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Numerical Approximation of Volterra Integro-Differential Equations of the Second Kind Using Boole's Quadrature Rule Method

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Abstract This article presents the numerical approximation of Volterra integro-differential equations (VIDEs) of the second kind using the quadrature rule in the modified block method. The new implementation of new block method which considers the closest point to approximate two solutions of $y(x_{n+1})$ and $y(x_{n+2})$ concurrently was taken into account. This method is said to have an advantage in reducing the number of total steps and function evaluations compared to the classical multistep method. The techniques of quadrature rule which consist of the trapezoidal rule, Simpson's 1/3 rule, Simpson's 3/8 rule and Boole's quadrature rule have been used to approximate the integral parts of Kernel function, z_n for $n = 1, 2, 3, 4, 5$ for the case of $K(x, s) = 1$. The analysis of the order, error constant, consistency and convergence of VIDEs in the proposed method has also been presented. The stability analysis is derived using the specified linear test equation for both approximate solutions until obtained the stability polynomial. To validate the efficiency of the developed method, some of the numerical results are presented and compared with the existing method. It is shown that the modified block method has given better accuracy and efficiency in terms of maximum error and number of steps and function calls.

Keywords Volterra Integro-differential Equations, Boole's Quadrature Rule Method, Predictor-corrector Block Method

1 Introduction

An integral equation is the equation in which the unknown function occurs under the integral sign [1]. There are two different ways used to distinguish the integral equation that depends on the integration limit which is the Fredholm integral equation (FIE) if the upper limit and lower limit of integration are fixed and second, the Volterra integral equation (VIE) if at least one the limit of integration is variable. The general form of Volterra integro-differential equations (VIDEs) can be expressed in the form of

$$\begin{aligned} y'(x) &= f(x, y(x), z(x)), \qquad 0 < x < a, \\ y(0) &= y_0, \end{aligned} \tag{1}$$

where

$$z(x) = \int_0^x K(x, s)y(s)dx. \tag{2}$$

The $z(x)$ is the integral part of VIDEs with the limits of integration $0 < x < a$ and contains $K(x, s)$ that is called the kernel of integration. VIDEs can be classified into two kinds which are the first kind and the second kind. For the first kind of VIDEs, the unknown function $f(x)$ only appears inside the integral sign in the form

$$\begin{aligned} &f(x) = \int_0^x K_1(x, s)y(s)ds + \int_0^x K_2(x, s)y(s)ds, \\ &K_2(x, s)ds \neq 0. \end{aligned} \tag{3}$$

that the kernels $K_1(x, s)$ and $K_2(x, s)$ for the first kind of VIDEs are difference kernels meaning that each kernel depends on the difference $(x - s)$. However, for the second kind, the unknown function appears inside and outside of the integral sign which the form can represent

$$y'(x) = f(x) + \int_0^x K(x, s) y(s) ds. \tag{4}$$

The equation can be linear if the kernel of integration, $K(x, s)$ is equal to 1 and nonlinear if $K(x, s)$ is not equal to 1.

There are various numerical and analytical methods that have been used to solve VIDEs, for example, the application of He's homotopy perturbation method [2], the Adomian decomposition method [3] and the variational iteration method (VIM) [4]. In [5], the new method of Laplace transform–Adomian decomposition method was introduced for solving the nonlinear VIDEs. From their research, it is stated that VIDEs are very hard to solve in analytical methods compared to numerical methods. Thus, in the previous decade, the study of VIDEs in numerical methods has gained much attention from all researchers. For example, Chebyshev spectral methods [6], the one-step hybrid block method [7], the Runge-Kutta method [8], the collocation method [9] and the Robust method [10]. Meanwhile, there are several quadrature rules that have been used to solve the integral part of VIDEs, such as repeated Simpson's and trapezoidal quadrature rule [11], Boole's approximation method [12] and open Newton-Cotes formula [13] to solve the integral part of VIDEs. Based on the literature, the two-point predictor-corrector block method is better than the one-point method using the integration coefficients in terms of the maximum error in solving first-order ordinary differential equations (ODEs) [14].

Therefore, in this paper, our intention is to solve the linear VIDEs of the second kind using Boole's quadrature rule in the new formulation of the modified block method. This paper is organized into 6 Sections. The section "Formulation of the Method" contains the formulation of the modified block method. In the Section "Implementation of Quadrature Rule", the implementation of trapezoidal quadrature rule, Simpson's quadrature 1/3 rule, Simpson's 3/8 quadrature rule and Boole's quadrature rule have been discussed when adapting with the block method. The following sections describe the convergence analysis and the stability analysis of the proposed method. The numerical approximation of VIDEs using Boole's quadrature rule is presented in the Section "Numerical Results" and the last section is "Conclusion".

2 Formulation of the Method

This section presents the formulation of the modified block method which is based on the concept of predictor (explicit) and corrector (implicit) from the idea of [15].

Let $y(x_{n+1})$ and $y(x_{n+2})$ be an approximate solution to the theoretical solution at x_{n+1} and x_{n+2}, then the solutions $y(x_{n+1})$ and $y(x_{n+2})$ can be determined by integrating (5) over the interval $[x_n, x_{n+1}]$ and $[x_{n+1}, x_{n+2}]$ as

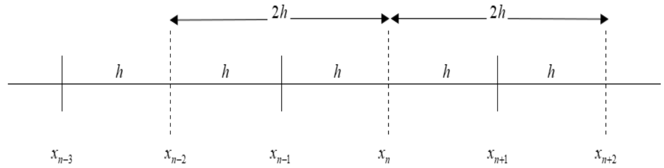

Figure 1. Modified Two-Point Block Method.

$$\begin{aligned} \int_{x_n}^{x_{n+1}} y'(x)dx &= \int_{x_n}^{x_{n+1}} f(x, y, z)dx, \\ \int_{x_{n+1}}^{x_{n+2}} y'(x)dx &= \int_{x_{n+1}}^{x_{n+2}} f(x, y, z)dx. \end{aligned} \tag{5}$$

By the definition of Lagrange interpolation polynomial

$$L_{n,j} = \prod_{k=0, k\neq j}^{n} \frac{x - x_k}{x_j - x_k}, \tag{6}$$

it follows that P_n can be written in the form

$$P_n(x) = \sum_{j=0}^{n} y_j L_{n,j}(x), \tag{7}$$

where $P_n(x)$ is the unique polynomial of degree at most n that pass through the $(n + 1)$ points. The predictor and corrector formula can be obtained by interpolating through the points of $\{(x_{n-3}, f_{n-3}), (x_{n-2}, f_{n-2}), (x_{n-1}, f_{n-1}), (x_n, f_n)\}$ and $\{(x_{n-2}, f_{n-2}), (x_{n-1}, f_{n-1}), (x_n, f_n), (x_{n-1}, f_{n-1}), (x_{n-2}, f_{n-2})\}$, respectively. Hence, the Lagrange polynomials for both predictor and corrector can be written as follows,

Predictor:

$$\begin{aligned} P_3(x) = {} & L_{3,0}(x) f_n + L_{3,1}(x) f_{n-1} \\ & + L_{3,2}(x) f_{n-2} + L_{3,3}(x) f_{n-3}. \end{aligned} \tag{8}$$

where

$$\begin{aligned} L_{3,0}(x) &= \frac{(x - x_{n-3})(x - x_{n-2})(x - x_{n-1})}{(x_n - x_{n-3})(x_n - x_{n-2})(x_n - x_{n-1})}, \\ L_{3,1}(x) &= \frac{(x - x_{n-3})(x - x_{n-2})(x - x_n)}{(x_{n-1} - x_{n-3})(x_{n-1} - x_{n-2})(x_{n-1} - x_n)}, \\ L_{3,2}(x) &= \frac{(x - x_{n-3})(x - x_{n-1})(x - x_n)}{(x_{n-2} - x_{n-3})(x_{n-2} - x_{n-1})(x_{n-2} - x_n)}, \\ L_{3,3}(x) &= \frac{(x - x_{n-2})(x - x_{n-1})(x - x_n)}{(x_{n-3} - x_{n-2})(x_{n-3} - x_{n-1})(x_{n-3} - x_n)}. \end{aligned} \tag{9}$$

Corrector:

$$\begin{aligned} P_4(x) = {} & L_{4,0}(x) f_{n+2} + L_{4,1}(x) f_{n+1} \\ & + L_{4,2}(x) f_n + L_{4,3}(x) f_{n-1} \\ & + L_{4,4}(x) f_{n-2}. \end{aligned} \tag{10}$$

where

$$L_{4,0}(x) = \frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+1})}{(x_{n+2}-x_{n-2})(x_{n+2}-x_{n-1})(x_{n+2}-x_n)(x_{n+2}-x_{n+1})},$$
$$L_{4,1}(x) = \frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+2})}{(x_{n+1}-x_{n-2})(x_{n+1}-x_{n-1})(x_{n+1}-x_n)(x_{n+2}-x_{n+2})},$$
$$L_{4,2}(x) = \frac{(x-x_{n-2})(x-x_n)(x-x_{n+1})(x-x_{n+2})}{(x_n-x_{n-2})(x_n-x_{n-1})(x_n-x_{n+1})(x_n-x_{n+2})},$$
$$L_{4,3}(x) = \frac{(x-x_{n-2})(x-x_n)(x-x_{n+1})(x-x_{n+2})}{(x_{n-1}-x_{n-2})(x_{n-1}-x_n)(x_{n-1}-x_{n+1})(x_{n-1}-x_{n+2})},$$
$$L_{4,4}(x) = \frac{(x-x_{n-1})(x-x_n)(x-x_{n+1})(x-x_{n+2})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)(x_{n-2}-x_{n+1})(x_{n-2}-x_{n+2})}. \quad (11)$$

Let s be the limit of integral in (5), then by substituting the considered lower and upper time step, the limit for the both solutions will be obtained as shown in Table 1.

Table 1. Limit for Predictor-Corrector Formula.

Limit	Predictor Formula	Corrector Formula
x_n	$s=\frac{x_n-x_n}{h}=0$	$s=\frac{x_n-x_{n+2}}{h}=-2$
x_{n+1}	$s=\frac{x_{n+1}-x_n}{h}=1$	$s=\frac{x_{n+1}-x_{n+2}}{h}=-1$
x_{n+2}	$s=\frac{x_{n+2}-x_n}{h}=2$	$s=\frac{x_{n+2}-x_{n+2}}{h}=0$

By taking $dx = hds$ and replacing $f(x, y)$ by $P_3(x)$ and $P_4(x)$ will gives the approximations:

Predictor:
First point,

$$\begin{aligned} & y(x_{n+1}) - y(x_n) \\ &= \int_0^1 P_3(x)\,dx \\ &= h\int_0^1 \frac{(s+3)(s+2)(s+1)}{(2)(2)(1)}\,dsf_n \\ &= h\int_0^1 \frac{(s+3)(s+2)(s)}{(2)(1)(-1)}\,dsf_{n-1} \\ &= h\int_0^1 \frac{(s+3)(s+1)(s)}{(1)(-1)(-2)}\,dsf_{n-2} \\ &= h\int_0^1 \frac{(s+2)(s+1)(s)}{(-1)(-1)(2)(-1)(3)}.\,dsf_{n-3}. \end{aligned} \quad (12)$$

Second point.

$$\begin{aligned} & y(x_{n+2}) - y(x_{n+1}) \\ &= \int_1^2 P_3(x)\,dx \\ &= h\int_1^2 \frac{(s+3)(s+2)(s+1)}{(2)(2)(1)}\,dsf_n \\ &= h\int_1^2 \frac{(s+3)(s+2)(s)}{(2)(1)(-1)}\,dsf_{n-1} \\ &= h\int_1^2 \frac{(s+3)(s+1)(s)}{(1)(-1)(-2)}\,dsf_{n-2} \\ &= h\int_1^2 \frac{(s+2)(s+1)(s)}{(-1)(-1)(2)(-1)(3)}\,dsf_{n-3}. \end{aligned} \quad (13)$$

Corrector:
First point,

$$\begin{aligned} & y(x_{n+1}) - y(x_n) \\ &= \int_{-2}^{-1} P_4(x)\,dx \\ &= h\int_{-2}^{-1} \frac{(s+4)(s+3)(s+2)(s+1)}{(4)(3)(2)}\,dsf_{n+2} \\ &= h\int_{-2}^{-1} \frac{(s+4)(s+3)(s+2)(s)}{(3)(2)(-1)}\,dsf_{n+1} \\ &= h\int_{-2}^{-1} \frac{(s+4)(s+3)(s+1)(s)}{(2)(-1)(-2)}\,dsf_n \\ &= h\int_{-2}^{-1} \frac{(s+4)(s+2)(s+1)(s)}{(1)(-1)(-1)(2)(-1)(3))}\,dsf_{n-1} \\ &= h\int_{-2}^{-1} \frac{(s+3)(s+2)(s+1)(s)}{(-1)(-2)(-1)(3)(-1)(4))}\,dsf_{n-2}. \end{aligned} \quad (14)$$

Second point.

$$\begin{aligned} & y(x_{n+2}) - y(x_{n+1}) \\ &= \int_{-1}^{0} P_4(x)\,dx \\ &= h\int_{-1}^{0} \frac{(s+4)(s+3)(s+2)(s+1)}{(4)(3)(2)}\,dsf_{n+2} \\ &= h\int_{-1}^{0} \frac{(s+4)(s+3)(s+2)(s)}{(3)(2)(-1)}\,dsf_{n+1} \\ &= h\int_{-1}^{0} \frac{(s+4)(s+3)(s+1)(s)}{(2)(-1)(-2)}\,dsf_n \\ &= h\int_{-1}^{0} \frac{(s+4)(s+2)(s+1)(s)}{(1)(-1)(-1)(2)(-1)(3))}\,dsf_{n-1} \\ &= h\int_{-1}^{0} \frac{(s+3)(s+2)(s+1)(s)}{(-1)(-2)(-1)(3)(-1)(4))}\,dsf_{n-2}. \end{aligned} \quad (15)$$

Since the proposed method is assumed to have a constant step size, thus the formula of the modified 2-point block method can be expressed as follows,

Predictor formula:

$$\begin{aligned} &y(x_{n+1}) - y(x_n) \\ &= \frac{h}{24}(-9f_{n-3} + 37f_{n-2} - 59f_{n-1} + 55f_n), \\ &y(x_{n+2}) - y(x_{n+1}) \\ &= \frac{h}{24}(-55f_{n-3} + 211f_{n-2} - 293f_{n-1} + 161f_n). \end{aligned} \tag{16}$$

Corrector formula:

$$\begin{aligned} &y(x_{n+1}) - y(x_n) \\ &= \frac{h}{720}(-19f_{n-1} + 346f_n + 456f_{n+1} - 74f_{n+2} + 11f_{n+3}), \\ &y(x_{n+2}) - y(x_{n+1}) \\ &= \frac{h}{720}(11f_{n-1} - 74f_n + 456f_{n+1} + 346f_{n+2} - 19f_{n+3}). \end{aligned} \tag{17}$$

3 Implementation of Quadrature Rule

In the present section, the modified two-point block method is used together with the quadrature rules to obtain the approximations methods of VIDEs. Several appropriate numerical quadrature rules are selected to be paired with the proposed method to solve the integral part in VIDEs, represented as

$$z_n(x) = \int_0^x K(x, s, y(s)), \quad n = 1, 2, 3, ... \tag{18}$$

For linear VIDEs, $K(x, s) = 1$, the value for z_1, z_2 and z_3 are calculated by using Newton-Cotes quadrature rule such as the trapezoidal quadrature rule, Simpson's 1/3 quadrature rule and Simpson's 3/8 quadrature rule. The formulas are given as

Trapezoidal quadrature rule:

$$z_1 = z_0 + \frac{h}{2}(y_0 + y_1). \tag{19}$$

Simpson's 1/3 quadrature rule:

$$z_2 = z_0 + \frac{h}{3}(y_2 + 4y_1 + y_0). \tag{20}$$

Simpson's 3/8 quadrature rule:

$$z_3 = z_0 + \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3). \tag{21}$$

Boole's quadrature rule method is paired with the proposed method when evaluating the integral part of z_4 and z_5 in VIDEs. The basic concept of the Boole's quadrature rule method can be illustrated by considering the following general quadrature formula

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_4} f(x)dx = \int_a^{a+4h} f(x)dx, \tag{22}$$

where

$$x_i = a + ih, \quad h = \frac{b-a}{4}. \tag{23}$$

Consider a function $y = f(x)$ over the interval $[x_0, x_4]$, the Boole's quadrature rule approximates the integral of $f(x)$ over $[x_0, x_1, x_2, x_3, x_4]$. Boole's quadrature rule method is being derived by putting $n = 4$ in the general quadrature formula means a polynomial of 4th degree can approximate $f(x)$, such as

$$\begin{aligned} \int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}[&7f(x_0) + 32f(x_1) + 12yf(x_2) \\ &+ 32f(x_3) + 7f(x_4)]. \end{aligned} \tag{24}$$

By taking derivative for solving integration,

$$\begin{aligned} \int_{x_0}^{x_4} f'(x)\,dx = \frac{2h}{45}[&7f'(x_0) + 32f'(x_1) + 12yf'(x_2) \\ &+ 32f'(x_3) + 7f'(x_4)]. \end{aligned} \tag{25}$$

The following statement results from the placement of specific integration.

$$\begin{aligned} f(x) = f(x_0) + \frac{2h}{45}[&7f'(x_0) + 32f'(x_1) + 12yf'(x_2) \\ &+ 32f'(x_3) + 7f'(x_4)]. \end{aligned} \tag{26}$$

Thus, the formula for Boole's quadrature rule method for implementation in z_4 and z_5 is given by

$$\begin{aligned} z_{n+4} = z_n + \frac{2h}{45}(&7y_{n+4} + 32y_{n+3} + 12y_{n+2} \\ &+ 32y_{n+1} + 7y_n), \quad n = 0, 1, 2, ... \end{aligned} \tag{27}$$

4 Convergence Analysis

The analysis of the order, error constant, consistency, convergence and stability polynomial of the proposed method is discussed in this section.

Definition 1 *The numerical method is said to be in order p if, $C_0 = C_1 = C_2 = ... = C_p = 0; C_{p+1} \neq 0$, where the error constant for the method is called as C_{p+1}, [16].*

Definition 2 *If the order of method is $p \geq 1$, then the numerical method is consistent. The numerical method will be consistent if and only if, $\sum_{j=0}^{k} \alpha_j = 0, \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$, [16].*

Definition 3 *A block method is said to be zero-stable if and only if providing the roots of $Rj, j = 1(1)k$ of the first characteristic polynomial, $p(R)$ specified as*

$$p(R) = det[\sum_{j=0}^{k} A^{(i)} R^{(k-i)}], \tag{28}$$

satisfies with $|Rj| \leq 1$ and those roots with $|Rj| = 1$, [16].

Theorem 1 *The method is said to be convergent if it is consistent and zero-stable, [16].*

The order of the method can be discovered by using the formula

$$\sum_{j=0}^{k}[\alpha_j y(x+jh)-h\beta_j y'(x+jh)] = C_p y^p + O(h^{p+1}), \quad (29)$$

where p is the order of the linear multistep method, $O(h^{p+1})$ is the local truncation error and C_p is developed as follows;

$$C_p = \sum_{j=0}^{k}(\frac{j^p\alpha_j}{p!} - \frac{j^{p-1}\beta}{(p-1)!}). \quad (30)$$

The coefficient for α_j and β_j are $\alpha_0 = \begin{bmatrix}0\\0\end{bmatrix}$, $\alpha_1 = \begin{bmatrix}0\\0\end{bmatrix}$, $\alpha_2 = \begin{bmatrix}0\\0\end{bmatrix}$, $\alpha_3 = \begin{bmatrix}-1\\0\end{bmatrix}$, $\alpha_4 = \begin{bmatrix}1\\-1\end{bmatrix}$, $\alpha_5 = \begin{bmatrix}0\\1\end{bmatrix}$, $\beta_0 = \begin{bmatrix}0\\0\end{bmatrix}$, $\beta_1 = \begin{bmatrix}\frac{11}{720}\\\frac{-19}{720}\end{bmatrix}$, $\beta_2 = \begin{bmatrix}\frac{-74}{720}\\\frac{106}{720}\end{bmatrix}$, $\beta_3 = \begin{bmatrix}\frac{456}{720}\\\frac{-264}{720}\end{bmatrix}$, $\beta_4 = \begin{bmatrix}\frac{346}{720}\\\frac{646}{720}\end{bmatrix}$ and $\beta_5 = \begin{bmatrix}\frac{-19}{720}\\\frac{251}{720}\end{bmatrix}$. Therefore, the order and error constant of the proposed method are

$$\begin{aligned}
C_0 &= \sum_{j=0}^{k}\frac{j^0.\alpha_j}{0!} = \begin{bmatrix}0\\0\end{bmatrix},\\
C_1 &= \sum_{j=0}^{k}\frac{j^1.\alpha_j}{1!} - \sum_{j=0}^{k}\frac{j^0.\beta_j}{0!} = \begin{bmatrix}0\\0\end{bmatrix},\\
C_2 &= \sum_{j=0}^{k}\frac{j^2.\alpha_j}{2!} - \sum_{j=0}^{k}\frac{j^1.\beta_j}{1!} = \begin{bmatrix}0\\0\end{bmatrix},\\
C_3 &= \sum_{j=0}^{k}\frac{j^3.\alpha_j}{3!} - \sum_{j=0}^{k}\frac{j^2.\beta_j}{2!} = \begin{bmatrix}0\\0\end{bmatrix},\\
C_4 &= \sum_{j=0}^{k}\frac{j^4.\alpha_j}{4!} - \sum_{j=0}^{k}\frac{j^3.\beta_j}{3!} = \begin{bmatrix}0\\0\end{bmatrix},\\
C_5 &= \sum_{j=0}^{k}\frac{j^5.\alpha_j}{5!} - \sum_{j=0}^{k}\frac{j^4.\beta_j}{4!} = \begin{bmatrix}0\\0\end{bmatrix},\\
C_6 &= \sum_{j=0}^{k}\frac{j^6.\alpha_j}{6!} - \sum_{j=0}^{k}\frac{j^5.\beta_j}{5!} = \begin{bmatrix}\frac{11}{1440}\\\frac{-3}{160}\end{bmatrix}.
\end{aligned} \quad (31)$$

Since $\alpha_j = \begin{bmatrix}0\\0\end{bmatrix}$, $j\alpha_j = \begin{bmatrix}1\\1\end{bmatrix}$ and $\beta_j = \begin{bmatrix}1\\1\end{bmatrix}$, so the method can be consistent because it satisfies the two conditions in Definition 2.

Since the roots of the method are $|R| \leq 1$, thus the associated method is zero stable concerning Definition. The verification of the zero stability of the derived predictor-corrector method is validated below

$$p(R) = det\begin{bmatrix}R+1 & R+1\\R+1 & R+1\end{bmatrix} = 0, \quad (32)$$

The method of order five is consistent, zero stable and then it converges on Definition 2.

5 Stability Analysis

The proposed method's stability on the VIDEs will be discussed in this section. A-stability type for linear multistep methods was used for the numerical solution of VIDEs. The linear test equation [16] of VIDEs is as follows:

$$y' = \xi y(x) = \eta\int_0^x y(s)\,ds, \quad (33)$$

where ξ, η are real constants. The solutions tend to zero as $x \to \infty$ if and only if $\xi < 0$ and $\eta < 0$.

The stability polynomial is analyzed using the proposed modified two-point block method with Boole's quadrature rule. The stability polynomial of the proposed methods is determined after substituting the polynomial of the first character into the general form of the VIDEs stability polynomial. The general form of the stability polynomial for VIDEs is as follows,

$$\begin{aligned}\pi(r, h\xi, h^2\eta) = \rho(r)[\rho(r) - h\xi\sigma(r)]\\ - h^2\eta\sigma(r)\sigma(r),\end{aligned} \quad (34)$$

where the parameter $h\xi; h^2\eta \in R$. Let $H_1 = h\xi$ and $H_2 = h^2\eta$, then the stability polynomial of VIDEs of the second kind will come to,

$$\begin{aligned}\pi(r, H_1, H_2) = \rho(r)[\rho(r)\\ - H_1\sigma(r)] - H_2\sigma(r)\sigma(r),\end{aligned} \quad (35)$$

where $p(r)$ is the first characteristics polynomial and $\sigma(r)$ is the second characteristic polynomial. Thus, the polynomial characteristics of the proposed method and Boole's quadrature rule are as the result,

i) First point of the corrector formula, y_{n+1}.

$$\begin{aligned}\rho(r) &= r^4 - r^3,\\ \sigma(r) &= \frac{11}{720}r - \frac{74}{720}r^2 + \frac{456}{720}r^3\\ &+ \frac{346}{720}r^4 - \frac{19}{720}r^5.\end{aligned} \quad (36)$$

ii) Second point of the corrector formula, y_{n+2}.

$$\begin{aligned}\rho(r) &= r^5 - r^4,\\ \sigma(r) &= -\frac{19}{720}r + \frac{106}{720}r^2 - \frac{264}{720}r^3\\ &+ \frac{646}{720}r^4 + \frac{251}{720}r^5.\end{aligned} \quad (37)$$

iii) Boole's quadrature rule.

$$\begin{aligned}\bar{\rho}(r) &= r^5 - r^4,\\ \bar{\sigma}(r) &= \frac{14}{45} + \frac{64}{45}r - \frac{24}{45}r^2\\ &+ \frac{64}{45}r^3 - \frac{14}{45}r^4.\end{aligned} \quad (38)$$

The stability polynomial for the modified two-point block method is evaluated by substituting the characteristics polynomial of the proposed method and Boole's quadrature rule into the stability polynomial of VIDEs. Thus, the stability polynomial can be attained as follows:

$$\begin{aligned}\pi(r,H_1,H_2) = &\frac{2891}{48600}t^8H_2H_1 + \frac{834841}{131220000}t^8H_2^2\\ &+\frac{43423}{259200}t^8H_1^2 - \frac{8787313}{11664000}t^8H_1\\ &-\frac{24266447}{262440000}t^8H_2 - \frac{14443}{16200}t^8\\ &+\frac{683}{360}t^7 + \frac{30643}{16200}t^6\\ &-\frac{323}{360}t^5 + \frac{137807}{131220000}t^2H_2^2\\ &-\frac{1699}{10800}t^2H_1^2 - \frac{5491}{131220000}tH_2^2\\ &-\frac{1}{2160}tH_1^2 + \frac{23}{720}t^7H_1^2\\ &-\frac{108446411}{131220000}t^7H_2^2 - \frac{49}{72}t^3H_1\\ &-\frac{3043}{2700}t^3H_2 - \frac{161}{720}t^2H_1\\ &-\frac{1}{450}t^2H_2 - \frac{23011}{5832000}t^7H_1\\ &-\frac{289087357}{131220000}t^7H_2 + \frac{466605577}{131220000}t^6H_2\\ &-\frac{284789}{5832000}t^5H_1 + \frac{411283499}{131220000}t^5H_2\\ &+\frac{10639831}{5832000}t^4H_2 - \frac{5380657}{2916000}t^6H_1\\ &+\frac{12017}{43200}t^6H_1^2 - \frac{88873}{259200}t^4H_1\\ &+\frac{111217}{259200}t^4H_1^2 - \frac{35274139}{14580000}t^6H_2^2\\ &+\frac{101053459}{43740000}t^5H_2^2 + \frac{73181039}{43740000}t^4H_2^2\\ &+\frac{12978641}{131220000}t^3H_2^2 + \frac{1609}{2160}t^5H_1^2\\ &-\frac{4589}{10800}t^3H_1^2 - \frac{2454203}{1944000}t^6H_1H_2\\ &+\frac{6247}{5832000}tH_1H_2 + \frac{90721}{1944000}t^2H_1H_2\\ &-\frac{23837}{129600}t^3H_1H_2 - \frac{19871}{54000}t^4H_1H_2\\ &-\frac{179189}{72000}t^5H_1H_2 - \frac{23779}{43200}t^7H_1H_2\\ &-t^3+t^4.\end{aligned} \tag{39}$$

6 Numerical Results

In order to assess the performance and efficiency of the proposed method, we compared it with the classical numerical method of the same order. Some numerical examples for VIDE are provided in this section to demonstrate the result.

Problem 1 [17]:

$$y'(x) = -\int_0^x y(s)ds, \qquad y(0)=1, \qquad 0 \le x \le 1.$$

Exact solution: $y'(x) = \cos x$.

Problem 2 [17]:

$$y'(x) = 1-\int_0^x y(s)ds, \qquad y(0)=0, \qquad 0 \le x \le 1.$$

Exact solution: $y'(x) = \sin x$.

Problem 3 [18]:

$$y'(x) = 1+\int_0^x y(s)ds, \qquad y(0)=1, \qquad 0 \le x \le 1.$$

Exact solution: $y'(x) = e^x$.

Notation used in the following tables are:

MAXERR	:	Maximum error.
h	:	Step size.
TS	:	Total steps.
TFC	:	Total function calls.
2PV5	:	Modified two-point block method order 5 proposed in this paper.
ABM5	:	Classical Adam-Bashfort Moulton method order 5.

Table 2. Numerical Result for Problem 1.

h	Method	MAXERR	TS	TFC
$\frac{1}{40}$	2PV5	1.7881E-06	22	40
	ABM5	1.8145E-05	41	77
$\frac{1}{80}$	2PV5	2.4199E-07	42	80
	ABM5	2.1387E-06	80	157
$\frac{1}{160}$	2PV5	3.1565E-08	82	160
	ABM5	1.0036E-07	160	318
$\frac{1}{320}$	2PV5	4.0325E-09	162	320
	ABM5	1.2456E-08	320	638
$\frac{1}{640}$	2PV5	5.0962E-10	322	640
	ABM5	1.5514E-09	640	1278
$\frac{1}{1280}$	2PV5	6.4054E-11	642	1280
	ABM5	1.9358E-10	1280	2558

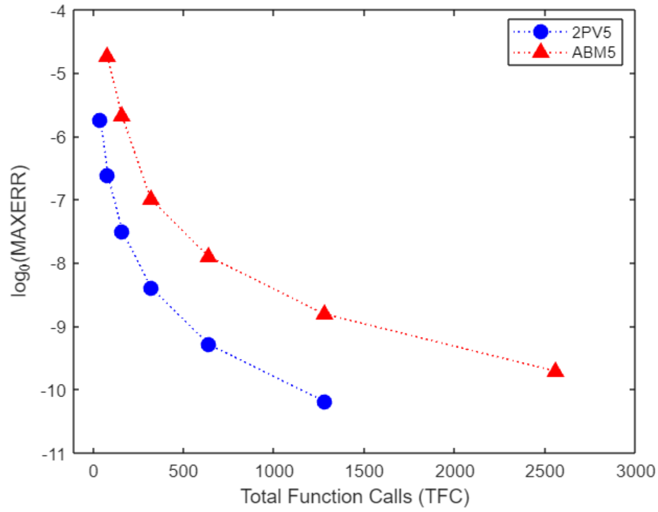

Figure 2. Numerical Result for Problem 1

Table 3. Numerical Result for Problem 2.

h	Method	MAXERR	TS	TFC
$\frac{1}{40}$	2PV5	3.89735E-07	22	40
	ABM5	1.30469E-05	41	77
$\frac{1}{80}$	2PV5	2.21379E-08	42	80
	ABM5	4.4238E-05	80	157
$\frac{1}{160}$	2PV5	1.3111E-09	82	160
	ABM5	1.0122E-05	160	318
$\frac{1}{320}$	2PV5	7.9632E-11	162	320
	ABM5	7.0424E-08	320	638
$\frac{1}{640}$	2PV5	4.9040E-12	322	640
	ABM5	1.8882E-08	640	1278
$\frac{1}{1280}$	2PV5	3.0421E-13	642	1280
	ABM5	4.8795E-09	1280	2558

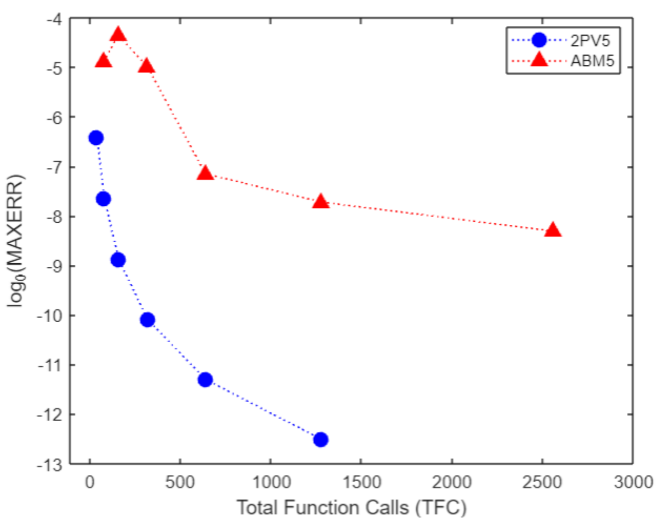

Figure 3. Numerical Result for Problem 2

Table 4. Numerical Result for Problem 3.

h	Method	MAXERR	TS	TFC
$\frac{1}{40}$	2PV5	5.1739E-07	22	40
	ABM5	2.6987E-04	41	77
$\frac{1}{80}$	2PV5	1.3828E-07	42	80
	ABM5	9.9842E-05	80	157
$\frac{1}{160}$	2PV5	3.5862E-08	82	160
	ABM5	2.5038E-05	160	318
$\frac{1}{320}$	2PV5	9.1368E-09	162	320
	ABM5	6.2692E-06	320	638
$\frac{1}{640}$	2PV5	2.3062E-09	322	640
	ABM5	1.5685E-06	640	1278
$\frac{1}{1280}$	2PV5	3.6848E-10	642	1280
	ABM5	3.9227E-07	1280	2558

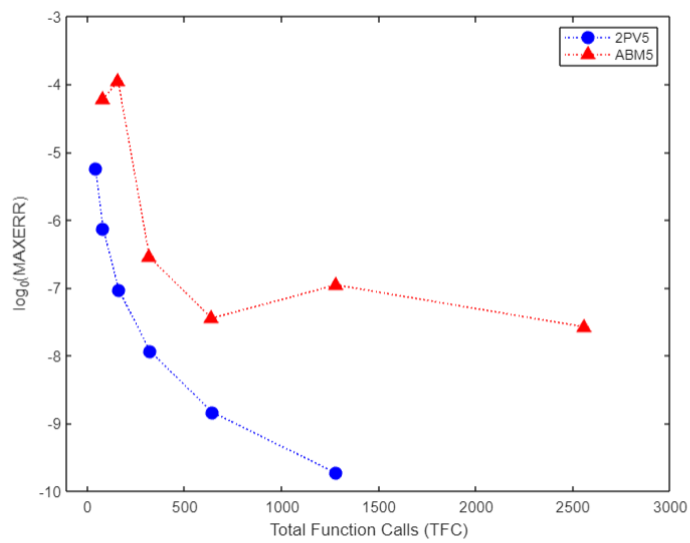

Figure 4. Numerical Result for Problem 3

The numerical results from Table 2 to Table 4 demonstrated that the maximum error of 2PV5 produced better accuracy as the step size decreased from $\frac{1}{40}$, $\frac{1}{80}$, $\frac{1}{160}$, $\frac{1}{320}$, $\frac{1}{640}$ to $\frac{1}{1280}$. For Figure 2 until Figure 4, the proposed 2PV5 method was more reliable then classical ABM5 method as 2PV5 obtained the lower number of steps and total function calls. Consequently, the approximate solution converged faster to the exact solution when compared to classical ABM5 method.

7 Conclusion

In this study, the proposed method of 2PV5 is suitable for solving VIDEs of the second kind with the implementation of Boole's quadrature rule using modified two-point block method. The total function calls, total number steps and maximum error are clearly illustrated by the given applications.

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