\alpha_3^5\Delta_4$$
$$+2574\left(\frac{B}{A}\right)^5\alpha_3^7 + 429\left(\frac{B}{A}\right)^6\alpha_3^7 - 2574\frac{B}{A}\alpha_3^7$$
$$-6435\left(\frac{B}{A}\right)^2\alpha_3^7 - 8580\left(\frac{B}{A}\right)^3\alpha_3^7 - 6435\left(\frac{B}{A}\right)^4\alpha_3^7$$
$$-2574\left(\frac{B}{A}\right)^5\alpha_3^7 - 429\left(\frac{B}{A}\right)^6\alpha_3^7\Big]$$
$$= \frac{1}{A_7}\left(1+\frac{B}{A}\right)\Big[\Delta_9 - 9\frac{B}{A}\left(\alpha_3\Delta_8 + \Delta_4\Delta_7 + 42\alpha_3^5\Delta_4\right.$$
$$\left.-2\alpha_3^2\Delta_7 + \Delta_5\Delta_6 - 14\alpha_3^4\Delta_5 + 5\alpha_3^3\Delta_6\right) + 45\left(\frac{B}{A}\right)^2$$
$$\cdot\left(\alpha_3\Delta_5^2 + 14\alpha_3^4\Delta_5 + \alpha_3^2\Delta_7 + 2\alpha_3\Delta_4\Delta_6 - 48\alpha_3^5\Delta_4\right.$$
$$\left.-4\alpha_3^3\Delta_6 + \Delta_4^2\Delta_5 + 5\alpha_3^3\Delta_4^2 - 4\alpha_3^2\Delta_4\Delta_5\right) - 165\left(\frac{B}{A}\right)^3$$
$$\cdot\left(\alpha_3^3\Delta_6 + \alpha_3\Delta_4^3 - 6\alpha_3^3\Delta_4^2\right.$$
$$\left.+15\alpha_3^5\Delta_4 - 6\alpha_3^4\Delta_5\right) + 495\left(\frac{B}{A}\right)^4\left(\alpha_3^4\Delta_5 + 2\alpha_3^3\Delta_4^2\right.$$
$$\left.-8\alpha_3^5\Delta_4\right) - 1287\left(\frac{B}{A}\right)^5\alpha_3^5\Delta_4\Big]$$
$$= \frac{1}{A_7}\left(1+\frac{B}{A}\right)\Big[\Delta_9 - 9\frac{B}{A}\left(\Delta_3\Delta_8 + \Delta_4\Delta_7\right.$$
$$+42\Delta_3^5\Delta_4 - 2\Delta_3^2\Delta_7 + \Delta_5\Delta_6 - 14\Delta_3^4\Delta_5$$
$$\left.+5\Delta_3^3\Delta_6\right) + 45\left(\frac{B}{A}\right)^2\left(\Delta_3\Delta_5^2 + 14\Delta_3^4\Delta_5\right.$$
$$+\Delta_3^2\Delta_7 + 2\Delta_3\Delta_4\Delta_6 - 48\Delta_3^5\Delta_4 - 4\Delta_3^3\Delta_6$$
$$\left.+\Delta_4^2\Delta_5 + 5\Delta_3^3\Delta_4^2 - 4\Delta_3^2\Delta_4\Delta_5\right) - 165\left(\frac{B}{A}\right)^3$$
$$\cdot\left(\Delta_3^3\Delta_6 + \Delta_3\Delta_4^3 - 6\Delta_3^3\Delta_4^2 + 12\Delta_3^5\Delta_4\right.$$
$$\left.+3\Delta_3^2\Delta_4\Delta_5 + 15\Delta_3^5\Delta_4 - 6\Delta_3^4\Delta_5\right) + 495\left(\frac{B}{A}\right)^4$$
$$\cdot\left(\Delta_3^4\Delta_5 + 2\Delta_3^3\Delta_4^2 - 8\Delta_3^5\Delta_4\right) - 1287\left(\frac{B}{A}\right)^5\Delta_3^5\Delta_4\Big]$$

and

$$\Theta' = \frac{1}{A^7}\Theta$$

Criterion 2 $\Longrightarrow$ Criterion 3

$$\psi_9(-1;\boldsymbol{\Delta})$$
$$= f_9(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9)$$
$$\frac{1}{A_7}\left(1+\frac{B}{A}\right)\Big[\Delta_9 - 9\frac{B}{A}\left(\Delta_3\Delta_8 + \Delta_4\Delta_7 + 42\Delta_3^5\Delta_4\right.$$
$$\left.-2\Delta_3^2\Delta_7 + \Delta_5\Delta_6 - 14\Delta_3^4\Delta_5 + 5\Delta_3^3\Delta_6\right)$$
$$+45\left(\frac{B}{A}\right)^2\left(\Delta_3\Delta_5^2 + 14\Delta_3^4\Delta_5 + \Delta_3^2\Delta_7 + 2\Delta_3\Delta_4\Delta_6\right.$$
$$\left.-48\Delta_3^5\Delta_4 - 4\Delta_3^3\Delta_6 + \Delta_4^2\Delta_5 + 5\Delta_3^3\Delta_4^2 - 4\Delta_3^2\Delta_4\Delta_5\right)$$
$$-165\left(\frac{B}{A}\right)^3\left(\Delta_3^3\Delta_6 + \Delta_3\Delta_4^3 - 6\Delta_3^3\Delta_4^2 + 12\Delta_3^5\Delta_4\right.$$
$$\left.+3\Delta_3^2\Delta_4\Delta_5 + 15\Delta_3^5\Delta_4 - 6\Delta_3^4\Delta_5\right) + 495\left(\frac{B}{A}\right)^4\left(\Delta_3^4\Delta_5\right.$$
$$\left.+2\Delta_3^3\Delta_4^2 - 8\Delta_3^5\Delta_4\right) - 1287\left(\frac{B}{A}\right)^5\Delta_3^5\Delta_4\Big]$$
$$\Delta_9' = \frac{1}{A^7}\left(1+\frac{B}{A}\right)\Big[\Delta_9 - 9\left(1+\frac{B}{A}-1\right)\left(\Delta_3\Delta_8 + \Delta_4\Delta_7\right.$$
$$\left.+42\Delta_3^5\Delta_4 - 2\Delta_3^2\Delta_7 + \Delta_5\Delta_6 - 14\Delta_3^4\Delta_5 + 5\Delta_3^3\Delta_6\right)$$
$$+45\left(1+\frac{B}{A}-1\right)^2\left(\Delta_3\Delta_5^2 + 14\Delta_3^4\Delta_5 + \Delta_3^2\Delta_7 + 2\Delta_3\Delta_4\Delta_6\right.$$
$$\left.-48\Delta_3^5\Delta_4 - 4\Delta_3^3\Delta_6 + \Delta_4^2\Delta_5 + 5\Delta_3^3\Delta_4^2 - 4\Delta_3^2\Delta_4\Delta_5\right)$$
$$-165\left(1+\frac{B}{A}-1\right)^3\left(\Delta_3^3\Delta_6 + \Delta_3\Delta_4^3 - 6\Delta_3^3\Delta_4^2 + 12\Delta_3^5\Delta_4\right.$$
$$\left.+3\Delta_3^2\Delta_4\Delta_5 + 15\Delta_3^5\Delta_4 - 6\Delta_3^4\Delta_5\right) + 495\left(1+\frac{B}{A}-1\right)^4$$
$$\cdot\left(\Delta_3^4\Delta_5 + 2\Delta_3^3\Delta_4^2 - 8\Delta_3^5\Delta_4\right) - 1287\left(1+\frac{B}{A}-1\right)^5\Delta_3^5\Delta_4\Big]$$
$$= \frac{1}{A^7}\left(1+\frac{B}{A}\right)\Big[\Delta_9 - 9\left(1+\frac{B}{A}\right)\left(\Delta_3\Delta_8 + \Delta_4\Delta_7\right.$$
$$\left.+42\Delta_3^5\Delta_4 - 2\Delta_3^2\Delta_7 + \Delta_5\Delta_6 - 14\Delta_3^4\Delta_5 + 5\Delta_3^3\Delta_6\right)$$
$$+9\left(\Delta_3\Delta_8 + \Delta_4\Delta_7 + 42\Delta_3^5\Delta_4 - 2\Delta_3^2\Delta_7 + \Delta_5\Delta_6\right.$$
$$\left.-14\Delta_3^4\Delta_5 + 5\Delta_3^3\Delta_6\right) + 45\left(1+\frac{B}{A}\right)^2\left(\Delta_3\Delta_5^2 + 14\Delta_3^4\Delta_5\right.$$
$$+\Delta_3^2\Delta_7 + 2\Delta_3\Delta_4\Delta_6 - 48\Delta_3^5\Delta_4 - 4\Delta_3^3\Delta_6 + \Delta_4^2\Delta_5$$
$$\left.+5\Delta_3^3\Delta_4^2 - 4\Delta_3^2\Delta_4\Delta_5\right) - 90\left(1+\frac{B}{A}\right)\left(\Delta_3\Delta_5^2 + 14\Delta_3^4\Delta_5\right.$$
$$+\Delta_3^2\Delta_7 + 2\Delta_3\Delta_4\Delta_6 - 48\Delta_3^5\Delta_4 - 4\Delta_3^3\Delta_6 + \Delta_4^2\Delta_5$$
$$\left.+5\Delta_3^3\Delta_4^2 - 4\Delta_3^2\Delta_4\Delta_5\right) + 45\left(\Delta_3\Delta_5^2 + 14\Delta_3^4\Delta_5 + \Delta_3^2\Delta_7\right.$$
$$+2\Delta_3\Delta_4\Delta_6 - 48\Delta_3^5\Delta_4 - 4\Delta_3^3\Delta_6 + \Delta_4^2\Delta_5 + 5\Delta_3^3\Delta_4^2$$
$$\left.-4\Delta_3^2\Delta_4\Delta_5\right) - 165\left(1+\frac{B}{A}\right)^3\left(\Delta_3^3\Delta_6 + \Delta_3\Delta_4^3 - 6\Delta_3^3\Delta_4^2\right.$$
$$\left.+12\Delta_3^5\Delta_4 + 3\Delta_3^2\Delta_4\Delta_5 + 15\Delta_3^5\Delta_4 - 6\Delta_3^4\Delta_5\right)$$
$$+495\left(1+\frac{B}{A}\right)^2\left(\Delta_3^3\Delta_6 + \Delta_3\Delta_4^3 - 6\Delta_3^3\Delta_4^2\right.$$
$$+12\Delta_3^5\Delta_4 + 3\Delta_3^2\Delta_4\Delta_5 + 15\Delta_3^5\Delta_4 + 3\Delta_3^2\Delta_4\Delta_5 + 15\Delta_3^5\Delta_4$$
$$\left.-6\Delta_3^4\Delta_5\right) - 495\left(1+\frac{B}{A}\right)\left(\Delta_3^3\Delta_6 + \Delta_3\Delta_4^3 - 6\Delta_3^3\Delta_4^2\right.$$
$$\left.+12\Delta_3^5\Delta_4 - 6\Delta_3^4\Delta_5\right) + 165\left(\Delta_3^3\Delta_6 + \Delta_3\Delta_4^3 - 6\Delta_3^3\Delta_4^2\right.$$
$$\left.+12\Delta_3^5\Delta_4 + 3\Delta_3^2\Delta_4\Delta_5 + 15\Delta_3^5\Delta_4 - 6\Delta_3^4\Delta_5\right)$$
$$+495\left(1+\frac{B}{A}\right)^4\left(\Delta_3^4\Delta_5 + 2\Delta_3^3\Delta_4^2 - 8\Delta_3^5\Delta_4\right)$$
$$-1980\left(1+\frac{B}{A}\right)^3\left(\Delta_3^4\Delta_5 + 2\Delta_3^3\Delta_4^2 - 8\Delta_3^5\Delta_4\right)$$
$$+2970\left(1+\frac{B}{A}\right)^2\left(\Delta_3^4\Delta_5 + 2\Delta_3^3\Delta_4^2 - 8\Delta_3^5\Delta_4\right)$$
$$-1980\left(1+\frac{B}{A}\right)\left(\Delta_3^4\Delta_5 + 2\Delta_3^3\Delta_4^2 - 8\Delta_3^5\Delta_4\right)$$
$$+495\left(\Delta_3^4\Delta_5 + 2\Delta_3^3\Delta_4^2 - 8\Delta_3^5\Delta_4\right) - 1287\left(1+\frac{B}{A}\right)^5\Delta_3^5\Delta_4$$

$+6435\left(1+\frac{B}{A}\right)^4\Delta_3^5\Delta_4-12870\left(1+\frac{B}{A}\right)^3\Delta_3^5\Delta_4$
$+12870\left(1+\frac{B}{A}\right)^2\Delta_3^5\Delta_4-6435\left(1+\frac{B}{A}\right)\Delta_3^5\Delta_4$
$+1287\Delta_3^5\Delta_4]$
$=\frac{1}{A^7}\left(1+\frac{B}{A}\right)[\Delta_9+9\Delta_3\Delta_8+9\Delta_4\Delta_7+378\Delta_3^5\Delta_4-18\Delta_3^2\Delta_7$
$+9\Delta_5\Delta_6-126\Delta_3^4\Delta_5+45\Delta_3^3\Delta_6+45\Delta_3\Delta_5^2+630\Delta_3^4\Delta_5$
$+45\Delta_3^2\Delta_7+90\Delta_3\Delta_4\Delta_6-2160\Delta_3^5\Delta_4-180\Delta_3^3\Delta_6+45\Delta_4^2\Delta_5$
$+225\Delta_3^3\Delta_4^2-180\Delta_3^2\Delta_4\Delta_5+165\Delta_3^3\Delta_6+165\Delta_3\Delta_4^3$
$-990\Delta_3^3\Delta_4^2+1980\Delta_3^5\Delta_4+495\Delta_3^2\Delta_4\Delta_5+2475\Delta_3^5\Delta_4$
$-990\Delta_3^4\Delta_5+495\Delta_3^4\Delta_5+990\Delta_3^3\Delta_4^2-3960\Delta_3^5\Delta_4+1287\Delta_3^5\Delta_4]$
$-9\frac{1}{A^7}\left(\frac{B}{A}\right)^2[\Delta_3\Delta_8+\Delta_4\Delta_7+42\Delta_3^5\Delta_4-2\Delta_3^2\Delta_7+\Delta_5\Delta_6$
$-14\Delta_3^4\Delta_5+5\Delta_3^3\Delta_6+10\Delta_3\Delta_5^2+140\Delta_3^4\Delta_5+10\Delta_3^2\Delta_7$
$+20\Delta_3\Delta_4\Delta_6-480\Delta_3^5\Delta_4-330\Delta_3^3\Delta_4^2+660\Delta_3^5\Delta_4$
$-40\Delta_3^3\Delta_6+10\Delta_4^2\Delta_5+50\Delta_3^3\Delta_4^2-40\Delta_3^2\Delta_4\Delta_5+55\Delta_3^3\Delta_6$
$+55\Delta_3\Delta_4^3+165\Delta_3^2\Delta_4\Delta_5+825\Delta_3^5\Delta_4-330\Delta_3^4\Delta_5$
$+220\Delta_3^4\Delta_5+440\Delta_3^3\Delta_4^2-1760\Delta_3^5\Delta_4+715\Delta_3^5\Delta_4]$
$+45\frac{1}{A^7}\left(1+\frac{B}{A}\right)^3[\Delta_3\Delta_5^2+14\Delta_3^4\Delta_5+\Delta_3^2\Delta_7+2\Delta_3\Delta_4\Delta_6$
$-48\Delta_3^5\Delta_4-4\Delta_3^3\Delta_6+\Delta_4^2\Delta_5+5\Delta_3^3\Delta_4^2-4\Delta_3^5\Delta_4$
$+11\Delta_3^3\Delta_6+11\Delta_3\Delta_4^3-66\Delta^3\Delta_4^2+132\Delta_3^5\Delta_4+33\Delta_3^2\Delta_4\Delta_5$
$+165\Delta_3^5\Delta_4+132\Delta_3^3\Delta_4^2-528\Delta_3^5\Delta_4+286\Delta_3^5\Delta_4]$
$-165\frac{1}{A^7}\left(1+\frac{B}{A}\right)^4[\Delta_3^3\Delta_6+\Delta_3\Delta_4^3-6\Delta_3^3\Delta_4^2+12\Delta_3^5\Delta_4$
$+3\Delta_3^2\Delta_4\Delta_5+15\Delta_3^5\Delta_4-6\Delta_3^4\Delta_5+12\Delta_3^4\Delta_5+24\Delta_3^3\Delta_4^2$
$-96\Delta_3^5\Delta_4+78\Delta_3^5\Delta_4]+495\frac{1}{A^7}\left(1+\frac{B}{A}\right)^5[\Delta_3^4\Delta_5+2\Delta_3^3\Delta_4^2$
$-8\Delta_3^5\Delta_4+13\Delta_3^5\Delta_4]-1287\frac{1}{A^7}\left(1+\frac{B}{A}\right)^6\Delta_3^5\Delta_4$
$=\frac{1}{A^7}\left(1+\frac{B}{A}\right)[\Delta_9+9\Delta_3\Delta_8+9\Delta_4\Delta_7+9\Delta_5\Delta_6+45\Delta_3^2\Delta_7$
$+90\Delta_3\Delta_4\Delta_6+45\Delta_3\Delta_5^2+45\Delta_4^2\Delta_5+165\Delta_3^3\Delta_6$
$+495\Delta_3^2\Delta_4\Delta_5+165\Delta_3\Delta_4^3+495\Delta_3^4\Delta_5+990\Delta_3^3\Delta_4^2$
$+1287\Delta_3^5\Delta_4]-9\frac{1}{A^7}\left(1+\frac{B}{A}\right)^2\Delta_3[\Delta_8+8\Delta_3\Delta_7+16\Delta_3^4\Delta_5$
$+20\Delta_3^2\Delta_6]-9\frac{1}{A^7}\left(1+\frac{B}{A}\right)^2\Delta_4[\Delta_7+2\Delta_3^5+55\Delta_3\Delta_4^3$
$+160\Delta_3^3\Delta_4]-9\frac{1}{A^7}\left(1+\frac{B}{A}\right)^2\Delta_5[\Delta_6+10\Delta_4^2+125\Delta_3^2\Delta_4]$

$$-45\frac{1}{A^7}\left(1+\frac{B}{A}\right)^3\Delta_3[\Delta_5^2+7\Delta_3^4\Delta_4]+45\frac{1}{A^7}\left(1+\frac{B}{A}\right)^3$$

$\cdot\Delta_3^2[\Delta_7+7\Delta_3\Delta_6+14\Delta_3^2\Delta_5]-45\frac{1}{A^7}\left(1+\frac{B}{A}\right)^3\Delta_4$
$\cdot[\Delta_4\Delta_5+11\Delta_3\Delta_4^2+29\Delta_3^2\Delta_5+71\Delta_3^3\Delta_4]$
$-90\frac{1}{A^7}\left(1+\frac{B}{A}\right)^3\Delta_3\Delta_4\Delta_6-165\frac{1}{A^7}\left(1+\frac{B}{A}\right)^4\Delta_3$
$\cdot[\Delta_3^2\Delta_6+6\Delta_3^3\Delta_5]-495\frac{1}{A^7}\left(1+\frac{B}{A}\right)^4\Delta_3^2[\Delta_4\Delta_5+3\Delta_3^3\Delta_4]$
$-165\frac{1}{A^7}\left(1+\frac{B}{A}\right)^4\Delta_4[\Delta_3\Delta_4^2+18\Delta_3^3\Delta_4]$
$-495\frac{1}{A^7}\left(1+\frac{B}{A}\right)^5\Delta_3[\Delta_3^2\Delta_6+5\Delta_3^4\Delta_4]$

$$-990\frac{1}{A^7}\left(1+\frac{B}{A}\right)^5\Delta_3^3\Delta_4^2-1287\frac{1}{A^7}\left(1+\frac{B}{A}\right)^6\Delta_3^5\Delta_4$$

$=\frac{1}{A^7}\left(1+\frac{B}{A}\right)[\Delta_9+9\Delta_3\Delta_8+9\Delta_4\Delta_7+9\Delta_5\Delta_6+45\Delta_3^2\Delta_7$
$+90\Delta_3\Delta_4\Delta_6+45\Delta_3\Delta_5^2+45\Delta_4^2\Delta_5+165\Delta_3^3\Delta_6$
$+495\Delta_3^2\Delta_4\Delta_5+165\Delta_3\Delta_4^3+495\Delta_3^4\Delta_5+990\Delta_3^3\Delta_4^2$
$+1287\Delta_3^5\Delta_4]-9\Delta_3'\Delta_8'-9\Delta_4'\Delta_7'-9\Delta_5'\Delta_6'-45\Delta_3'^2\Delta_7'$
$-90\Delta_3'\Delta_4'\Delta_6'-45\Delta_3'\Delta_5'^2-45\Delta_4'^2\Delta_5'-165\Delta_3'^3\Delta_6'$
$-495\Delta_3'^2\Delta_4'\Delta_5'-165\Delta_3'\Delta_4'^3-495\Delta_3'^4\Delta_5'-990\Delta_3'^3\Delta_4'^2$
$-1287\Delta_3'^5\Delta_4'$.

This means

$\Delta_9'=\frac{1}{A^7}\left(1+\frac{B}{A}\right)f_9(\mathbf{\Delta})-9\Delta_3'\Delta_8'-9\Delta_4'\Delta_7'-9\Delta_5'\Delta_6'$
$-45\Delta_3'^2\Delta_7'-90\Delta_3'\Delta_4'\Delta_6'-45\Delta_3'\Delta_5'^2-45\Delta_4'^2\Delta_5'-165\Delta_3'^3\Delta_6'$
$-495\Delta_3'^2\Delta_4'\Delta_5'-165\Delta_3'\Delta_4'^3-495\Delta_3'^4\Delta_5'-990\Delta_3'^3\Delta_4'^2$
$-1287\Delta_3'^5\Delta_4'$,

i.e.,

$$f_9(\mathbf{\Delta}')=\frac{1}{A^7}\left(1+\frac{B}{A}\right)f_9(\mathbf{\Delta}).$$

Criterion 3 $\Longrightarrow$ Criterion 1

$\Delta_9'=\alpha_9'-429\alpha_3'^7$
$=\frac{1}{A^7}\left(1+\frac{B}{A}\right)[\Delta_9+9\Delta_3\Delta_8+9\Delta_4\Delta_7+9\Delta_5\Delta_6$
$+45\Delta_3^2\Delta_7+90\Delta_3\Delta_4\Delta_6+45\Delta_3\Delta_5^2+45\Delta_4^2\Delta_5$
$+165\Delta_3^3\Delta_6+495\Delta_3^2\Delta_4\Delta_5+165\Delta_3\Delta_4^3+495\Delta_3^4\Delta_5$
$+990\Delta_3^3\Delta_4^2+1287\Delta_3^5\Delta_4]-9\Delta_3'\Delta_8'-9\Delta_4'\Delta_7'-9\Delta_5'\Delta_6'$
$-45\Delta_3'^2\Delta_7'-90\Delta_3'\Delta_4'\Delta_6'-45\Delta_3'\Delta_5'^2-45\Delta_4'^2\Delta_5'-165\Delta_3'^3\Delta_6'$
$-495\Delta_3'^2\Delta_4'\Delta_5'-165\Delta_3'\Delta_4'^3-495\Delta_3'^4\Delta_5'$
$-990\Delta_3'^3\Delta_4'^2-1287\Delta_3'^5\Delta_4'$
$=\frac{1}{A^7}\left(1+\frac{B}{A}\right)[\alpha_9-429\alpha_3^7+9\alpha_3(\alpha_8+132\alpha_3^6)$
$+9(\alpha_4+2\alpha_3^2)(\alpha_7-142\alpha_3^3)+9(\alpha_5-5\alpha_3^3)(\alpha_6+14\alpha_3^4)$
$+45\alpha_3^2(\alpha_7-42\alpha_3^5)+90\alpha_3(\alpha_4+2\alpha_3^2)(\alpha_6+14\alpha_3)$
$+45\alpha_2(\alpha_5-5\alpha_3^3)^2+45(\alpha_4+2\alpha_3^2)^2(\alpha_5-5\alpha_3^3)$
$+165\alpha_3^3(\alpha_6+14\alpha_4^4)+495\alpha_3^4(\alpha_5-5\alpha_3^3)$
$+990\alpha_3^3(\alpha_4+2\alpha_3^2)+1287\alpha_3^5(\alpha_4+2\alpha_3^2)]$
$-\alpha_9'-9\alpha_3'(\alpha_8'+132\alpha_3'^6)-9(\alpha_4'+2\alpha_3'^2)(\alpha_7'-142\alpha_3'^3)$
$-9(\alpha_5'-5\alpha_3'^3)(\alpha_6'+14\alpha_3'^4)-45\alpha_3'^2(\alpha_7'-42\alpha_3'^5)$
$-90\alpha_3'(\alpha_4'+2\alpha_3'^2)(\alpha_6'+14\alpha_3')-45\alpha_3'(\alpha_5'-5\alpha_3'^3)^2$
$-45(\alpha_4'+2\alpha_3'^2)^2(\alpha_5'-5\alpha_3'^3)-165\alpha_3'^3(\alpha_6'+14\alpha_3'^4)$
$-495\alpha_3'^4(\alpha_5'-5\alpha_3'^3)-990\alpha_3'^3(\alpha_4'+2\alpha_3'^2)$
$-1287\alpha_3'^5(\alpha_4'+2\alpha_3'^2)$
$=\frac{1}{A^7}\left(1+\frac{B}{A}\right)[\alpha_9-429\alpha_3^7+9\alpha_3\alpha_8+132\alpha_3^6$
$+9(\alpha_4+2\alpha_3^2)(\alpha_7-142\alpha_3^3)+9(\alpha_5-5\alpha_3^3)(\alpha_6+14\alpha_3^4)$
$+45\alpha_3^2(\alpha_7-42\alpha_3^5)+90\alpha_3(\alpha_4+2\alpha_3^2)(\alpha_6+14\alpha_3)$
$+45\alpha_2(\alpha_5-5\alpha_3^3)^2+45(\alpha_4+2\alpha_3^2)^2(\alpha_5-5\alpha_3^3)$
$+165\alpha_3^3(\alpha_6+14\alpha_4^4)+495\alpha_3^4(\alpha_5-5\alpha_3^3)$
$+990\alpha_3^3(\alpha_4+2\alpha_3^2)+1287\alpha_3^5(\alpha_4+2\alpha_3^2)]$
$-9\frac{1}{A}\left(1+\frac{B}{A}\right)\alpha_3\left(\frac{1}{A^6}\left(1+\frac{B}{A}\right)[\alpha_8-8\frac{B}{A}\alpha_3\alpha_7\right.$
$+36\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_6+72\left(\frac{B}{A}\right)^2\alpha_3\alpha_4\alpha_5-8\frac{B}{A}\alpha_4\alpha_6+12\left(\frac{B}{A}\right)^2\alpha_4^3$
$-180\left(\frac{B}{A}\right)^3\alpha_3^2\alpha_4^2-120\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_5+330\left(\frac{B}{A}\right)^4\alpha_3^4\alpha_4$
$-4\frac{B}{A}\alpha_5^2-132\left(\frac{B}{A}\right)^5\alpha_3^6+132\frac{1}{B}\left(1+\frac{B}{A}\alpha_3^6\right)$
$-9\left(\frac{1}{A^2}\left(1+\frac{B}{A}\right)[\alpha_4-2\frac{B}{A}\alpha_3^2]+2\alpha_3^2\right)$
$\cdot\left(\frac{1}{A^5}\left(1+\frac{B}{A}\right)[\alpha_7-7\frac{B}{A}\alpha_3\alpha_6+28\left(\frac{B}{A}\right)^2\alpha_3\alpha_4^2\right.$
$+28\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_5-7\frac{B}{A}\alpha_4\alpha_5-84\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_4+42\left(\frac{B}{A}\right)^4\alpha_3^5]$
$\left.-84\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_4+42\left(\frac{B}{A}\right)^4\alpha_3^5]-142\alpha_3^3\right)$
$-9\left(\frac{1}{A^3}\left(1+\frac{B}{A}\right)[\alpha_5-5\frac{B}{A}\alpha_3\alpha_4+5\left(\frac{B}{A}\right)^2\alpha_3^3]\right.$
$\left.-5\alpha_3^3\right)\left(\frac{1}{A^4}\left(1+\frac{B}{A}\right)[\alpha_6-6\frac{B}{A}\alpha_3\alpha_5+21\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_4\right.$
$\left.-3\frac{B}{A}\alpha_4^2-14\left(\frac{B}{A}\right)^3\alpha_3^4]+14\alpha_3^4\right)$
$-45\alpha_3^2\left(\frac{1}{A^5}\left(1+\frac{B}{A}\right)[\alpha_7-7\frac{B}{A}\alpha_3\alpha_6+28\left(\frac{B}{A}\right)^2\alpha_3\alpha_4^2\right.$
$+28\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_5-7\frac{B}{A}\alpha_4\alpha_5-84\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_4$
$\left.+2\alpha_3^2)+42\left(\frac{B}{A}\right)^4\alpha_3^5]-42\alpha_3^5\right)$ $-$
$90\alpha_3\left(\frac{1}{A^2}\left(1+\frac{B}{A}\right)[\alpha_4-2\frac{B}{A}\alpha_3^2]\right.$
$\cdot\left(\frac{1}{A^4}\left(1+\frac{B}{A}\right)[\alpha_6-6\frac{B}{A}\alpha_3\alpha_5+21\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_4-3\frac{B}{A}\alpha_4^2\right.$
$\left.-14\left(\frac{B}{A}\right)^3\alpha_3^4]+14\alpha_3\right)$
$-45\alpha_3\left(\frac{1}{A^3}\left(1+\frac{B}{A}\right)[\alpha_5-5\frac{B}{A}\alpha_3\alpha_4+5\left(\frac{B}{A}\right)^2\alpha_3^3]-5\alpha_3^3\right)^2$
$-45\left(\frac{1}{A^2}\left(1+\frac{B}{A}\right)[\alpha_4-2\frac{B}{A}\alpha_3^2]+2\alpha_3^2\right)^2$
$\cdot\left(\frac{1}{A^3}\left(1+\frac{B}{A}\right)[\alpha_5-5\frac{B}{A}\alpha_3\alpha_4+5\left(\frac{B}{A}\right)^2\alpha_3^3]-5\alpha_3^3\right)$
$-165\alpha_3^3\left(\frac{1}{A^4}\left(1+\frac{B}{A}\right)[\alpha_6-6\frac{B}{A}\alpha_3\alpha_5+21\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_4\right.$

$-3\frac{B}{A}\alpha_4^2-14\left(\frac{B}{A}\right)^3\alpha_3^4\Big]+14\alpha_3^4\Big)$
$-495\alpha_3^4\left(\frac{1}{A^3}\left(1+\frac{B}{A}\right)\Big[\alpha_5-5\frac{B}{A}\alpha_3\alpha_4+5\left(\frac{B}{A}\right)^2\alpha_3^3\Big]-5\alpha_3^3\right)$
$-990\alpha_3^3\left(\frac{1}{A^2}\left(1+\frac{B}{A}\right)[\alpha_4-2\frac{B}{A}\alpha_3^2]+2\alpha_3^2\right)$
$-1287\alpha_3^5\left(\frac{1}{A^2}\left(1+\frac{B}{A}\right)\left[\alpha_4-2\frac{B}{A}\alpha_3^2\right]+2\alpha_3^2\right).$

We find α_9' as follows:

$\alpha_9'=\frac{1}{A^7}\left(1+\frac{B}{A}\right)\left[\alpha_9-429\alpha_3^7+9\alpha_3\alpha_8+132\alpha_3^6\right.$
$+9\left(\alpha_4+2\alpha_3^2\right)\left(\alpha_7-142\alpha_3^3\right)+9\left(\alpha_5-5\alpha_3^3\right)\left(\alpha_6+14\alpha_3^4\right)$
$+45\alpha_3^2\left(\alpha_7-42\alpha_3^5\right)+90\alpha_3\left(\alpha_4+2\alpha_3^2\right)\left(\alpha_6+14\alpha_3\right)$
$+45\alpha_2\left(\alpha_5-5\alpha_3^3\right)^2+45\left(\alpha_4+2\alpha_3^2\right)^2\left(\alpha_5-5\alpha_3^3\right)$
$+165\alpha_3^3\left(\alpha_6+14\alpha_4^4\right)+495\alpha_3^4\left(\alpha_5-5\alpha_3^3\right)$
$\left.+990\alpha_3^3\left(\alpha_4+2\alpha_3^2\right)+1287\alpha_3^5\left(\alpha_4+2\alpha_3^2\right)\right]$
$-9\frac{1}{A}\left(1+\frac{B}{A}\right)\alpha_3\left(\frac{1}{A^6}\left(1+\frac{B}{A}\right)\Big[\alpha_8-8\frac{B}{A}\alpha_3\alpha_7+36\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_6\right.$
$+72\left(\frac{B}{A}\right)^2\alpha_3\alpha_4\alpha_5-8\frac{B}{A}\alpha_4\alpha_6+12\left(\frac{B}{A}\right)^2\alpha_4^3-180\left(\frac{B}{A}\right)^3\alpha_3^2\alpha_4^2$
$-120\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_5+330\left(\frac{B}{A}\right)^4\alpha_3^4\alpha_4-4\frac{B}{A}\alpha_5^2-132\left(\frac{B}{A}\right)^5\alpha_3^6$
$+132\frac{1}{B}\left(1+\frac{B}{A}\alpha_3^6\right)-9\left(\frac{1}{A^2}\left(1+\frac{B}{A}\right)\left[\alpha_4-2\frac{B}{A}\alpha_3^2\right]+2\alpha_3^2\right)$
$\cdot\left(\frac{1}{A^5}\left(1+\frac{B}{A}\right)\Big[\alpha_7-7\frac{B}{A}\alpha_3\alpha_6+28\left(\frac{B}{A}\right)^2\alpha_3\alpha_4^2+28\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_5\right.$
$\left.-7\frac{B}{A}\alpha_4\alpha_5-84\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_4+42\left(\frac{B}{A}\right)^4\alpha_3^5\Big]-142\alpha_3^3\right)$
$-9\left(\frac{1}{A^3}\left(1+\frac{B}{A}\right)\Big[\alpha_5-5\frac{B}{A}\alpha_3\alpha_4+5\left(\frac{B}{A}\right)^2\alpha_3^3\Big]-5\alpha_3^3\right)$
$\cdot\left(\frac{1}{A^4}\left(1+\frac{B}{A}\right)\Big[\alpha_6-6\frac{B}{A}\alpha_3\alpha_5+21\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_4-3\frac{B}{A}\alpha_4^2\right.$
$\left.-14\left(\frac{B}{A}\right)^3\alpha_3^4\Big]+14\alpha_3^4\right)-45\alpha_3^2\left(\frac{1}{A^5}\left(1+\frac{B}{A}\right)\Big[\alpha_7-7\frac{B}{A}\alpha_3\alpha_6\right.$
$+28\left(\frac{B}{A}\right)^2\alpha_3\alpha_4^2+28\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_5-7\frac{B}{A}\alpha_4\alpha_5$
$\left.-84\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_4+42\left(\frac{B}{A}\right)^4\alpha_3^5\Big]-42\alpha_3^5\right)$
$-90\alpha_3\left(\frac{1}{A^2}\left(1+\frac{B}{A}\right)[\alpha_4-2\frac{B}{A}\alpha_3^2]+2\alpha_3^2\right)\left(\frac{1}{A^4}\left(1+\frac{B}{A}\right)\right.$
$\cdot\Big[\alpha_6-6\frac{B}{A}\alpha_3\alpha_5+21\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_4-3\frac{B}{A}\alpha_4^2-14\left(\frac{B}{A}\right)^3\alpha_3^4\Big]$
$\left.+14\alpha_3\right)-45\alpha_3\left(\frac{1}{A^3}\left(1+\frac{B}{A}\right)\Big[\alpha_5-5\frac{B}{A}\alpha_3\alpha_4+5\left(\frac{B}{A}\right)^2\alpha_3^3\Big]\right.$
$\left.-5\alpha 3_3\right)^2-45\left(\frac{1}{A^2}\left(1+\frac{B}{A}\right)\left[\alpha_4-2\frac{B}{A}\alpha_3^2\right]+2\alpha_3^2\right)^2$
$\cdot\left(\frac{1}{A^3}\left(1+\frac{B}{A}\right)\Big[\alpha_5-5\frac{B}{A}\alpha_3\alpha_4+5\left(\frac{B}{A}\right)^2\alpha_3^3\Big]-5\alpha_3^3\right)$
$-165\alpha_3^3\left(\frac{1}{A^4}\left(1+\frac{B}{A}\right)\cdot\Big[\alpha_6-6\frac{B}{A}\alpha_3\alpha_5+21\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_4\right.$
$\left.-3\frac{B}{A}\alpha_4^2-14\left(\frac{B}{A}\right)^3\alpha_3^4\Big]+14\alpha_3^4\right)$
$-495\alpha_3^4\left(\frac{1}{A^3}\left(1+\frac{B}{A}\right)\Big[\alpha_5-5\frac{B}{A}\alpha_3\alpha_4+5\left(\frac{B}{A}\right)^2\alpha_3^3\Big]-5\alpha_3^3\right)$

$-990\alpha_3^3\left(\frac{1}{A^2}\left(1+\frac{B}{A}\right)\left[\alpha_4-2\frac{B}{A}\alpha_3^2\right]+2\alpha_3^2\right)$

$-1287\alpha_3^5\left(\frac{1}{A^2}\left(1+\frac{B}{A}\right)\left[\alpha_4-2\frac{B}{A}\alpha_3^2\right]+2\alpha_3^2\right)+429\alpha_3'^7.$

Then the substitution $\alpha_3'=\frac{1}{A}(1+\frac{B}{A})\alpha_3$ along with a simplification gives

$\alpha_9'=\frac{1}{A^7}\left(1+\frac{B}{A}\right)\Big[\alpha_9-9\frac{B}{A}\alpha_3\alpha_8+45\left(\frac{B}{A}\right)^2\alpha_3^2\alpha_7$
$-165\left(\frac{B}{A}\right)^3\alpha_3^3\alpha_6+495\left(\frac{B}{A}\right)^4\alpha_3^4\alpha_5-9\frac{B}{A}\alpha_4\alpha_7$
$+45\left(\frac{B}{A}\right)^2\alpha_4^2\alpha_5+90\left(\frac{B}{A}\right)^2\alpha_3\alpha_4\alpha_6-495\left(\frac{B}{A}\right)^3\alpha_3^2\alpha_4\alpha_5$
$+990\left(\frac{B}{A}\right)^4\alpha_3^3\alpha_4^2-9\frac{B}{A}\alpha_5\alpha_6+45\left(\frac{B}{A}\right)^2\alpha_3\alpha_5^2$
$-165\left(\frac{B}{A}\right)^3\alpha_3\alpha_4^3-1287\left(\frac{B}{A}\right)^5\alpha_3^5\alpha_4+429\left(\frac{B}{A}\right)^6\alpha_3^7\Big].$

This completes the proof.

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