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Monte Carlo Algorithms for the Solution of Quasi-Linear Dirichlet Boundary Value Problems of Elliptical Type

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Abstract The application of Monte Carlo methods in various fields is constantly growing due to increases in computer capabilities. Increasing speed and memory, and the wide availability of multiprocessor computers, allow us to solve many problems using the "method of statistical sampling", better known as the Monte Carlo method. Monte Carlo methods are known to have particular strengths. These include: Algorithmic simplicity with a strong analogy to the underlying physical processes, solve complex realistic problems that include sophisticated geometry and many physical processes, solve problems with high dimensions, the ability to obtain point solutions or evaluation linear functional of the solution, error estimates can be empirically obtained for all types of problems in parallel way, and ease of efficient parallel implementation. A shortcoming of the method is slow rate of convergence of the error, namely $O(n^{-1/2})$ where n is the number of numerical experiments or realizations of the random variable. In this paper, we will propose Monte Carlo algorithms for the solution of the interior Dirichlet boundary value problem (BVP) for the Helmholtz operator with a polynomial nonlinearity on the right-hand side. The statistical algorithm is justified and complexity of the proposed algorithms is investigated, also the ways of decreasing the computational work are considered.

Keywords Monte Carlo Algorithms, Helmholtz Operatos, Polynomial Nonlinearity, Branching Markov Processes, Unbiased Estimators, Variance Decreasing

1 Introduction

We will consider the Dirichlet problem for quasi linear elliptic equations in the convex domain D, which is bounded and has a smooth boundary Γ

$$\Delta u(x) = -f(x, u(x)), u(x)|_\Gamma = \varphi(x). \tag{1}$$

Here $x \in R^3$, Δ is the Laplace operator, the function $f(x, u)$ has continuous derivatives for all $x \in \overline{D}$ and for all u, $\varphi(x) \in C^1(\Gamma)$. Let f(x,u) be $|f(x), u(x))| \leq L$, where $L = const$, and let it satisfy the following conditions $\dfrac{\partial f(x,u)}{\partial u} \geq 0$. Under these conditions, the problem (1) has a unique solution [1]. We consider the case when

$$f(x, u(x)) = \sum_{i=1}^{n} a_i(x)u^i(x) + a_0(x)$$

here all $a_i(x)$ are smooth functions.

The Monte Carlo algorithms for the solution (BVP) with Helmholtz operator in linear case were investigated in work [2]. Using the results of [2], Rasulov and Sipin [3] proposed an appoach connected with branching Markov processes and applied this approach to the solution of the following nonlinear Dirichlet problem

$$\Delta u(x) = u^2(x) + f(x), \quad u(x)|_\Gamma = \varphi(x), \quad x \in D \subset R^3$$

Futher in [4], Mikhaylov proposed a new approach Monte Carlo solution nonlinear elliptic equations. Below we generalized methods used in [3], [9].

Let $c = \sup\limits_{x\in D} \dfrac{\partial f(x,u)}{\partial u}$.

After transforming (1) we have

$$\begin{aligned} &\Delta u(x) - cu(x) = -(f(x, u(x) + cu(x)), \\ &u(x)|_\Gamma = \varphi(x). \end{aligned} \tag{2}$$

Let introduce the parameters $\alpha_i, \quad i = 0, .., n+1$ and require that they satisfy the conditions

$$\sum_{i=0}^{n+1} \alpha_i = 1, \quad \text{where } 0 < \alpha_i < 1 \quad i = 0, .., n+1 \tag{3}$$

Futher, we will use the parameters for constrution of our Monte Carlo algorithms.

2 Construction of Statistical Algorithms and Justification

Now we will construct unbiased estimators for the solution, $u(x)$, of problem (2). Using definition from [2] for the point solution, we determine a ball, $K_\rho(x)$, and a sphere $S_\rho(x)$ with maximal radius ρ and center at the point x: $K_\rho(x) \subset D$. From the theory of fundamental solutions for Helmholtz operator, it's known that we can use Green's formula to obtain the following integral equation

$$\begin{aligned} u(x) = (1 - q(x)) \int_{S_\rho(x)} u(y_1) \frac{dy_1 s}{4\pi\rho^2} + \\ + \frac{1}{c} q(x) \int_{K_\rho(x)} p(x, y_2) \left[f(y_2, u(y_2)) + cu(y_2) \right] dy_2. \end{aligned} \tag{4}$$

Here

$$q(x) = 1 - \frac{\rho\sqrt{c}}{\sin h(\rho\sqrt{c})}, \quad p(x, y_2) = \frac{c \sin h((\rho - r)\sqrt{c})}{4\pi r q(x) \sin h(\rho\sqrt{c})}$$

Now we transform the equation (4) to the following way

$$\begin{aligned} u(x) = (1 - q(x)) \int_{S_\rho(x)} u(y_1) \frac{dy_1 S}{4\pi\rho^2} + \\ \alpha_0 q(x) \int_{K_\rho(x)} p(x, y_2) \frac{a_0(y_2)}{c\alpha_0} dy_2 + \\ + \sum_{i=1}^{n} \alpha_i q(x) \int_{K_\rho(x)} p(x, y_2) \frac{a_i(y_2)}{c\alpha_i} u^i(y_2) dy_2 + \\ + \alpha_{n+1} q(x) \int_{K_\rho(x)} p(x, y_2) \frac{u(y_2)}{\alpha_{n+1}} dy_2, r = |x - y_2|. \end{aligned}$$

Furthermore, the results of work [5] are used. In the abovementioned books, a connection branching Markov process with one type of particle and the solution of polynomial nonlinear integral equation are investigated in detail. Using this connection, in a domain D, we define a branching Markov process. On its trajectory we construct the sequence $\xi_t(x)\}_{t=1}^{\infty}$ of unbiased estimators at the point $x \in D$ for the unknown function $u(x)$.

Suppose initially we have a particle at the point x. For a one step transition, the particle moves with probability $1 - q(x)$ to the point y_1 which is uniformly distributed on the sphere $S_\rho(x)$. In this case, the estimator of the solution takes the form of $\xi_1(x) = 1 * u(y_1)$. Further, with probability $\alpha_0 q(x)$, the particle is absorbed at the point y_2, which is distributed in $K_\rho(x)$ with density $p(x, y_2)$. In this case, $\xi_1(x) = \frac{a_0(y_2)}{\alpha_0 c} u(y_2)$. With probability $\alpha_i q(x), i = 1, .., n$ at the point y_2 , i particles are generated. In this case, the estimator has a form $\xi_1(x) = \frac{a_i(y_2)}{\alpha_i c} u^i(y_2)$. With probability $\alpha_{n+1} q(x)$, at the point y_2, one particle is generated, and $\xi_1(x) = \frac{1}{\alpha_{n+1}} u(y_2)$. New particles behave the same as their parents. The estimator then takes the form $\xi_t(x) = \xi_{t-1}(x)$, if the process terminates up to time t. Otherwise it is obtained from $\xi_{t-1}(x)$ by substituting $u(y)$ with their estimators. If the particle hits the boundary, Γ, it is absorbed. In this case, $\xi_1(x) = \varphi(y)$, where $y \in \Gamma$.

We constructed a branching Markov process with one type of particle. The fact is that the proposed algorithm is similar to the argument for equation (1) (see [3], [6]). We state this without proof, as the proof appears to be a simple generalization of the results in above mentioned works. However, the current results are more general from work [3], [6].

For the constructing finite random process "random walks on the sphere with branching" we derive an expression for the average number of branches. Let's call $M(x)$ the average number of branches in the trajectory beginning at the point $x \in D$. Then for $M(x)$ the following statement will be valid.

Theorem 1 . *The average number of branches consructed processes, $M(x)$, in D is a solution to the following Dirichlet boundary value problem:*

$$\begin{aligned} \Delta M(x) + c \left(\alpha_{n+1} + \sum_{i=1}^{n} i\alpha_i - 1 \right) M(x) = -c\alpha_0, \\ M(x)|_\Gamma = 0. \end{aligned} \tag{5}$$

Proof. From the constructed process, it is easy to see that $M(x)$ satisfies the following equation

$$\begin{aligned} M(x) = (1 - q(x)) \int_{S_\rho(x)} M(y) \frac{dyS}{4\pi\rho^2} + \\ \alpha_0 q(x) \int_{K_\rho(x)} p(x, y) dy + \\ + \sum_{i=1}^{n} \alpha_i q(x) \int_{K_\rho(x)} p(x, y) i M(y) dy \\ + \alpha_{n+1} q(x) \int_{K_\rho(x)} p(x, y) M(y) dy, \end{aligned} \tag{6}$$

For the equation (6), the method of Picard iteration converges from the initial function $\alpha_0 q(x)$, because $\int_{K_\rho(x)} p(x, y) dy = 1$.

Then it is easy to see that $M(x)$ coincides with solution of the Dirichlet problem (5), which proves the theorem.

The constructed process may be embedded into the branching process with diffusion of a single type of particle. Then we can derive easily the condition for absorption of such a particle.

Suppose that a particle in one unit time, with probability $\alpha_i\, q(x), \quad i = 1, .., n$ goes from the point x to the point $y \in K_\rho(x)$, which is distributed in $K_\rho(x)$ with transition density $p(x, y)$ and generates i particles of the same type. With probability $1 - q(x)$, a particle moves to the point y, which is uniformly distributed on the sphere $S_\rho(x)$ and generates one particle. We will denote $p(x, y, t)$ as the density function of diffusing particle, at time t, we assume that a particle begins diffusing at time zero from the point x.

It is known from [6] that the density function for such diffusing particles satisfies the equation

$$\frac{\partial p(x,y,t)}{\partial t} = \Delta_x p(x,y,t), \qquad p(x,y,0) = \delta(x-y), \quad p(x,y,t)\,|_{x\to\Gamma} = 0 \tag{7}$$

Denote by $\tau_{x,b(x)}$ the time of first branching, $b(x)$ is the average number of particles, which will appear in one transition over time Δt.

If $\tau_x = \min\{\tau_{x,k}, \tau_{x,b(x)}\}$, where $\tau_{x,k}$ is the first exit time of the diffusing particle from the sphere $S_\rho(x)$. It is clear that the distribution of a branching spherical random walk starting from x over one time step coincides with the distribution corresponding to branching diffusion at time τ_x.

Let $M(x,t)$ be the average number branches for the diffusion processes at time t. It is easy to show, that $M(x) = \lim_{t\to\infty} \inf M(x,t)$, where $M(x)$ is the average number branches for the trajectory begins at the point $x \in D$. That is the way that we investigate the behavior of $M(x,t)$ as $t \to \infty$.

From the equation (6) we get that

$$b(x) = \left(\sum_{i=1}^{n} i\alpha_i + \alpha_{n+1} - 1\right) q(x) + 1.$$

Theorem 2 *To have a bounded average branching number in our constructed branching random walk on the sphere, it is necessary and sufficient that* $\sup_{x\in D}\{c(b(x)-1\} < \lambda_1$*; here* $\lambda_1 > o$ *is the first (minimal) eigenvalue to the following boundary value problem in* D*:*

$$\Delta u(x) + \lambda u(x) = 0, \quad u(x)_\Gamma = 0.$$

Proof. It's easy to see [6], that $M(x,0) = 0$ and $M(x,t)|_{x\to\Gamma} = 0$, so $M(x,t) = \int_D \int_0^t p(x,y,t) c \exp(-cs)\left[1 + b(x)M(y,t-s)\right] dyds$.

Taking that derivative of the above expression, and using (7), we derive

$$\frac{\partial M(x,t)}{\partial t} = \Delta_x M(x,) + a(x)M(x,t) + c, \qquad M(x,0) = 0, M(x,t)\,|_{x\to\Gamma} = 0, \tag{8}$$

here a(x)=c(b(x)-1). We will solve problem (8) by a Fourier series method. We will seek a solution of the form

$$M(x,t) = \sum_{m=1}^{\infty} \varepsilon_m(t)\varphi_m(x) \tag{9}$$

Here $\varphi_m(x)$ is the eigen function of the problem

$$\Delta \quad \varphi_m(x) + \lambda_m \varphi_m(x) = 0, \quad \varphi_m(x)|_\Gamma = 0, \quad x \in D.$$

It is known, that $0 < \lambda_1 \le \lambda_2 \le \lambda_3, \ldots$, and $\varphi_1(x) > 0$ in D. By substituting (9) into (8) and equating the coefficients of $\varphi_m(x)$ we obtain $\frac{\partial \varepsilon_m(t)}{\partial t} = -\lambda_m \varepsilon_m(t) + a(x)\varepsilon_m(t) + c$, $\varepsilon_m(0) = 0$. From here $\varepsilon_m(t) = \int_0^t c \exp\{\vartheta_m(t-s)\}\, ds$, $\vartheta_m = (a(x) - \lambda_m)$. So, for $M(x,t)$ we have $M(x,t) = \sum_{m=1}^{\infty} (c \exp\{v_m(t-s)\}\, ds)\varphi_m(x)$. We can conclude that the series (9) is uniformly convergent when $t > 0$, since the eigenvalues, λ_m, tends to infinity and absolute value of the eigenfunctions $|\varphi_m(x)|$ are bounded above by a polynomial λ_m [7]. That is why the first term in (9) controls then asymptotic as $t \to \infty$. Thus, the constructed process will terminate (or the process will be degenerate), if $\vartheta_1 < 0$, which implies $\sup_{x\in D}\{c(b(x)-1\} < \lambda_1$, which proved the theorem.

Now we study the behavior of the constructed unbiased estimators $\{\xi_t(x)\}_{t=1}^{\infty}$. This sequence of estimators forms a martingale. Let $\{F_t\}_{t=1}^{\infty}$ be a family of $\sigma-$ algebras, where F_t is generated by the branching process up to time t, if s $\le t$ then $F_s \subseteq F_t$. The family of random variables, $\{\xi_t(x), F_t\}_{t=1}^{\infty}$ is F_t measurable for each t, $\xi_n(x) = E_x\{\xi_\infty(x)/F_n\}$ and $E_x|\xi_t(x)| < +\infty$. It can be easily shown [9] that if $s < t$ and $A \in F_s$, then $\int_A \xi_t dx \le \int_A \xi_s dx$. Since $\xi_t(x) \ge 0$ for each t, the family of random variables is a supermartingale.

Theorem 3 *The sequence of random variables,* $\xi_t(x)$*, tends to the integrable random variable* $\xi_\infty(x)$ *with the probability one, as* $t \to \infty$*.*

Proof. We notice that $E_x\xi_t(x) \ge E_x\xi_\infty(x)$ since $\xi_t(x)$ is a supermartingale. The above inequality will be an equality, if $\xi_t(x)$ is uniformly integrable. To show uniform integrability, it is enough to show that the following condition $\sup_t E_x\varepsilon_t^2(x) \le +\infty$ (see:[10]). Consider the following boundary value problem in D

$$\Delta\omega(x) - c\omega(x) = -\tilde{f}_1(x,\omega(x)), \qquad \omega(x)|_\Gamma = \varphi^2(x), \tag{10}$$

where $\tilde{f}_1(x,\omega(x)) = \frac{a_0^2(x)}{c\alpha_0} + \sum_{i=1}^{n} \frac{a_i^2(x)}{c\alpha_i}\omega^{2i}(x) + \frac{c}{\alpha_{n+1}}\omega(x)$.

There is the unique solution of this problem, and moreover $|\omega(x)| < \vartheta(x)$, where $\vartheta(x)$ is the solution of the problem $\Delta\vartheta(x) = -L$, $\vartheta(x)|_\Gamma = 0$. Since condition (3) is satisfied [1], from the maximum principle, it follows that $\vartheta(x) \ge 0$ and is bounded. Now we will construct a sequence of unbiased estimators $\{\eta_t(x)\}_{t=1}^{\infty}$ for problem (10). Since this sequence is analogous to what we did for problem (2), the sequence $\{\eta_t(x)\}_{t=1}^{\infty}$ is a supermartingale $\vartheta(x) \ge E_x\eta(x) \ge E_x\eta_\infty(x)$. Let's say $E_x\eta_t(x) = E_x\xi_t^2(x)$, so that $\sup_t E_x\xi_t^2(x) \le \vartheta(x)$. This implies that $\{\xi_t(x)\}_{t=1}^{\infty}$ is uniformly integrable, and in this case $E_x\xi_\infty(x) = u(x)$, which proved the theorem.

The estimator $\xi_\infty(x)$ is non realizable computationally because the process cannot hit the boundary in a finite number of steps. That is why we construct an ε -biased estimator [2], [3].

For that purpose, we define the ε-neighborhood of the boundary Γ, $\Gamma_\varepsilon = \{x \in \overline{D} : \rho(x) < \varepsilon\}$, where $\rho(x) = \min_{y\in\Gamma} |x - y|$, $\overline{D} = D \cup \Gamma$.

Let $u(x)$ satisfy the Lipschitz condition $|u(x) - u(y)| \leq A|x - y|$. The "random walk with branching" process is absorbed when the trajectory hits Γ_ε. The difference between the new $\zeta_\varepsilon(x)$ and previous estimator, $\xi_t(x)$, is the function $\varphi(y)$, where $y \in \Gamma$, will be replaced with $\varphi(y^*)$ where $y^* \in \Gamma_\varepsilon$. We then obtain an ε - biased estimator. Repeating the above methodology, one can show that

$$E_x\varsigma_\varepsilon(x) = u(x),\ E_x\varsigma_\varepsilon^2(x) \leq v(x)$$

Let $u_\delta(x)$ be the solution of problem (2) with conditions $u_\delta(x)|_\Gamma = \varphi(x) - \delta$, $u_{-\delta}(x)|_\Gamma = \varphi(x) + \delta$, where δ is sufficiently small and $u_\delta(x) \geq 0$, $u_{-\delta}(x) \geq 0$ on the border Γ.

From the maximum principle we get

$$u_\delta(x) \leq u(x) \leq u_\delta(x) + \delta, \quad u_\delta(x) - \delta \leq u(x) - \delta \leq u_\delta(x).$$

Let $\varepsilon(\delta)$ be chosen, so that $|u_\delta(x) - u_\delta(y)| \leq \delta$, $|u_{-\delta}(x) - u_{-\delta}(y)| \leq \delta$, when $|x - y| < \varepsilon$. Then we have $u_\delta(x) - \delta < u_\delta(x) < u_\delta(x) + \delta$. If $y = y^*$, then $u_\delta(x) < u_\delta(y^*) = \varphi(y^*)$. Then $E_x\zeta_\varepsilon(x) \geq u(x) - \delta$ and so $E_x\zeta_\varepsilon(x) \leq u(x) + \delta$. This means that $\zeta_\varepsilon(x)$ is an ε-biased estimator. One can easily show that $E_x(\zeta_\varepsilon(x))^2$ is bounded.

3 The Case of Quadratic Nonlinearity

A quasi-linear second-order partial differential equation plays an important role in the theory of non-linear waves, in particular, in non-linear optics and plasma physics. Taking into account this, we shall consider only the simplest quasilinear equation for the problem (1) and more deeply investigate this case.

Let D be a bounded domain in R^3 with smooth boundary, Γ. We will study the following problem

$$\begin{aligned} &\Delta u(x) = u^2(x) + f(x), \\ &u(x)\,|_\Gamma = \varphi(x) \geq 0, \quad x \in D. \end{aligned} \tag{11}$$

It assumes that $f(x) \in C(\overline{D})$, $\varphi(x) \in C(\Gamma)$. In [8] it was proved that when $f(x) = 0$, the solution of the problem exists and is non-negative. Using that idea, we could obtain necessary and sufficient conditions for the existence of a non-negative solution of (11).

Theorem 4 *The problem (11) will have a unique non-negative solution if and only if the following problem*

$$\Delta\tilde{u}(x) = f(x), \quad \tilde{u}(x)|_\Gamma = \varphi(x) \geq 0, \quad x \in D$$

has a non-negative solution.

Proof. To prove existence of the solution to problem (11) when $\tilde{u}(x) \geq 0$, we consider the following iterative method

$$\begin{aligned} &\Delta u_{n+1}(x) = 2u_{n+1}(x)u_n(x) - u_n^2(x) + f(x), \\ &u_{n+1}(x)\,|_\Gamma = \varphi(x). \end{aligned}$$

Let $u_0(x) = 0$. Then $u_1(x) = \tilde{u}(x)$, and if we denote $\omega_n(x) = u_n(x) - u_{n+1}(x)$, $n = 1, 2, ...$ We derive $\Delta\omega_n(x) = 2u_n(x)\omega_n(x) - \omega_{n-1}^2(x), \omega_n(x)|_\Gamma = 0$. From this we obtain

$$\Delta\omega_1(x) = 2\vec{u}(x)\omega_1(x) - \tilde{u}^2(x), \omega_1(x)|_\Gamma = 0.$$

From the maximum principle, it follows that $0 \leq \omega_1(x) \leq \tilde{u}(x)/2$. Since $u_n(x) = \tilde{u}(x) - \omega_1(x) - ... - \omega_{n-1}(x)$, for $n = 2$, we get, that $u_2(x) \geq \tilde{u}(x)/2$. From this we get that $\omega_2(x) \geq 0$. Taking into account $2u_2(x)\omega_2(x) - \omega_1^2(x) \leq 0$, we have $\omega_2(x) \leq \tilde{u}(x)/4 = \tilde{u}(x)/2^2$. According to the principle of mathematical induction $0 \leq \omega_n(x) \leq \tilde{u}(x)/2^n$. Since $\omega_n(x)$ converges uniformly, there exists a positive solution of problem (11).

Necessity follows from equation

$$\Delta(u(x) - \tilde{u}(x)) = u^2(x), \quad (u(x) - \tilde{u}(x))|_\Gamma = 0.$$

Since the solution of the above problem is $u(x) - \tilde{u}(x) \leq 0$, we will get $\tilde{u}(x) \geq 0$. For proving uniqueness of the solution to (11) we assume that there exist two non negative solutions $u_1(x)$ and $u_2(x)$. We denote $v(x) = u_1(x) - u_2(x)$. It is easy to show that $v(x)$ satisfies

$$\Delta v(x) = (u_1(x) + u_2(x))v(x) = a(x)v(x), \quad v(x)|_\Gamma = 0.$$

When $a(x) \geq 0$ this problem will have only the zero solution. This means $u_1(x) = u_2(x)$, and the theorem is proved.

Now we propose Monte Carlo algorithm to solve problem (11). Let $c \geq \|\tilde{u}(x)\|_{C(\bar{D})}$, and define $\nu(x) = c - u(x)$. This (11) becomes

$$\begin{aligned} &-\Delta\nu(x) + 2c\nu(x) = c^2 + f(x) + \nu^2(x) \\ &\nu(x)\,|_\Gamma = c - \varphi(x). \end{aligned}$$

We will construct a sequence of unbiased estimators for the solution of the new problem at the point $x \in D$. Using Green's formula, we get

$$\begin{aligned} \nu(x) = {} & (1 - q(x)) \int\limits_{S_\rho(x)} \nu(y_1)\frac{dy_1 s}{4\pi\rho^2} + \\ & + \frac{1}{2c}q(x) \int\limits_{K_\rho(x)} p(x, y_2)(f(y_2) + c^2 + \nu^2(y_2))dy_2. \end{aligned} \tag{12}$$

Here

$$q(x) = 1 - \rho\sqrt{2c}/\sinh(\rho\sqrt{2c}), \quad r = |x - y_2|\,,$$

$$p(x, y_2) = 2\cosh((\rho - r)\sqrt{2c})/4\pi r q(x)\sinh(\rho\sqrt{2c}).$$

In the domain, D, we define the branching random process, and on the trajectory of that process we construct the sequence $\{\xi_t(x)\}_{t=1}^\infty$ of unbiased estimators for $v(x)$ in $x \in D$. At the initial point x we have only one particle. In one step, with probability $q(x)/2$, it is absorbed at y_2, where y_2 is distributed with probability density $p(x, y_2)$. In this case, the estimator will be $\xi_1(x) = c + f(y_2)/c$. With probability $1 - q(x)$ the particle moves to the point y_1. The point y_1 is uniformly distributed on the sphere, $S_\rho(x)$, and the estimator will be $\xi_1(x) = v(y_1)$.

With probability $q(x)/2$ the particle will generate two particles at the point y_2, in this case $\xi_1(x) = (1/c)v^2(y_2)$.

New particles behave the same as their parents. If the process terminates up to time t, the estimator will be $\xi_t(x) = \xi_{t-1}(x)$. Otherwise $\xi_{t-1}(x)$ will be calculated replacing $v(y)$ to their estimators $\xi_1(x)$. If $x \in \Gamma_\varepsilon$ then the processes will terminate, and $\xi_1(x) = c - \varphi(x)$.

Based on the above, the following holds.

Theorem 5 *With probability one, the constructed branching random process will be absorbed or hit the boundary with finite number of generated particles.*

Proof. Let ν be the quantity of branching in the constructed process. We will show that the probability $P_x\{\nu < +\infty\} = 1$. Let $k > 1$, and m be the first branching time. After the first branching, the new particles will generate new branches. For the quantity of these new branches, we will denote as ν_1 and ν_2 respectively. Thus

$$P_x\{\nu > k\} = E_x\left[\sum_{m=1}^{\infty}\prod_{i=1}^{m-1}(1-q(x_i))\tfrac{1}{2}q(x_m)P_{x_{m+1}}\{(\nu_1+\nu_2) > (k-1)\}\right]$$

$$\le \psi(x)\max_{x\in D} P_x\{\nu > (k-1)/2\}$$

where $\psi(x) = E_x\left[\sum_{m=1}^{\infty}\prod_{i=1}^{m-1}(1-q(x_i))q(x_m)\right] \le \delta < 1$.

Therefore $P_x\{\nu > k\} \le \delta^{\ln 2^k}$, then $P_x\{\nu > k\}_{k\to\infty} \to 0$, and the theorem is proved.

Let $M(x)$ be the average number of branches for trajectories starting at $x \in D$. Analogous to theorem 1, we can prove the following assertion.

Theorem 6 *The average number of branches will be* $M(x) = c\int_D G(x,y)dy$, *where* $G(x,y)$ *is a Green's function of the following Dirichlet's problem*

$$-\Delta M(x) = c, \quad M(x)|_\Gamma = 0.$$

If we show that $\sup_t E_x\xi_t^2(x) < +\infty$, than the constructed sequence of unbiased estimators is uniformly integrable, and $E_x\xi_t(x) = E_x\xi_\infty(x) = v(x)$. We will choose the constant c, and show that $\sup_t E_x\xi_t^2(x) < +\infty$ in next assertion.

Theorem 7 *Let*

$$c \ge (\|\tilde{u}(x)\|_{C(\bar{D})} + (\|\tilde{u}(x)\|^4_{C(\bar{D})} + 2m^2d^2u_0)^{\frac{1}{2}}/u_0,$$

where $m = \|f(x)\|_{C(\bar{D})}$, *$d$ is the diameter of the domain D and* $u_0 = \min_{x\in D}|\tilde{u}(x)| \ge 0$. *Then* $\sup_t E_x\xi_t^2(x) \le c^2$.

The prove of this theorem is the same as the theorem on existence and uniqueness for problem (11) (see:[8]). Let x^* be point in the ε -neighborhood of Γ. Replace x in the estimators using $\varphi(x)$ with the $\varphi(x^*)$ to get the estimator $\xi_t^*(x)$. Because of ε -bias, we can easily prove the following assertion:

Theorem 8 *If* $|x^* - x| < \varepsilon$, *then* $|v(x) - E_x\xi_t^*(x)| < k\varepsilon$, *where k is some constant.*

Analogously for problem (11), we can derive an algorithm for solution to the following problem

$$\begin{aligned}\Delta u(x) &= a(x)u^2(x) + f(x),\\ u(x)\,|_\Gamma &= \varphi(x), \quad a(x) \ge 0.\end{aligned} \tag{13}$$

In this case, we use the same transformations as with (11) and derive the probability density for the transition of particles the following

$p(x, y_2) =$
$2ca(x)\sinh((\rho - r)\sqrt{2ca(x)})/4\pi rq(x)\sinh(\rho\sqrt{2ca(x)})$,
where, $q(x) = 1 - \rho\sqrt{2ca(x)}/\sinh(\rho\sqrt{2ca(x)})$,
$r = |x - y_2|$.

4 Conclusion

Thus, using priori information about solutions of the equations, of interest, one can give recommendations on choosing optimal estimators. The results of the computational experiment show that with the discussed algorithms we can obtain practically efficient estimators. The parameter, which is used in calculating the probabilities of the branching, gives us the opportunity to control the average quantity of branches and the average number of the particles in the tree. Thus the estimators were constructed on trees with a minimal number of branches. We can consider the case of when f (x,u)= k exp(u),f (x,u)=k sin(u) and other functions, where k is constant, and after expanding to the Taylor series f (x,u) above proposed algorithms could be applied for the solution corresponding (BVP) boundary value problems. In our methodology, we have an estimator via absorption. In the furure, we could build an estimator via collision and hope will get new results. The results of numerical experiments and applications to concret problems will be given in our next publications.

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