Mathematics and Statistics 11(3): 586-591, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110316

Statistical Convergence on Intuitionistic Fuzzy Normed Spaces over Non-Archimedean Fields

N. Saranya, **K. Suja***

Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur, Chennai 603203, India

Received February 14, 2023; Revised April 18, 2023; Accepted May 9, 2023

Cite This Paper in the following Citation Styles

***(a)**: [1] N. Saranya, K. Suja, "Statistical Convergence on Intuitionistic Fuzzy Normed Spaces over Non-Archimedean Fields," Mathematics and Statistics, Vol.11, No.3, pp. 586-591, 2023. DOI: 10.13189/ms.2023.110316*

***(b)**: N. Saranya, K. Suja, (2023). Statistical Convergence on Intuitionistic Fuzzy Normed Spaces over Non-Archimedean Fields. Mathematics and Statistics, 11(3), 586-591. DOI: 10.13189/ms.2023.110316*

Abstract This paper aims to explore the fundamental properties of statistical convergence sequences within non-Archimedean fields. In pure mathematics, statistical convergence plays a fundamental role. The idea of statistical convergence is an extension of the concept of convergence. Statistical convergence has been discussed in various fields of mathematics namely ergodic theory , fuzzy set theory , approximation theory, measure theory, probability theory, trigonometric series, number theory, and banach spaces, where problems were resolved using the concept of statistical convergence. Summability theory and functional analysis are two disciplines that heavily rely on the idea of statistical convergence. The study of analysis over non-Archimedean fields is called non-Archimedean analysis. The theory of statistical convergence plays a significant role in the functional analysis and summability theory. The objective of this paper is to expand upon the concepts of statistical convergence and statistically Cauchy sequences in non-Archimedean intuitionistic fuzzy normed spaces, and obtain some relevant results related to them. This article proves that some properties of statistically convergent sequences, which are not true classically, are true in a non-Archimedean field. Furthermore, in these spaces, we defined statistically complete and statistically continuous and established some fundamental facts.Throughout this paper, $\mathcal{K}$ denotes a complete, non-trivially valued, non-Archimedean field.

Keywords Non-Archimedean Fields, Statistically Convergent, Statistically Cauchy Sequence, Intuitionistic Fuzzy Normed Spaces

1 Introduction

Statistical convergence of a real number sequence was originated by Fast [7] and it was further investigated by Fridy [8]. After the work of Fridy, it becomes a significant topic in summability theory. Quite a few researchers see for instance [5, 9–12] have extended and generalized this concept and applied different fields of mathematics. The fuzzy set characterized by a membership function was first introduced by Zadeh [16]. Later on, many researchers like [3, 10, 13] applied this theory to the classical set theory. The concept of an intuitionistic fuzzy set was introduced by Atanassov [2]. Saadati and Park [14] gave the idea of intuitionistic fuzzy normed space. The concepts of fuzzy minimality, fuzzy basicity, fuzzy biorthogonality, and fuzzy space of coefficients are introduced by Bilalov et al. [4]. The Non-Archimedean analysis is the study of analysis over non-Archimedean fields. Recently, Suja and Srinivasan [15] introduced the concept of statistically convergent and statistically Cauchy sequences in non-Archimedean fields. Eghbali and Ganji [6] studied the generalized statistical convergence in the non-Archimedean L-fuzzy normed spaces. The paper demonstrates the presence of statistical convergence in non-Archimedean intuitionistic fuzzy normed spaces (NA-IFN Spaces) and establishes that certain features of statistical convergence of real sequences remain applicable in non-Archimedean fields. The article specifically focuses on the analysis of sequences within the non-Archimedean field $\mathcal{K}$.

A sequence $x = \{a_k\}$ is said to be statistically convergent to a limit ℓ if for any $\epsilon > 0$,
$\lim\limits_{n\to\infty} \frac{1}{n}\{k \leq n : |a_k - \ell| \geq \epsilon\} = 0.$
In this case, we write
$stat - \lim\limits_{k\to\infty} a_k = \ell.$

Example 1.1 Consider the sequence $x = \{a_k\}$ defined by

$$a_k = \begin{cases} \frac{k-1}{k^2}, & \text{k is a perfect square.} \\ 0, & \text{otherwise;} \end{cases}$$

Choosing the non-Archimedean valuation to be 2-adic, the terms of the sequence are $(0, 0, 0, 1, 0, 0, 0, 0, 1/8, 0, 0, \dots)$. Thus, it is statistically convergent to zero .

The sequence $x = \{x_i\}$ is said to be statistically Cauchy sequence if for every $\varepsilon > 0$, there exists a number $n \in \mathcal{N}$ such that

$$\lim_{n\to\infty} \frac{1}{n}\Big|\{i \le n; n \in \mathcal{N} : |x_{i+1} - x_i| \ge \varepsilon\}\Big| = 0.$$

Let $\mathcal{K}$ be a non-Archimedean fields. A valuation on $\mathcal{K}$ is said to be non-Archimedean if satisfies the following axioms:[1]

(i) $|x| \ge 0$ and $|x| = 0$ iff $x = 0$,

(ii) $|xy| = |x||y|$,

(iii) $|x + y| \le \max[|x|, |y|]$ for all $x, y \in \mathcal{K}$ (Ultrametric Inequality).

2 Preliminaries

In this section, we will go through the notations and definitions that will be utilized throughout the paper in order to ensure a common understanding of the terminology and symbols used.

Definition 2.1 A binary operations $* : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a continuous t-norm, if it satisfies the following conditions:

(a) * is associative and commutative,

(b) * is continuous,

(c) $a * 1 = a$ for all $a \in [0, 1]$,

(d) $a*b \le c*d$, whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0, 1]$.

Definition 2.2 A binary operation $\diamond : [0, 1] \times [0, 1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(a') $\diamond$ is associative and commutative,

(b') $\diamond$ is continuous,

(c') $a \diamond 0 = a$ for all $a \in [0, 1]$,

(d') $a\diamond b \le c\diamond d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0, 1]$.

Definition 2.3 The five tuple $(V, \mu, \phi, *, \diamond)$ is said to be a NA-IFN space if V is a vector space over a field $\mathcal{K}$, $\diamond$ is a continuous t-conorm, $*$ is a continuous t-norm, and μ, ϕ are functions from $V \times \mathbb{R}$ to $[0, 1]$ satisfying the following coditions. For every $a, b \in V$ and $u, v \in \mathcal{K}$.

(i) $\mu(a, u) + \phi(a, u) \le 1$,

(ii) $\mu(a, u) > 0$,

(iii) $\mu(a, u) = 1 \Longleftrightarrow a = 0$,

(iv) $\mu(\alpha a, u) = \mu(a, \frac{1}{|\alpha|})$ for each $\alpha \ne 0$,

(v) $\mu(a, u) * \mu(b, v) \le \mu(a + b, max\{u, v\})$,

(vi) $\mu(a, \cdot) : (0, \infty) \to [0, 1]$ is continuous,

(vii) $\lim_{u\to\infty} \mu(a, u) = 1$ and $\lim_{u\to 0} \mu(a, u) = 0$,

(viii) $\phi(a, u) < 1$,

(ix) $\phi(a, u) = 0 \Longleftrightarrow a = 0$,

(x) $\phi(\alpha a, u) = \phi(a, \frac{1}{|\alpha|})$ for each $\alpha \ne 0$,

(xi) $\phi(a, u) \diamond \phi(b, v) \ge \phi(a + b, max\{u, v\})$,

(xii) $\phi(a, \cdot) : (0, \infty) \to [0, 1]$ is continuous and

(xiii) $\lim_{u\to\infty} \phi(a, u) = 0$ and $\lim_{u\to 0} \phi(a, u) = 1$.

Here, (μ, ϕ) is called a non-Archimedean intuitionistic fuzzy norm.
A sequence $\{a_k\}$ is said to be convergent in NA-IFN space $(V, \mu, \phi, *, \diamond)$ or simply (μ, ϕ)-convergent to $x \in V$ if for every $u > 0$ and $\epsilon > 0$, there exist $k_0 \in \mathbb{N}$ such that $k \ge k_0$,

$$\mu(a_k - x, u) > 1 - \epsilon \; and \; \phi(a_k - x, u) < \epsilon$$

In this case, we write $(\mu, \phi) - \lim_k a_k = x$.
Example 2.1 Let $(V, \mu, \phi, *, \diamond)$ be a non-Archimedean normed space, $a * b = ab$ and $a \diamond b = min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in X$, every $t > 0$ and $k = 1, 2, ...$ Consider the following,

$$\mu_k(x, t) = \begin{cases} \frac{t}{t+k\|x\|}, & \text{if } t > 0 \\ 0, & t \le 0; \end{cases}$$

$$\phi_k(x, t) = \begin{cases} \frac{k\|x\|}{t+k\|x\|}, & \text{if } t > 0 \\ 1, & t \le 0; \end{cases}$$

Then $(V, \mu, \phi, *, \diamond)$ is a non-Archimedean intuitionistic fuzzy normed space.

Definition 2.4 A sequence $\{a_k\}$ in a NA-IFN space $(V, \mu, \phi, *, \diamond)$ is said to be statistically convergent to a limit $x \in V$ with respect to the non-Archimedean fuzzy norm (μ, ϕ) if for every $\epsilon > 0$ and $u > 0$,

$\lim_n \frac{1}{n}|k \leq n : \mu(a_k - x, u) < 1-\epsilon \; or \; \phi(a_k - x, u) \geq \epsilon| = 0$. In this case, we write

$$stat_{\mu,\phi} - \lim_k a_k = x$$

Where x is the $stat_{\mu,\phi} - limit$.

Example 2.2 Let $(\mathcal{Q}_p, |.|)$ denote the space of p-adic numbers with the usual norm, and let $a * b = ab$ and $a \diamond b = min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in \mathcal{Q}_p$ and every $t > 0$, consider
$\mu_0(x,t) = \frac{t}{t+|x|}$ and $\phi_0(x,t) = \frac{|x|}{t+|x|}$. In this case observe that $(\mathcal{Q}_p, \mu, \phi, *, \diamond)$ is an IFNS.
Now define a sequence $x = \{a_k\}$ whose terms are given by

$$a_k = \begin{cases} 1, & \text{if } k = m^2 (m \in \mathcal{N}) \\ 0, & \text{otherwise;} \end{cases}$$

Then for every $0 < \epsilon < 1$ and for any $t > 0$, let $K_n(\epsilon, t) = k \leq n : \mu_0(a_k, t) \leq 1-\epsilon \; or \; \phi_0(a_k, t) \geq \epsilon$.
Since
$k_n(\epsilon, t) = \left\{k \leq n : \frac{t}{t+|a_k|} \leq 1-\epsilon \; or \; \frac{|a_k|}{t+|a_k|} \geq \epsilon\right\}$
$= \left\{k \leq n : |a_k| \geq \frac{\epsilon t}{1-\epsilon} > 0\right\} = \left\{k \leq n : |a_k| = 1\right\}$
$= \left\{k \leq n : k = m^2 \; and \; m \in \mathcal{N}\right\}$
We have,
$\frac{1}{n}|k_n(\epsilon, t)| = \frac{1}{n}\left\{k \leq n : k = m^2 \; and \; m \in \mathcal{N}\right\} \leq \frac{\sqrt{n}}{n}$
Which yields that
$\lim_n \frac{1}{n}|k_n(\epsilon, t)| = 0$
Hence by the above definition, $stat_{\mu,\phi} - lim a_k = 0$.

3 Statistical Convergence on Intuitionistic Fuzzy Normed Spaces

The aim of this section is to establish theorems concerning convergence and statistical convergence within the context of intuitionistic fuzzy normed spaces over non-Archimedean fields $\mathcal{K}$.

Lemma 3.1 Let $(V, \mu, \phi, *, \diamond)$ be a NA-IFN space. Then the followimg statements are equivalent for every $\epsilon > 0$ and $u > 0$:
(i) $Stat_{\mu,\phi} - \lim_n a_k = x$.
(ii) $\lim_n \frac{1}{n}|\{k \leq n : \mu(a_k - x, u) \leq 1-\epsilon\}| = \lim_n \frac{1}{n}|\{k \leq n : \phi(a_k - x, u) \geq \epsilon\}| = 0$.
(iii) $\lim_n \frac{1}{n}|\{k \leq n : \mu(a_k - x, u) > 1-\epsilon \; and \; \phi(a_k - x, u) < \epsilon\}| = 1$.
(iv) $\lim_n \frac{1}{n}|\{k \leq n : \mu(a_k - x, u) > 1-\epsilon\}| = \lim_n \frac{1}{n}|\{k \leq n : \phi(a_k - x, u) < \epsilon\}| = 1$.
(v) $stat - \lim \mu(a_k - x, u) = 1 \; and \; stat - lim \phi(a_k - x, u) = 0$.

Theorem 3.2 Let $(V, \mu, \phi, *, \diamond)$ be an NA-IFN space. If a sequence $\{a_k\}$ is statistically convergent with respect to the intuitionistic fuzzy norms (μ, ϕ), then $stat_{\mu,\phi}$ - limit is unique.

Proof: Assume that $stat_{\mu,\phi} - \lim_k a_k = x_1$ and $stat_{\mu,\phi} - \lim_k a_k = x_2$.
For a given $\epsilon > 0$, choose $t > 0$ such that $(1-t)*(1-t) > 1-\epsilon$ and $t \diamond t < \epsilon$. Then,for any $u > 0$, define the following sets :

$$\begin{aligned} k_{\mu,1}(t,u) &:= \{k \in \mathbb{N} : \mu(a_k - x_1, u) \leq 1-t\}, \\ k_{\mu,2}(t,u) &:= \{k \in \mathbb{N} : \mu(a_k - x_2, u) \leq 1-t\}, \\ k_{\phi,1}(t,u) &:= \{k \in \mathbb{N} : \phi(a_k - x_1, u) \geq t\}, \\ k_{\phi,2}(t,u) &:= \{k \in \mathbb{N} : \phi(a_k - x_2, u) \geq t\}. \end{aligned}$$

Since $stat_{\mu,\phi} - \lim_k a_k = x_1$, we have

$$\lim_n \frac{1}{n}\{k_{\mu,1}(\epsilon, u)\} = \lim_n \frac{1}{n}\{k_{r,1}(\epsilon, u)\} = 0 \; for \; all \; u > 0.$$

Furthermore, using $stat_{\mu,\phi} - \lim_k a_k = x_2$, we get

$$\lim_n \frac{1}{n}\{k_{\mu,2}(\epsilon, u)\} = \lim_n \frac{1}{n}\{k_{r,2}(\epsilon, u)\} = 0 \; for \; all \; u > 0.$$

Now let,

$$k_{\mu,\phi}(\epsilon, u) := \{k_{\mu,1}(\epsilon, u) \cup k_{\mu,2}(\epsilon, u)\} \cap \{k_{\phi,1}(\epsilon, u) \cup k_{\phi,2}(\epsilon, u)\}.$$

If $k_{\mu,\phi}(\epsilon, u) = k_{\mu,\phi}$, $\{k_{\mu,1}(\epsilon, u) \cup k_{\mu,2}(\epsilon, u)\} = k_\mu$ and $\{k_{\phi,1}(\epsilon, u) \cup k_{\phi,2}(\epsilon, u)\} = k_\phi$, then
$k_{\mu,\phi} = k_\mu \cap k_\phi$.
Then observe that, $\lim_n \frac{1}{n}\{k_{\mu,\phi}\} = 0$.
which implies, $\lim_n \frac{1}{n}\{k^C_{\mu,\phi}\} = 1$.
If $k \in k^C_{\mu,\phi}$, then there are two possibilities to consider:
The first case is $k \in \{k^C_\mu\}$ and the later is $k \in \{k^C_\phi\}$.
we first consider that $k \in \{k^C_\mu\}$.
Then we have,

$$\begin{aligned} \mu(x_1 - x_2, u) &= \mu(x_1 - a_k + a_k - x_2, u) \\ &\geq \mu(x_1 - a_k, u) * \mu(a_k - x_2, u) \\ &= \mu(a_k - x_1, u) * \mu(a_k - x_2, u) \\ &> (1-t) * (1-t). \end{aligned}$$

Since $(1-t) * (1-t) > 1-\epsilon$, it follows that

$$\mu(x_1 - x_2, u) > 1-\epsilon.$$

Since $\epsilon > 0$ was arbitrary,
$\mu(x_1 - x_2, u) > 1$ for all $u > 0$, which gives $x_1 = x_2$.
On the other hand, if $k \in \{k^C_\phi\}$, then we may write that,
$\phi(x_1 - x_2, u) \leq \phi(a_k - x_1, u) \diamond \phi(a_k - x_2, u) < t \diamond t$.
Now using the fact that $t \diamond t < \epsilon$, we see that $\phi(x_1 - x_2, u) < \epsilon$.
Again, since $\epsilon > 0$ was arbitrary, we have $\phi(x_1 - x_2, u) = 0$ for all $u > 0$. which implies $x_1 = x_2$.

Thus, $stat_{\mu,\phi}$ - limit is unique.

Theorem 3.3 If a sequence $\{a_k\}$ in a NA-IFN space $(V, \mu, \phi, *, \diamond)$ is (μ, ϕ)-convergent to $x \in V$, then it is $stat_{\mu,\phi}$ - convergent to $x \in V$.

Proof: Since $\{a_k\}$ is (μ, ϕ)-convergent to $x \in V$, for every $\epsilon > 0$ and $u > 0$, there exist $n_0 \in N$ such that for all $n \geq n_0$,

$$\mu(a_k - x, u) > 1 - \epsilon \, and \, \phi(a_k - x, u) < \epsilon.$$

This implies that the set $\{k \in \mathbb{N} : \mu(a_k - x, u) \leq 1 - \epsilon$ or $\phi(a_k - x, u) \geq \epsilon\}$ has at the most a finite number of terms.
ie, $\lim_n \frac{1}{n}|\{k \leq n : \mu(a_k - x, u) \leq 1 - \epsilon$ or $\phi(a_k - x, u) \geq \epsilon\}| = 0$.
ie, $stat_{\mu,\phi} - lim a_k = x$.

Note: It is interesting to note that the converse of this, which is not true classically, is true in a NA-IFN space as shown below.
Let $\{a_k\}$ be $stat_{\mu,\phi}$ - convergent to $x \in V$. Then for every $\epsilon > 0$ and $u > 0$,
$\lim_n \frac{1}{n}|\{k \leq n : \mu(a_k - x, u) \leq 1 - \epsilon \, or \, \phi(a_k - x, u) \geq \epsilon\}| = 0$
Now to prove that $\{a_k\}$ is (μ, ϕ) convergent to $a \in X$. ie, to prove that for every $\epsilon > 0$ and $u > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\mu(a_k - x, u) > 1 - \epsilon \, and \, \phi(a_k - x, u) < \epsilon.$$

Let us assume the contrary that,

$$\mu(a_k - x, u) \leq 1 - \epsilon \, and \, \phi(a_k - x, u) \geq \epsilon.$$

This implies that the set
$\{k \in \mathbb{N} : \mu(a_k - x, u) \leq 1 - \epsilon \, or \, \phi(a_k - x, u) \geq \epsilon\}$ has infinitely many terms.
ie, $\lim_n \frac{1}{n}|\{k \leq n : \mu(a_k - x, u) \leq 1 - \epsilon \, or$
$\phi(a_k - x, u) \geq \epsilon\}| \neq 0$.
which is a contradiction.
Therefore, $\{a_k\}$ is (μ, ϕ) convergent to $x \in V$.

Theorem 3.4 Let $\{a_k\}$ and $\{b_k\}$ be sequences in a NA-IFN space $(V, \mu, \phi, *, \diamond)$ such that $stat_{\mu,\phi} - \lim_{n\to\infty} a_k = a$ and $stat_{\mu,\phi} - \lim_{n\to\infty} b_k = b$ where $a, b \in V$. Then we have $stat_{\mu,\phi} - \lim_{k\to\infty}(a_k + b_k) = a + b$.

Proof:
Let $stat_{\mu,\phi} - \lim_{n\to\infty} a_k = a$ and $stat_{\mu,\phi} - \lim_{n\to\infty} b_k = b$.
Choose $t > 0$ such that $(1 - t) * (1 - t) > 1 - \epsilon$ and $t \diamond t < \epsilon$ for a given $\epsilon > 0$. Then, for $u > 0$, define

$$\begin{aligned} k_{\mu,1}(t, u) &:= \{k \in \mathbb{N} : \mu(a_k - a, u) \leq 1 - t\}, \\ k_{\mu,2}(t, u) &:= \{k \in \mathbb{N} : \mu(b_k - b, u) \leq 1 - t\}, \\ k_{\phi,1}(t, u) &:= \{k \in \mathbb{N} : \phi(a_k - a, u) \geq t\}, \\ k_{\phi,2}(t, u) &:= \{k \in \mathbb{N} : \phi(b_k - b, u) \geq t\}. \end{aligned}$$

Since $stat_{\mu,\phi} - \lim_{k\to\infty} a_k = a$ and $stat_{\mu,\phi} - \lim_{k\to\infty} b_k = b$,
$\lim_n \frac{1}{n}\{k_{\mu,1}(\epsilon, u)\} = \lim_n \frac{1}{n}\{k_{\phi,1}(\epsilon, u)\} = 0$,
$\lim_n \frac{1}{n}\{k_{\mu,2}(\epsilon, u)\} = \lim_n \frac{1}{n}\{k_{\phi,2}(\epsilon, u)\} = 0$.
Now, let

$$k_{\mu,\phi}(\epsilon, u) := \{k_{\mu,1}(\epsilon, u) \cup k_{\mu,2}(\epsilon, u)\} \cap \{k_{\phi,1}(\epsilon, u) \cup k_{\phi,2}(\epsilon, u)\}$$

ie, if $K = K_{\mu,\phi}(\epsilon, u)$, $K_1 = \{K_{\mu,1}(\epsilon, u) \cup K_{\mu,2}(\epsilon, u)\}$ and $K_2 = \{k_{\phi,1}(\epsilon, u) \cup k_{\phi,2}(\epsilon, u)\}$ then $K = K_1 \cap K_2$.
Since K^C is a non-empty set. Let $k \in K^C$ then we have two possible cases. The former is $k \in K_1^C$ and the later is $k \in K_2^C$.
First consider, $k \in K_1^C$, then we have,
$\mu(a_k - a, u) > 1 - t \, and \, \mu(b_k - b, u) > 1 - t$.
Now, we have,

$$\begin{aligned} \mu(a_k + b_k - a - b, u) &> \mu(a_k - a, u) * \mu(b_k - b, u) \\ &> (1 - t) * (1 - t). \end{aligned}$$

Since $(1 - t) * (1 - t) > 1 - \epsilon$, it follows that, $\mu(a_k + b_k - a - b, u) > 1 - \epsilon$.
Since ϵ is arbitrary, $\mu(a_k + b_k - a - b, u) = 1$ for all $u > 0$.
which yields, $\mu(a_k + b_k - (a + b), u) = 1$.
Similarlly, if $k \in K_2^C$ then, $\phi(a_k - a, u) < t$ and $\phi(b_k - b, u) < t)$.

$$\begin{aligned} \Rightarrow \phi(a_k + b_k - a - b, u) &\leq \phi(a_k - a, u) \diamond \phi(b_k - b, u) < t \\ &< t \diamond t \\ &< \epsilon. \end{aligned}$$

Since ϵ is arbitrary,
$\phi(a_k + b_k - a - b, u) = 0$, for all $u > 0$
$\Rightarrow \phi(a_k + b_k - (a + b), u) = 0$.
Thus, $stat_{\mu,\phi} - \lim_{k\to\infty}(a_k + b_k) = a + b$.

Theorem 3.5 Let $(V, \mu, \phi, *, \diamond)$ be an NA-IFN space over $\mathcal{K}$.If $\lim_{k\to\infty} \mu(a_k - a, u) = 1$ and $\lim_{k\to\infty} \phi(a_k - a, u) = 1$ then $stat_{\mu,\phi} - \lim_{k\to\infty} a_k = a$.

Proof:
Let $\lim_{k\to\infty} \mu(a_k - a, u) = 1$ and $\lim_{k\to\infty} \phi(a_k - a, u) = 1$. Then for every $t > 0$ and $\epsilon > 0$, there is a number $k_0 \in \mathbb{N}$ such that, $\mu(a_k - a, u) > 1 - \epsilon$ and $\phi(a_k - a, u) < \epsilon$ for all $k \geq k_0$.
Hence the set, $\{k \in \mathbb{N} : \mu(a_k - a, u) \leq 1 - \epsilon \, or \, \phi(a_k - a, u) \geq \epsilon\}$ has a finite number of terms.
So, $\lim_n \frac{1}{n}|\{k \leq n : \mu(a_k - a, u) \leq 1 - \epsilon \, or \, \phi(a_k - a, u) \geq \epsilon\}| = 0$.
Thus, $stat_{\mu,\phi} - \lim_{k\to\infty} a_k = a$.

4 Statistically Cauchy Sequences on Intuitionistic Fuzzy Normed Spaces

Definition 4.1 Let $(V, \mu, \phi, *, \diamond)$ be an NA-IFN space over $\mathcal{K}$. Then, a sequence $\{a_k\}$ is said to be statistically Cauchy if for every $\epsilon > 0$ and $u > 0$ there exists $\mathbb{N}$ such that for all $k, m \geq N$,
$\lim_n \frac{1}{n}|\{k, m \leq n : \mu(a_k - a_m, u) \leq 1 - \epsilon \; or \; \phi(a_k - a_m, u) \geq \epsilon\}| = 0.$

Definition 4.2 Let $(V, \mu, \phi, *, \diamond)$ be an NA-IFN space. A sequence $\{a_k\}$ is called a Cauchy sequence if for each $\epsilon > 0$ and $u > 0$, there exists a number $k_0 \in \mathbb{N}$ such that, for all $k, m \geq k_0$,
$\mu(a_k - a_m, u) > 1 - \epsilon$ and $\phi(a_k - a_m, u) < \epsilon$.

Theorem 4.1 Every Cauchy sequence with respect to (μ, ϕ) in NA-IFN space $(V, \mu, \phi, *, \diamond)$ over $\mathcal{K}$ is statistically Cauchy.
Proof:
If $\{a_k\}$ is a Cauchy sequence with respect to (μ, ϕ), then there exists $k_0 \in N$ for all $\epsilon > 0$ and $u > 0$ and let p be an arbitrary constant, we have
$\mu(a_{k+p} - a_k, u) > 1 - \epsilon$ and $\phi(a_{k+p} - a_k, u) < \epsilon$.
The number of terms in the set $\{k \in \mathbb{N} : \mu(a_{k+p} - a_k, u) \leq 1 - \epsilon \; or \; \phi(a_{k+p} - a_k, u) \geq \epsilon\}$ is limited. So
$\lim_n \frac{1}{n}|\{k + p, k \leq n : \mu(a_{k+p} - a_k, u) \leq 1 - \epsilon \; or \; \phi(a_{k+p} - a_k, u) \geq \epsilon\}| = 0.$

Theorem 4.2 If a sequence is statistically convergent in a NA-IFN space $(V, \mu, \phi, *, \diamond)$ over $\mathcal{K}$, then it is statistically Cauchy.
Proof:
If the sequence $\{a_k\}$ is statistically convergent to x then,
$\lim_n \frac{1}{n}|\{k \leq n : \mu(a_k - x, u) \leq 1 - \epsilon \; or \; \phi(a_k - x, u) \geq \epsilon\}| = 0.$
Now, we have

$$\begin{aligned}
&\lim_n \frac{1}{n}|\{k, m \leq n : \mu(a_k - a_m, u) \leq 1 - \epsilon \; or \; \phi(a_k - a_m, u) \geq \epsilon\}| \\
&= \lim_n \frac{1}{n}|\{k, m \leq n : \mu(a_k - x, u) * \mu(a_m - x, u) \leq 1 - \epsilon \; or \\
&\qquad \phi(a_k - x, u) \diamond \phi(a_m - x, u) \geq \epsilon\}| \\
&= 0.
\end{aligned}$$

5 Statistically complete and statistically continuous on Intuitionistic Fuzzy Normed Spaces

A NA-IFN space $(V, \mu, v, *, \diamond)$ is said to be complete if every (μ, ϕ)-Cauchy is (μ, ϕ)-convergent.

Definition 5.1 A NA-IFN space $(V, \mu, \phi, *, \diamond)$ over $\mathcal{K}$ is said to be statistically complete if every statistically Cauchy sequence with respect to (μ, ϕ) is statistically convergent with respect to (μ, ϕ).

Theorem 5.1 Every NA-IFN space $(V, \mu, \phi, *, \diamond)$ over $\mathcal{K}$ is statistically complete with respect to (μ, ϕ).
Proof:
Let $\{a_k\}$ be statistically Cauchy. If it is not statistically convergent to $x \in V$, then we have,

$$\begin{aligned}
&\lim_n \frac{1}{n}|\{k, m \leq n : \mu(a_k - a_m, u) \leq 1 - \epsilon \; or \\
&\qquad \phi(a_k - a_m, u) \geq \epsilon\}| \\
&= \lim_n \frac{1}{n}|\{k, m \leq n : \mu(a_k - x, u) * \mu(a_m - x, u) \leq 1 - \epsilon \\
&\qquad or \; \phi(a_k - x, u) \diamond \phi(a_m - x, u) \geq \epsilon\}| \\
&= 0,
\end{aligned}$$

which is a contradiction.

Definition 5.2 Let $(V, \mu, \phi, *, \diamond)$ be a NA-IFN space over $\mathcal{K}$. A map $f : V \to V$ is called (μ, v) continuous at a point $a \in V$, if the convergence of the sequence in the NA-IFN space implies the convergence of $f(a_k)$ to $f(a)$ in the NA-IFNS.

Definition 5.3 Let $(V, \mu, \phi, *, \diamond)$ be a NA-IFN space over $\mathcal{K}$. A map $f : V \to V$ is called statistically continuous at a point $a \in X$, if $stat_{\mu,\phi} - \lim_{k\to\infty} a_k = a$ implies that $stat_{\mu,\phi} - \lim_{k\to\infty} f(a_k) = f(a)$.

Theorem 5.2 Let $(V, \mu, \phi, *, \diamond)$ be a NA-IFN space over $\mathcal{K}$. If $f : V \to V$ is continuous with respect to (μ, ϕ), then it is statistically continuous.
Proof:
Let $\{a_k\} \in V$ and $stat_{\mu,\phi} - \lim_{k\to\infty} a_k = a$. Then for every $\epsilon > 0$ and $u > 0$, the inequality,
$\mu(a_k - a, u) > 1 - \epsilon$ and $\phi(a_k - a, u) < \epsilon$ implies that
$\mu(f(a_k) - f(a), u) > 1 - \epsilon$ and $\phi(f(a_k) - f(a), u) < \epsilon$.
Since f is continuous with respect to (μ, ϕ) at $a \in V$. Thus,

$$\begin{aligned}
&\{k \in \mathbb{N} : \mu(f(a_k) - f(a), u) \leq 1 - \epsilon \; or \\
&\qquad v(f(a_k) - f(a), u) \geq \epsilon\} \\
&\subset \{k \in \mathbb{N} : \mu(a_k - a, u) \leq 1 - \epsilon \; and \; \phi(a_k - a, u) \geq \epsilon\}.
\end{aligned}$$

Since, $stat_{\mu,\phi} - \lim_{k\to\infty} a_k = a.$
we have $\lim_n \frac{1}{n}|\{k \leq n : \mu(a_k - a, u) \leq 1 - \epsilon \; or \; \phi(a_k - a, u) \geq \epsilon\}| = 0.$
This implies that,
$\lim_n \frac{1}{n}|\{k \leq n : \mu(f(a_k) - f(a), u) \leq 1 - \epsilon \; or \; \phi(f(a_k) - f(a), u) \geq \epsilon\}| = 0.$
Which means that, $stat_{\mu,\phi} - \lim_{n\to\infty} f(a_k) = f(a)$.
Hence, f is statistically continuous.

6 Conclusions

Known results in Archimedean fields have been extended to non-Archimedean fields. Some inclusion relations in statistical convergence and statistically Cauchy sequences on intuitionistic fuzzy normed spaces over non-Archimedean fields have been proved in this article.

REFERENCES

[1] Bachman G., "Introduction to p-adic numbers and valuation theory," Academic Press, 1964.

[2] Atanassov K. T., Intuitionistic fuzzy sets, Fuzzy Sets and Systems vol. 20, pp. 87-96, 1986.

[3] Bag T., Samanta S. K., "Fuzzy bounded linear operators," Fuzzy sets and systems, vol. 151, pp.513-547, 2005.

[4] Bilalov B. T., Farahani S. M., Guliyeva F. A., "The intuitionistic fuzzy normed space of co-efficients," Abstract and Applied Analysis, vol. 2012, 11pages, 2012.

[5] Bilalov B. T., Nazarova T.Y., "On statistical convergence in metric spaces," Journal of mathematics research, vol. 7, no. 1, pp. 37-43, 2015.

[6] Eghbali N., Ganji M., "Generalized statistical convergence in the Non-Archimedean L-fuzzy normed spaces," Azerbaijam Journal of Mathematics, vol. 6, no. 1, 2016.

[7] Fast H.,"Sur la convergence statistique," Colloquium Mathematicae, vol. 2, no. 3-4, pp.,241-244, 1951. DOI:10.4064/cm-2-3-4-241-244

[8] Fridy J. A.,"On statistical convergence," Analysis, vol. 5, no. 4, pp. 301-314, 1985. https://doi.org/10.1524/anly.1985.5.4.301

[9] Karakaya V., Simsek N., Erturk M., Gursoy F., "Statistical convergence of sequences of functions in intuitionistic fuzzy normed spaces," Abstract and Applied Analysis, vol. 2012, 19pages, 2012.

[10] Karakus S., Demirci K., Duman O., "Statistical convergence on intuitionistic fuzzy normed spaces," Choas, Solitons and Fractals, vol. 35, pp.763-769, 2008.

[11] Mohiuddine S.A., Danish Lohani Q. M., "On generalized statistical convergence in intuitionistic fuzzy normed space," Choas,Solitons and Fractals, vol. 42, pp.1731-1737, 2009.

[12] Mohiuddine S. A., Sevli H., Cancan M., "Statistical convergence of double sequences in fuzzy normed spaces," Filomat, vol. 26, no.4, pp.673-681, 2012.

[13] Mursaleen M.,"λ-Statistical convergence," Mathematica Slovaca, vol. 50, no. 1, pp. 111-115, 2000. http://dml.cz/dmlcz/136769

[14] Saadati R., Park J. H., "On the intuitionistic fuzzy topological spaces," Chaos,Solitons and Fractals, vol. 27, pp.331-44, 2006.

[15] Suja K., Srinivasan V., "On Statistically Convergent and Statistically Cauchy Sequences in Non-Archimedean Fields," Journal of Advances in Mathematics, vol. 6, no. 3, pp.1038-1043, 2014. https://doi.org/10.24297/jam.v6i3.3659

[16] Zadeh L. A., "Fuzzy Sets," Inform and Control, vol. 8, pp.338-353, 1965.