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Upper Bound for Partition Dimension of Comb Product of a Wheel Graph and Tree

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Abstract The concept of partition dimension in graph theory was first introduced by Chartrand et al. [1] as a variation of metric dimension. Since then, numerous studies have attempted to determine the partition dimensions of various types of graphs. However, for many types of graphs, their partition dimensions remain unknown as determining a general graph's partition dimension is an NP-complete problem. In this study, we aim to determine the partition dimension of a specific graph, namely the comb product of a wheel and a tree. One approach to finding the partition dimension of a graph is to determine its upper and lower bounds. In this article, we propose an upper bound for the partition dimension of the comb product using number representation for certain bases. We divide the problem into two cases based on the path graph. For the first case, which is the comb product with a path of a single vertex, Tomescu et al. [2] have already provided an upper bound. In the other case, we utilize the bijection property of a number system on the number copy of the tree to find an upper bound. Our results show that the partition dimension of the second case has a smaller upper bound compared to the general upper bound proposed by Chartrand et al. [1].

Keywords Partition Dimension, Graph Theory, Comb Product, Wheel, Tree, Upper bound, NP-complete Problem

1. Introduction

Partition dimension is a research topic in graph theory that emerged from research on metric dimensions introduced by [3] and further explored by [4] in a journal titled "On the metric dimension of a graph". In brief, metric dimension focuses on finding a resolving set, which is a collection of points called landmarks on a graph that defines the shape of a graph based on the distance of each point to the landmark. Given a connected graph G, the metric dimension of G is the smallest integer k such that for every pair of distinct vertices u and v in G, there exists a set of k vertices (called a resolving set) that distinguishes u and v, i.e., there exists a vertex in the set that is closer to u than to v or vice versa. In other words, for any two vertices in G, there must be at least one vertex in the resolving set that is closer to one vertex than the other. Then, [5] proposed in a journal titled "The partition dimension of a graph" to divide the graph into several partitions referred to as resolving partitions, with all points being divisible into each partition. Formally, given a graph G, a resolving partition is a partition of the vertices of G such that for any two distinct vertices u and v in G, there exists a partition that contains one of them and not the other. The partition dimension of G, denoted $pd(G)$, is the minimum number of sets in a resolving partition of G. In other words, $pd(G)$ is the smallest number of subsets of vertices that can distinguish between any two vertices in G. Finding the partition dimension of a graph is known to be an NP-complete problem in general, which means that it is computationally difficult to solve for large graphs. Therefore, researchers often focus on finding upper and lower bounds on the partition dimension for specific types of graphs or classes of graphs.

During its development, researchers have attempted to determine partition dimensions using various experimental and analytical methods. [5] presented several partition dimension results on special graphs such as path graphs, complete graphs, bipartite graphs, and others. Calculating partition dimension is regarded as an NP-complete task because [6] claimed that computing metric dimensions on a general graph is an NP-complete problem. Therefore, many researchers have presented their research findings on special graphs, including [2], [7-19].

Partition dimensions have found numerous applications in various fields. For instance, they have been used to represent the shape of chemical compounds and for robot navigation [6]. Additionally, [20] proposed a method for constructing and verifying computer networks using partition dimensions. Moreover, research on partition dimensions has led to the development of advanced topics such as connected partition dimensions [21] and star partition dimensions [22]. Therefore, research on partition dimensions can provide significant benefits in various applications.

2. Results and Discussion

A wheel graph W_n has $n+1 \geq 4$ vertices and a tree graph T has $|V(T)| \geq 1$ vertices. A single vertex is also included as a tree graph and can be denoted as a path graph P_1.

Trivially, the graph resulting from the comb product between the wheel graph W_n and the single vertex P_1 will be generates the wheel graph W_n, denoted as $W_n \rhd_o P_1 = W_n$. Then, it can be concluded that calculating the partition dimensions of this graph depends on calculating the partition dimensions of the wheel graph itself.

In general, if G and H are connected graph that have at least two vertices, [23] suggest that the metric dimensions of the comb product of G and H can be formulated in the following equation

$$\beta(G \rhd_o H) = \begin{cases} m \cdot (\beta(H) - 1), \text{if } \exists o \text{ in a basis of H} \\ m \cdot \beta(H), \quad \text{otherwise} \end{cases}$$

Chartrand et al. via Theorem 1.1. mentions that $pd(G) \leq \beta(G) + 1$. Therefore, the upper-bound of the partition dimension of the graph $W_n \rhd_o T$ can be formulated in

$$pd(W_n \rhd_o T) \leq \beta(W_n \rhd_o T) = (n+1)\beta(T) + 1.$$

For each vertex $v \in V$ of a tree $T = (V, E)$, the legs at v are the bridges which are paths. To calculate the metric dimension of a tree graph T, Khuller et al. formulate partition dimension of T in the following equation.

$$\beta(T) = \sum_{v \in V: l_v > 1} (l_v - 1)$$

where l_v is the number of legs at v for every vertex in T. In particular, a graph G has a metric dimension $\beta(G) = 1$ if and only if graph G is a path graph P_n.

The explanation above leads to the following conclusions for the circumstances in which the upper-bound dimensions partitions are calculated:

1. The comb product of the wheel graph W_n and P_1
2. The comb product of the wheel graph W_n and the tree graph $T \neq P_1$

2.1. The Comb Product of W_n and P_1

Note that the result of the comb product between the wheel graph W_n and the single vertex P_1 will produce a wheel graph W_n. Therefore, the calculation of the partition dimensions of this graph depends on calculating the partition dimensions of the wheel graph W_n itself.

Tomescu et al.[2] formulated a bound of the partition dimension on a wheel graph W_n with $n \geq 3$ on the following inequalities and have calculated the value of the partition dimension for some values of n.

$$\lceil (2n)^{1/3} \rceil \leq pd(W_n) \leq 2\left\lceil n^{\frac{1}{2}} \right\rceil + 1.$$

2.2. The Comb Product W_n and $T \neq P_1$

Suppose a tree graph T has $S(T)$ where $S(T)$ is a minimum resolving set of T. Based on Saputro et al., a resolving set $W_n \rhd_o T$ is $S(W_n \rhd_o T) = \{(i,s) | i \in V(W_n), s \in S(T)\}$ with cardinality $|S(W_n \rhd_o T)| = (n+1)\beta(T)$.

Therefore, the upper-bound of the partition dimension $pd(W_n \rhd_o T) \leq (n+1)\beta(T) + 1$. This upper-bound is formed from the resolving set $S(W_n \rhd_o T)$ joined with one non-landmark set $S' = \{(i,v) | i \in V(W_n), v \in V(T), v \notin S(T)\}$. Formally, we get a resolving partition of the graph $W_n \rhd_o T$ which is given in Theorem 1.

Theorem 1 (Chartrand-Saputro) Given a wheel graph W_n and a tree T. The graph $W_n \rhd_o T$ has a resolving partition $\Pi_0 = \{\{(i,s)\} | i \in V(W_n), s \in S(T)\} \cup S'$ with cardinality $(n+1)\beta(T) + 1$. In other words, $pd(W_n \rhd_o T) \leq (n+1)\beta(T) + 1$.

Here is an illustrative example of Theorem 1 on graphs W_4 and a tree T with $V(T) = \{0,1,2,\ldots,6\}$.

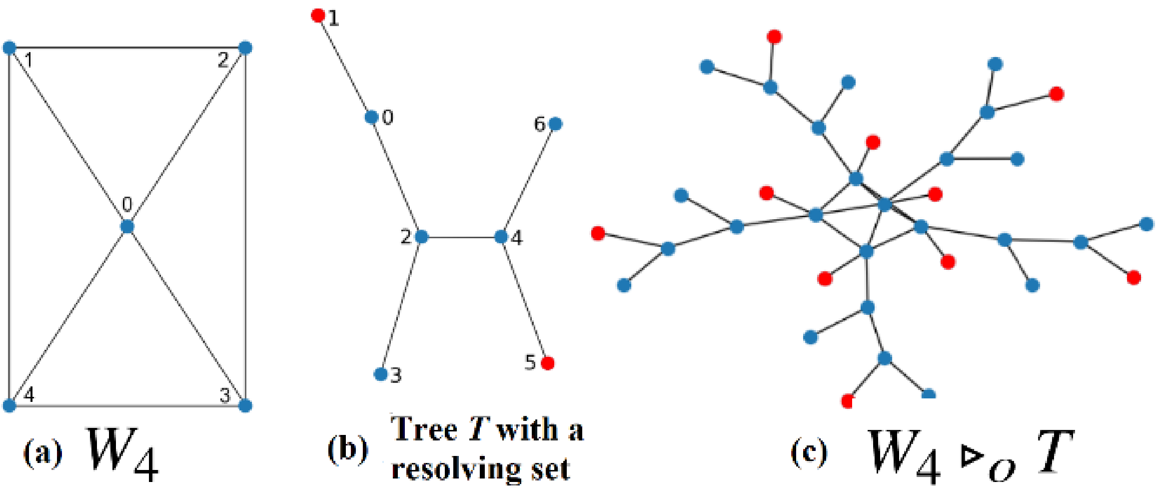

Figure 1. $W_4 \triangleright_o T$ with a resolving set

We have that $S(T) = \{1,5\}$ is a minimum resolving set of T and $S' = \left\{\{(i,v)|0 \le i \le 4, v \in \{0,2,3,4,6\}\}\right\}$. The partition of $W_4 \triangleright_o T$ from Theorem 1 is

$$\Pi_0 = \left\{\begin{matrix}\{(0,1)\},\{(1,1)\},\{(2,1)\},\{(3,1)\},\{(4,1)\},\\ \{(0,5)\},\{(1,5)\},\{(2,5)\},\{(3,5)\},\{(4,5)\},\\ S'\end{matrix}\right\}$$

From this partition, we obtain that $pd(W_4 \triangleright_o T) \le 11$.

If we look further, a tree T that are duplicated in the comb product operation are isomorphic graphs. Hence, there are some elements that can be merged into one set to reduce the cardinality of a resolving partition.

Let s_j be the jth landmark of a tree T with $1 \le j \le \beta(T)$. For each landmark s_j, the vertex (i, s_j) can be grouped into one set $S_{(s_j)} = \{(i,s_j)|i \in V(W_n)\}$. Therefore, we can define a partition $\Pi_1 = \left\{S_{(s_j)}|s_j \in S(T)\right\} \cup \{S'\}$ with cardinality is $\beta(T)+1$. Based on the property of the resolving set $S(T)$, we can prove that $r((i,v),\Pi_1\ ) \ne r((i,u),\Pi_1\ )$ for any $u,v \in V(T)$ and $v \ne u$.

However, the partition Π_1 causes $r((i,v),\Pi_1\ ) = r((j,v),\Pi_1\ )$ for $0 \le i,j \le n$. It can be concluded that Π_1 is not a resolving partition. To overcome this, we make a one-to-one correspondence between T_i and (i,S_1) since $d((i,v),(i,s_1)) < d((i,v),(j,s_1))$ for $j \ne i$. This inequality is given by Lemma 1.

Lemma 1 For any $i,j \in V(W_n)$, the inequality $d((i,v),(i,S)) < d((i,v),(j,s))$ with $i \ne j, v \in V(T)$ and $s \in S(T)$.

Proof. Let $(i,v),(j,v) \in V(W_n \triangleright_o T)$ and let $(i,s),(j,s) \in S(W_n \triangleright_o T)$ for any i,j that satisfies $0 \le i,j \le n$ and $i \ne j$. Based on the isomorphic of trees T_i, we obtain that $d((i,v),(i,u)) = d((j,v),(j,u))$ for any $(i,u),(j,u) \in V(W_n \triangleright_o T)$ and $u \in V(T)$. By the general properties of graphs, we can also determine the inequality

$$d((i,v),(i,s)) \le d((i,v),(i,o)) + d((i,o),(i,s))$$

and $d((i,o),(j,o)) > 0$. Hence, we get

$$d((i,v),(i,s)) \le d((i,v),(i,o)) + d((i,o),(i,s))$$
$$< d((i,v),(i,o)) + d((i,o),(j,o)) + d((i,o),(i,s))$$
$$= d((i,v),(i,o)) + d((i,o),(j,o)) + d((j,o),(j,s))$$

and the corresponding path $P = (i,v) \ldots (i,o) \ldots (j,o) \ldots (j,s)$ is the shortest path from (i,v) to (j,s). We conclude that

$$d((i,v),(i,s)) < d((i,v),(j,s)).$$

Theorem 2 Given a wheel graph W_n and a tree T. The graph $W_n \triangleright_o T$ has a resolving partition with cardinality $\beta(T)+n+1$. In other words, $pd(W_n \triangleright_o T) \le \beta(T)+n+1$.

Proof. Suppose partition Π_2 is defined as

$$\Pi_2 = \left\{S_{(s_j)}|s_j \in S(T), 2 \le j \le \beta(T)\right\} \cup \left\{S_{(i,s_1)}|i \in V(W_n)\right\} \cup \{S'\}$$

with $S' = \{(i,v)|i \in V(W,W_n), v \in V(T), v \notin S(T)\}$. Since $S(T)$ is a resolving set, then $r((i,v),\Pi_2) \ne r((i,u),\Pi_2)$ for $u \ne v$. Since T_i is isomorphic to T and from Lemma 1 we have that

$$d\left((i,v),S_{(i,s_1)}\right) = d\left((j,v),S_{(j,s_1)}\right) < d\left((j,v),S_{(i,s_1)}\right).$$

Therefore, we conclude that $r((i,v),\Pi_2) \ne r((j,v),\Pi_2)$ for $i \ne j$.

We have proved that $r((i,v),\Pi_2) \ne r((j,v),\Pi_2)$ for any $(i,v),(j,v) \in V(W_n \triangleright_o T)$ with $i \ne j$ and $u \ne v$. It follows that Π_2 is a resolving partition for $W_n \triangleright_o T$. In other words, we have an upper bound of partition dimension of $W_n \triangleright_o T$,

$$pd(W_n \triangleright_o T) \le (\beta(T)-1)+(n+1)+1.$$

Equivalently $pd(W_n \rhd_o T) \leq \beta(T) + n + 1$.

Mathematically, Theorem 2 has a more optimal upper-bound partition dimension compared to Theorem 1. Indeed,

$$\begin{aligned}\beta(T) + n + 1 &\leq \beta(T) + n \cdot \beta(T) + 1 \\ &= (1+n)\beta(T) + 1 \\ &= (n+1)\beta(T) + 1.\end{aligned}$$

Consider an example of a partition of $W_4 \rhd_o T$ from Theorem 1. The partition Π_1 in Theorem 2 is

$$\Pi_1 = \begin{Bmatrix} \{(0,1)\},\{(1,1)\},\{(2,1)\},\{(3,1)\},\{(4,1)\}, \\ \{(0,5),(1,5),(2,5),(3,5),(4,5)\}, \\ S' \end{Bmatrix}$$

From this partition, we obtain that $pd(W_4 \rhd_o T) \leq 7$.

We know that every natural number n can be represented in a unique way in the number system with base $b > 1$ [24]. A base is a specific value used in positional notation to represent a numeric value. For instance, the base of the decimal system, which is the base that is most frequently used in daily life, is 10. This indicates that every digit in a decimal number represents a power of ten multiples. Binary (base 2), octal (base 8), and hexadecimal (base 16) are further frequent bases. The base dictates the collection of symbols used to represent numbers in each of these systems as well as how place value is allocated to each digit in a number. For every tree T_i, we will use the number representation of i to create a resolving partition.

According to number theory, every number in a set $H = \{0,1,2,\ldots,n\}$ can be represented in a number base $b > 1$. Trivially, there are b^d different numbers that can be formed in base b with length d.

Any natural number can be written as $\sum a_j b^j$ with $a_j \in \{0,1,\ldots,b-1\}$ and $b > 1$.We define function $f(b,i,j) = a_{j-1}$ as the value of the jth digit from the right of the number i represented in base $b > 1$. For example, the number $11 = 1 \cdot 3^2 + 0 \cdot 3^1 + 2 \cdot 3^0$ can be represented as 102 in base 3. Therefore, $f(3,11,1) = 2, f(3,11,2) = 0$, and $f(3,11,3) = 1$.

The above observations can be applied to the problem of the partition dimension of $W_n \rhd_o T$. Suppose $d = \beta(T)$, we will represent every vertex in $V(W_n)$ into a base b that satisfy $b^d \geq n + 1 > 1$. This inequality is equivalent to $b \geq \sqrt[d]{n+1} > 1$. For every $s_j \in S(T)$, define a set $S_{x,(s_j)} = \{(i,s_j) | i \in V(W_n), f(b,i,j=x)\}$ for $0 \leq x < b$. For every number $i \in H$, define

$$n_{b,j} = \max_i f(b,i,j) + 1$$

as the number of possible digits used in the jth digit from the right for every i. Formally, we obtain a resolving partition for $W_n \rhd_o T$ in the following theorem.

Theorem 3 (Number Base Correspondence) Given a wheel graph W_n and a tree T. The graph $W_n \rhd_o T$ has a resolving partition with cardinality $\left(\sum_j n_{b,j}\right) + 1$. In other words, $pd(W_n \rhd_o T) \leq \left(\sum_j n_{b,j}\right) + 1$. For all possible bases b, there exists b such that $\left(\sum_j n_{b,j}\right) + 1$ reaches a minimum.

Proof. Suppose partition Π_3 is defined as

$$\Pi_3 = \left\{S_{x,(s_j)} | s_j \in S(T), 0 \leq x < n_{b,j}\right\} \cup S'$$

with $S' = \{(i,v) | i \in V(W_n), v \in V(T), v \notin S(T)\}$. By Lemma 1 and using the one-to-one correspondence of numbers and its representation in base b, it is easy to prove that $r\big((i,v),\Pi_3\big) \neq r\big((j,u),\Pi_3\big)$ for any $(i,v),(j,u) \in V(W_n \rhd_o T)$ with $(i,v) \neq (j,u)$. It follows that Π_3 is a resolving partition.

Next, we will find a base b such that $\left(\sum_j n_{b,j}\right) + 1$ reaches a minimum. We know that $b^d \geq n + 1$. It follows that $b \geq \sqrt[d]{n+1}$. Note that $f(b,i,j) = f(n+1,i,j)$ for all $b \geq n + 1$ and $0 \leq i \leq n$. Let $B = \left\{b | \sqrt[d]{n+1} \leq b \leq n+1, b \in \mathbb{Z}\right\} \subseteq \mathbb{N}$. Since B is finite, then there exists a base $\hat{b} \in B$ such that $\left(\sum_j n_{\hat{b},j}\right) + 1 \leq \left(\sum_j n_{b,j}\right) + 1$ for all base b.

Theorem 3 has a more optimal upper-bound compared to Theorem 2. Let $\hat{b}$ be a base in Theorem 3. If we choose $b = n + 1$, then

$$\begin{aligned}\left(\sum_j n_{\hat{b},j}\right) + 1 &\leq \left(\sum_j n_{n+1,j}\right) + 1 \\ &= (\beta(T) - 1) + (n+1) + 1 \\ &= \beta(T) + n + 1\end{aligned}$$

For example, the resolving partition Π_3 is defined as

$$\Pi_3 = \begin{Bmatrix} \{(0,1),(3,1)\},\{(1,1),(4,1)\},\{(2,1)\}, \\ \{(0,5),(1,5),(2,5)\},\{(3,5),(4,5)\}, \\ S' \end{Bmatrix}$$

From this partition, we obtain that $pd(W_4 \rhd_o T) \leq 6$.

3. Conclusions

Based on the research that has been done, the results obtained can be concluded as follows: In the case of $W_n \rhd_o T$ with T is a single vertex, a bound of its partition dimension refers to the results of research conducted by Tomescu et al. (2007). In the case T is not a tree with a single vertex, then we give an upper bound of the partition dimension using the one-to-one correspondence between numbers and their representation in base b. We obtain that

$$pd(W_n \rhd_o T) \leq \left(\sum_j n_{b,j}\right) + 1$$

where $n_{b,j}$ is the number of possible digits used in the jth digit from the right for every i.

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REFERENCES

[1] G. Chartrand, E. Salehi, and P. Zhang, "On the partition dimension of a graph," Congr. Numer., vol. 130, pp. 157–168, 1998.

[2] I. Tomescu, I. Javaid, and Slamin, "On the partition dimension and connected partition dimension of wheels," Ars. Combin., vol. 84, pp. 311–317, 2007.

[3] P. J. Slater, "Leaves of trees," Proc. 6th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Congr. Numer., vol. 14, pp. 549–559, 1975.

[4] F. Harary and R. A. Melter, "On the metric dimension of graph," Ars. Combin., vol. 2, pp. 191–195, 1976.

[5] G. Chartrand, E. Salehi, and P. Zhang, "The partition dimension of a graph," Aequationes Math., vol. 59, pp. 45–54, 2000.

[6] S. Khuller, B. Raghavachari, and A. Rosenfeld, "Landmarks in graphs," Discrete Appl. Math., vol. 77, pp. 217–229, 1996.

[7] I. Javaid and S. Shokat, "The partition dimension of some wheel related graphs," J. Prime Res. Math., vol. 4, pp. 154–164, 2008.

[8] J. A. Rodr\'\iguez-Velázquez, I. G. Yero, and M. Lemańska, "On the partition dimension of trees," Discrete Appl. Math., vol. 166, pp. 204–209, 2014.

[9] Amrullah, E. T. Baskoro, R. Simanjuntak, and S. Uttunggadewa, "The Partition Dimension of a Subdivision of a Complete Graph," Procedia Comput Sci, vol. 74, pp. 53–59, 2015, doi: 10.1016/j.procs.2015.12.075.

[10] K. Q. Fredlina and E. T. Baskoro, "The Partition Dimension of Some Families of Trees," Procedia Comput Sci, vol. 74, pp. 60–66, 2015, doi: 10.1016/j.procs.2015.12.076.

[11] Amrullah, E. T. Baskoro, S. Uttunggadewa, and R. Simanjuntak, "The partition dimension of subdivision of a graph," 2016, p. 020001. doi: 10.1063/1.4940802.

[12] R. Alfarisi, "Dimensi partisi dan dimensi partisi bintang graf hasil operasi comb dua graf terhubung," Institut Teknologi Sepuluh Nopember, Surabaya, 2017.

[13] Darmaji and R. Alfarisi, "On the partition dimension of comb product of path and complete graph," 2017, p. 020038. doi: 10.1063/1.4994441.

[14] C. Grigorious, S. Stephen, B. Rajan, and M. Miller, "On the Partition Dimension of Circulant Graphs," Comput J, Oct. 2016, doi: 10.1093/comjnl/bxw079.

[15] J. Santoso and Darmaji, "The partition dimension of cycle books graph," J Phys Conf Ser, vol. 974, no. 1, p. 12070, Mar. 2018, doi: 10.1088/1742-6596/974/1/012070.

[16] Amrullah, S. Azmi, H. Soeprianto, M. Turmuzi, and Y. S. Anwar, "The partition dimension of subdivision graph on the star," J Phys Conf Ser, vol. 1280, no. 2, p. 22037, Nov. 2019, doi: 10.1088/1742-6596/1280/2/022037.

[17] Z. Hussain et al., "Bounds for partition dimension of M-wheels," Open Physics, vol. 17, no. 1, pp. 340–344, Jul. 2019, doi: 10.1515/phys-2019-0037.

[18] A. Amrullah, "The partition dimension of a subdivision of a homogeneous firecracker," Electronic Journal of Graph Theory and Applications, vol. 8, no. 2, p. 445, Oct. 2020, doi: 10.5614/ejgta.2020.8.2.20.

[19] A. Estrada-Moreno, "On the k-partition dimension of graphs," Theor Comput Sci, vol. 806, pp. 42–52, Feb. 2020, doi: 10.1016/j.tcs.2018.09.022.

[20] Z. Beerliova et al., "Network Discovery and Verification," IEEE Journal on Selected Areas in Communications, vol. 24, no. 12, pp. 2168–2181, 2006, doi: 10.1109/JSAC.2006.884015.

[21] V. Saenpholphat and P. Zhang, "Connected Partition Dimension of Graphs," Discussiones Mathematicae Graph Theory, vol. 22, pp. 305–323, 2002.

[22] R. Marinescu-Ghemeci, "ON STAR PARTITION DIMENSION OF TREES," 2012.

[23] S. W. Saputro, N. Mardiana, and I. A. Purwasih, "The metric dimension of comb product graphs," Matematicki Vesnik, vol. 69, no. 4, pp. 248–258, Dec. 2017.

[24] K. H. Rosen, Discrete Mathematics and its Applications, 8th ed. New York, USA: McGraw-Hill Education, 2019.