Civil Engineering and Architecture 11(4): 2033-2039, 2023 http://www.hrpub.org
DOI: 10.13189/cea.2023.110426

# Application of the Deformation Theory in Section Analysis of Reinforced Concrete Members with High-Strength Reinforcement

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*Received December 4, 2022; Revised February 14, 2023; Accepted March 12, 2023*

***Cite This Paper in the Following Citation Styles***

***(a):*** *[1] Andrii Pavlikov, Olha Harkava , "Application of the Deformation Theory in Section Analysis of Reinforced Concrete Members with High-Strength Reinforcement," Civil Engineering and Architecture, Vol. 11, No. 4, pp. 2033 - 2039, 2023. DOI: 10.13189/cea.2023.110426.*

***(b):*** *Andrii Pavlikov, Olha Harkava (2023). Application of the Deformation Theory in Section Analysis of Reinforced Concrete Members with High-Strength Reinforcement. Civil Engineering and Architecture, 11(4), 2033 - 2039. DOI: 10.13189/cea.2023.110426.*



**Abstract** The application of deformation theory in section analysis of reinforced concrete members is based on the use of ultimate values of concrete compressive strains at the time of failure. These strains may be determined analytically by examining the bearing capacity function at the extremum. In such a study, it is usually assumed that the tensile reinforcement reaches the yield point and the stress in it is constant. When using high-strength reinforcement with conditional yield strength, the stress in the reinforcement at the moment of failure is not a constant value. This is also characteristic of over-reinforced members, in which at the moment of destruction stress in tensile reinforcement is less than its yield stress. For such a case, an analytical criterion for determining the ultimate compressive strain of concrete of reinforced concrete members is proposed. This criterion, together with the extreme concrete strength criterion, which is used for common reinforced concrete members, may be used as the basis of the methodology for calculating the strength of reinforced concrete structures. The ultimate compressive strains of concrete are determined by using the derived criterion for a range of concrete classes that match coefficient $k$ from 1 to 5. Simple analytical dependencies for section analysis of reinforced concrete members with high-strength reinforcement are obtained on the basis of ultimate compressive strains of concrete. The application of the developed method is considered using examples.

**Keywords** Reinforced Concrete Member, Ultimate Concrete Strain, Section Analysis

## 1. Introduction

At present, the analysis of the section of reinforced concrete members is based on the deformation model. Determination of the design (ultimate) values of concrete compressive strain in the composition of the reinforced concrete member is carried out by different approaches. One of them is characterized by the application of ultimate values of concrete strain in the most compressed fiber determined experimentally [1–6]. According to the other approach, the introduced concept of the extreme strength criterion of a reinforced concrete member $\partial M/\partial \varepsilon_{cm} = 0$ is used. The ultimate values of concrete strain in the composition of flexural reinforced concrete members are determined analytically by studying the function $M_{Rd} = f(\varepsilon_{cm})$ on the extremum, where $\varepsilon_{cm}$ is the strain of concrete in the most compressed cross-sectional fiber of reinforced concrete member [7–10]. Analytical approaches to determining ultimate compressive strain based on the energy balance method [11–12] and others [13–14] are known.

The application of the extreme criterion is possible for normally reinforced concrete members, namely those in which at the moment of failure the stress in the reinforcement of the tensile area reaches the yield point, i.e. $\sigma_s = f_{yd}$. At the same time, for members in which, at the time of destruction, the strain of the reinforcement does not reach the yield point, the ultimate values of concrete strain will exceed the values determined by the extreme criterion. This is typical for over-reinforced members and prestressed members reinforced with high-strength reinforcement. The application of the extreme criterion $\partial M/\partial \varepsilon_{cm} = 0$ for such members is unacceptable, and therefore the ultimate strain of concrete for these cases may only be determined iteratively. This is especially characteristic of members reinforced with high-strength reinforcement, in the calculation of whose strength a two-line stress-strain diagram with an inclined upper branch is used. In such members, the stress in the reinforcement after reaching the value $\sigma_p = f_{pd}$ is a variable value depending on the position of the neutral axis in the section. In this case, the ultimate values of concrete strain may reach higher values compared to those calculated at a constant value of stress in the reinforcement. However, so far, there are no proposals for determining such ultimate values of concrete strain by calculation in over-reinforced and prestressed reinforced concrete members.

## 2. Materials and Methods

The calculation of the ultimate strain of concrete for reinforced concrete members is based on a nonlinear deformation model [1]. The essence of the calculation is that the growth of strains, not forces in the cross-section is taken into account. In this calculation, the growth of strain (Fig. 1) is considered in the most compressed concrete fiber (at the point with the maximum level of relative strain $\eta_m = \varepsilon_{cm} / \varepsilon_{c1}$).

The solution to the problem of determining the ultimate strain of concrete is carried out according to the design scheme shown in Figure 1. In the scheme, the distribution of stresses in the concrete of the compressed area is represented by the approximation function of the stress-strain diagram in the form of dependence (3.4) according to [1].

The problem is considered for a single reinforced beam of rectangular cross-section (Fig. 1). At the same time, the reinforcement in the cross-section will work with an incomplete design resistance $\sigma_s < f_{yd}$.

In the given problem, the cross-sectional area of the longitudinal reinforcement $A_s$, the dimensions of the cross-section of the beam $b\times h$, the physical and mechanical characteristics of concrete $f_{cd}$, $E_{cd}$, $\varepsilon_{c1}$ and the reinforcement $f_{yd}$, $E_s$ are taken as known values.

The values of the moment $M_{Rd}$, which may be perceived by the beam, and the corresponding value of the relative strains in the most compressed concrete fiber $\varepsilon_{cm} = \varepsilon_{cu}$ (their level $\eta_m = \eta_u$) are considered as unknown values.

To derive the desired dependence according to the accepted design scheme (Fig. 1), the following are used:

- equilibrium equation:

$$\Sigma X = 0; \quad N_S - N_C = 0; \tag{1}$$

$$\Sigma M_Z = 0; \quad M_{Ed} - N_c\left(d - x + y_{Nc}\right) = 0; \tag{2}$$

- the “stress-strain” diagram for concrete at axial loading according to [1] in the form

$$\sigma_c = \frac{f_{cd}(k\eta - \eta^2)}{(1 + (k-2)\eta)}, \tag{3}$$

where $k = 1{,}1(E_{cd}\,\varepsilon_{c1} / f_{cd})$, $\eta = (\varepsilon_c / \varepsilon_{c1})$;
$f_{cd}$, $E_{cd}$ – accordingly, design values of concrete strength under axial compression and its modulus of elasticity;

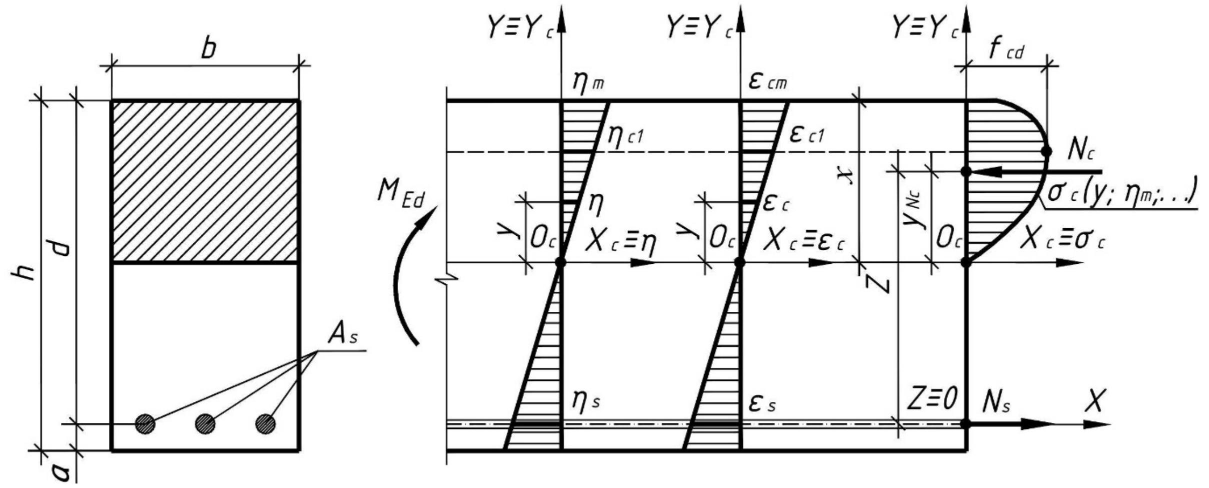


**Figure 1.** Design scheme of forces, stresses and strains in the cross-section of an over-reinforced concrete member

- the condition of compatibility of a strain of concrete and reinforcement according to the following law:

$$\varepsilon_C = \varepsilon_S; \tag{4}$$

- hypothesis of plane sections by condition

$$\frac{\varepsilon_{cm}}{x} = \frac{\varepsilon_S}{d-x}; \tag{5}$$

- the "stress-strain" diagram for the reinforcement with the physical yield point (Fig. 2) according to dependencies:

$$\sigma_S = E_S \varepsilon_S \text{ at } 0 \le \varepsilon_S \le f_{yd}/E_S; \tag{6}$$

$$\sigma_S = f_{yd} \text{ at } \varepsilon_S > f_{yd}/E_S. \tag{7}$$

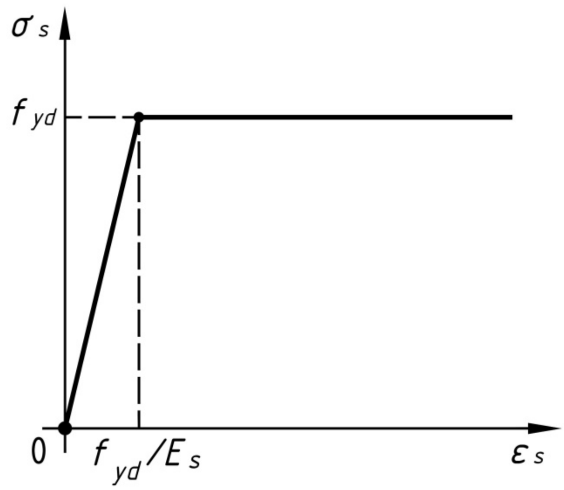


**Figure 2.** Design a stress-strain diagram for reinforcing steel by [1]

Since the destruction of over-reinforced members begins with a compressed area, it is obvious that the criterion for destruction, in this case, will be the achievement of the maximum strength of the compressed concrete area of the member in the form of conditions:

$$N_{cu}(\varepsilon_{cu}) = \max N_c(\varepsilon_{cm}), \tag{8}$$

or

$$N_{cu}(\eta_u) = \max N_c(\eta_m), \tag{9}$$

in which the ultimate value of the relative compression strains of concrete (or its level) $\varepsilon_{cu}$ ($\varepsilon_{cu} / \varepsilon_{c1} = \eta_u$) exceeds the strains corresponding to the extreme criterion of the strength of the given section in the beam at a constant value of the yield stress in the reinforcement [9].

It follows from the accepted prerequisites (1) – (9) that, using criteria (8) – (9), the dependence for determining the unknown value $\varepsilon_{cu}$ may be obtained by differentiating the function $N_c = f(x, \eta_m,\ldots)$. That is, the basis of the sought dependence is a criterion of the maximum strength of the concrete compressed area under the condition that the stresses in the longitudinal tensile reinforcement increase.

In order to obtain the desired dependence for determining the unknown quantity $\varepsilon_{cu}$ (or $\eta_u$) by differentiating the function $N_c$ from the variable $\eta_m$ in the form $N_c = f(\sigma_c, x, \varepsilon_{cm},\ldots)$, it is necessary to functionally express $N_c$ in equation (1) by $x$, $\varepsilon_{cm}$ or through $x$, $\eta_m$. At the same time, it should be taken into account that the function-equation in the form of $N_c = f(x, \eta_m,\ldots)$ will have a more convenient form for performing mathematical transformations.

Achieving the set goal needs to have the law of stress distribution in the concrete of the compressed area in the form $\sigma_c = f(y, \eta_m,\ldots)$. Such a law, which describes the distribution of stresses in the concrete of the compressed zone in the $Y_cO_cX_c$ coordinate system with its origin $O_c$ on the neutral axis (Fig. 1), is obtained in the following form:

$$\sigma_C = \frac{f_{cd}\eta_m y(kx - \eta_m y)}{x(x + (k-2)\eta_m y)}. \tag{10}$$

The peculiarity of law (10) is that it describes the distribution of stresses in concrete across the cross-section of the compressed area for any level of relative strains of concrete in the most compressed fiber, and thus for any level of loading of the beam member.

Applying dependence (10), the components of equations (1) and (2), after performing the necessary mathematical operations, are reduced to the following expressions:

$$N_C = b\int_0^x \frac{f_{cd}\eta_m y(Kx - \eta_m y)}{x(x + (K-2)\eta_m y)}dy = f_{cd}bx\omega; \tag{11}$$

$$y_{Nc} = S_C / N_C = x\frac{\varphi}{\omega}; \tag{12}$$

$$S_C = b\int_0^x \frac{f_{cd}\eta_m y(Kx - \eta_m y)y}{x(x + (K-2)\eta_m y)}dy = f_{cd}bx^2\varphi, \tag{13}$$

in which:

$$\left.\begin{aligned} \omega &= \frac{(k-1)^2(c - \ln c - 1)}{(k-2)^3\eta_m} - \frac{\eta_m}{2(k-2)} \text{ at } k \ne 2, \\ \omega &= \eta_m\left(1 - \frac{\eta_m}{3}\right) \text{ at } k = 2, \end{aligned}\right\}; \tag{14}$$

$$\left.\begin{aligned} \varphi &= \frac{(k-1)^2\left((c-2)^2 + 2\ln c - 1\right)}{2(k-2)^4\eta_m^2} - \frac{\eta_m}{3(k-2)} \text{ at } k \ne 2, \\ \varphi &= \eta_m\left(\frac{2}{3} - \frac{\eta_m}{4}\right) \text{ at } k = 2, \end{aligned}\right\}, \tag{15}$$

where $\omega$ is, as can be seen from formula (11), the coefficient of completeness of the stress diagram in the concrete compressed area.

Formulas (14) and (15) were obtained as a result of integration in the derivation of expressions (11) – (13), while the designation $c = 1 + (k-2)\eta_m$ was made.

Criterion (8 – 9) is used to calculate the values of the ultimate strains of concrete $\varepsilon_{cu}$ ($\eta_u$), which characterize the state of a reinforced concrete member when the concrete resistance in the compressed area reaches a maximum. Based on this criterion, in order to determine the values of $\varepsilon_{cu}$ ($\eta_u$), equation (11) was tested for extremum under the condition $dN_c / d\eta_m = 0$.

For example, for the value $k = 2$, the derivative of function (11) will have the form

$$\frac{dNc}{d\eta_m} = f_{cd}bx\left(1-\frac{2\eta_m}{3}\right). \tag{16}$$

Equating expression (16) to zero, $\eta_u = 1.5$ is got.

The results of the function (11) studying for the extremum are represented graphically (Fig. 3). The diagram of the ultimate levels of fiber strains (structural strains) of the concrete in the composition of the reinforced concrete member is plotted for the moment when the compressed concrete area reaches the maximum resistance to the action of the external load. It may be used in calculations.

After substituting the values of $N_c$ and $y_{Nc}$ functionally expressed in terms of $\eta_u$ into (1) and (2) and taking into account the operation of the reinforcement on the inclined section of the two-line strain diagram according to dependence (6), the equilibrium equations take the form:

$$\sigma_S A_S = f_{cd}bx\omega; \tag{17}$$

$$M_{Ed} - \sigma_S A_S\left(d - \chi\frac{\sigma_S A_S}{f_{cd}b}\right) = 0, \tag{18}$$

where the coefficient $\chi$ shows what part of the neutral axis depth $x$ the distance from the most compressed fiber of the cross-section to the point of the resultant $N_c$ application takes

$$\chi(\eta_m)=(\omega-\varphi)/\omega^2. \tag{19}$$

Since the values of the parameters $\omega$, $\varphi$, $\chi$ in the obtained dependencies (17) and (18) depend on the coefficient $k$ and the ultimate level of fiber strains of concrete $\eta_u$ in the composition of reinforced concrete members, they are summarized in Table 1 for the convenience of their use in calculations.

When solving the problem for a beam of rectangular cross-section reinforced with a single high-strength reinforcement, the design scheme shown in Figure 4 is considered.

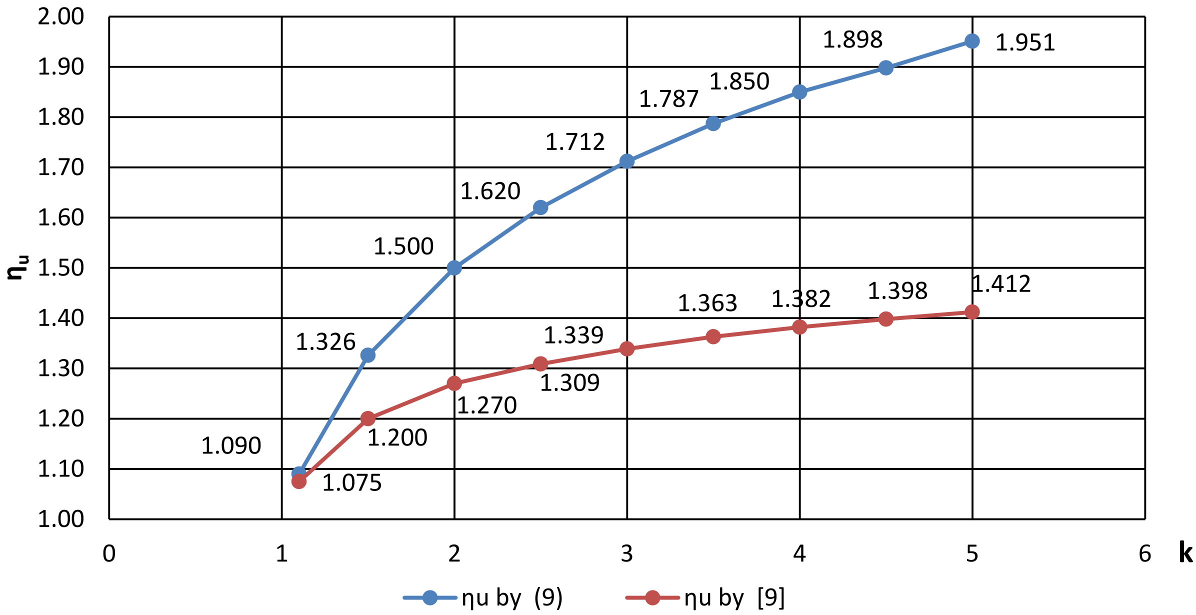


**Figure 3.** Graphs of the ultimate levels of fiber strains of concrete in the compressed area of a reinforced concrete member depending on the parameter $k$, obtained according to (9) and using the extreme criterion by [9]

**Table 1.** Parameter values $\omega$, $\varphi$, $\chi$

| | k | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **1.1** | **1.5** | **2** | **2.5** | **3** | **3.5** | **4** | **4.5** | **5** |
| ηu | 1.090 | 1.326 | 1.500 | 1.620 | 1.712 | 1.787 | 1.850 | 1.898 | 1.951 |
| ω | 0.568 | 0.685 | 0.750 | 0.787 | 0.813 | 0.832 | 0.846 | 0.858 | 0.868 |
| φ | 0.372 | 0.418 | 0.438 | 0.448 | 0.454 | 0.459 | 0.463 | 0.466 | 0.468 |
| χ | 0.609 | 0.569 | 0.555 | 0.547 | 0.543 | 0.539 | 0.535 | 0.532 | 0.531 |

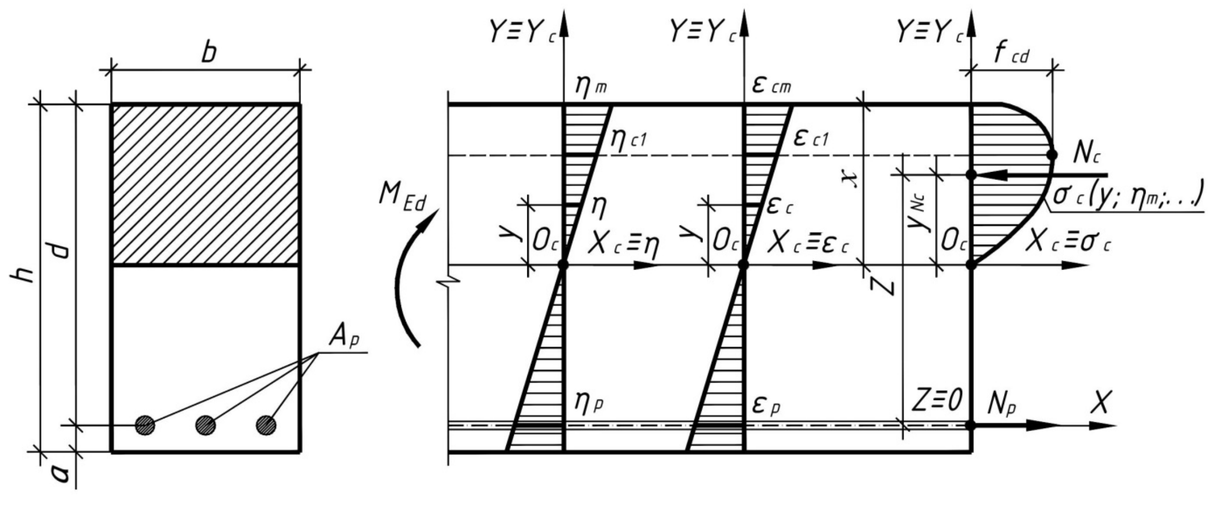


**Figure 4.** Design diagram of forces, stresses and strains in the cross-section of a concrete member reinforced with high-strength reinforcement

When using high-strength reinforcement with a conditional yield point in the calculation, as a rule, a two-line diagram with an inclined branch (Fig. 5) of limited length is used according to dependencies:

$$\sigma_p = E_p \varepsilon_p \text{ at } 0 \le \varepsilon_p \le f_{pd} / E_p ; \tag{20}$$

$$\sigma_p = f_{pd} + \frac{f_{pk}/\gamma_s - f_{pd}}{f_{pd}/E_p - \varepsilon_{ud}} \left( \frac{f_{pd}}{E_p} - \varepsilon_p \right) \tag{21}$$

at $f_{pd}/E_p < \varepsilon_p \le \varepsilon_{ud}$.

In this case, it is possible to use the proposed criterion (9) when the reinforcement is operating over the entire strain range $0 \le \varepsilon_p \le \varepsilon_{ud}$. At the same time, the equations (17) – (18) may be used by replacing $\sigma_s$ with $\sigma_p$ and $A_s$ by $A_p$.

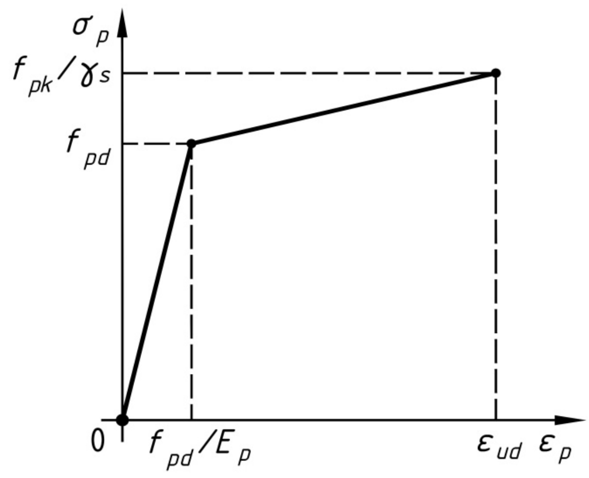


**Figure 5.** Design stress-strain diagram for prestressing steel by [1]

The application of the proposed method is considered using examples.

**Example 1.** Given: a reinforced concrete beam of a rectangular profile with cross-sectional dimensions $b = 200$ mm, $h = 400$ mm; concrete C30/35 ($f_{cd} = 19.5$ MPa, $E_{cd} = 27$ GPa, $\varepsilon_{c1,cd} = 1.70‰$, $k = 2.5$); reinforcement A500C ($f_{yd} = 417$ MPa, $E_s = 210$ GPa) is located at a distance from the lower face of the section $a = 50$ mm, area of the reinforcement $A_s = 1885$ mm$^2$. For the beam under consideration, according to the method [9], it was established that, at the moment of failure, the tensile reinforcement does not reach the yield point.

It is necessary to determine the bending moment that the beam may take.

The effective depth of the section

$$d = h - a = 400 - 50 = 350 \text{ mm}.$$

The neutral axis depth is defined from equation (17) at $\eta_u = 1.620$ and $\omega = 0.787$ (Table 1), $x = 217.19$ mm.

The bending moment is determined from equation (18) at $\sigma_s = 353.65$ MPa $< f_{yd} = 417$ MPa according to (6) and $\chi = 0.547$ (Table 1), $M_{Rd} = 171.0$ kNm.

**Example 2.** Given: a reinforced concrete beam of a rectangular profile with cross-sectional dimensions $b = 200$ mm, $h = 400$ mm; concrete C30/35 ($f_{cd} = 19.5$ MPa, $E_{cd} = 27$ GPa, $\varepsilon_{c1,cd} = 1.70‰$, $k = 2.5$); reinforcement A800 ($f_{pd} = 638$ MPa, $E_p = 190$ GPa) is located at a distance from the lower face of the cross-section $a = 30$ mm, reinforcement area $A_p = 308$ mm$^2$.

It is necessary to determine the bending moment that the beam may take.

The effective depth of the section

$$d = h - a = 400 - 30 = 370 \text{ mm}.$$

The neutral axis depth is defined from equation (17) at $\eta_u = 1.620$ and $\omega = 0.787$ (Table 1), $x = 68.29$ mm.

The bending moment is determined from equation (18) at $\sigma_p = 680.53$ MPa $> f_{pd} = 638$ MPa according to (21) and $\chi = 0.547$ (Table 1) $M_{Rd} = 71.4$ kNm.

When describing the operation of the reinforcement by the two-linear diagram with a horizontal section of unlimited length and an extreme strength criterion of strength, according to [9], the value of the bending moment $M_{Rd}$ = 67.5 kNm is obtained. Therefore, the strength reserve according to the proposed method is 5.8%.

## 3. Discussions

This paper demonstrates that the analysis of the section of flexural reinforced concrete members, when the prestressed reinforcement works in the strengthening stage or the tensile reinforcement works in the elastic stage, may be carried out by applying the proposed criterion (8) – (9). The unambiguity of the formulated criterion is justified by physical prerequisites regulated by current regulatory documents [1]. The joint application of the extreme criterion and the criterion of the maximum use of concrete of the compressed area in the calculations of the bearing capacity of flexural reinforced concrete members is implemented in a closed form. Formulas (8) – (9) may also be used to calculate the design ultimate value of fiber strain when determining the bearing capacity of members subjected to axial load and bending moment, the tensile reinforcement of which works in the elastic stage.

The analysis of formulas (11), (8) and (9) shows that the numerical values of fiber strain depend directly on the form of the compressed area, which in practice means their dependence on the form of the cross-section of the reinforced concrete member. Therefore, the obtained and given design values of fiber strains (Table 1) are acceptable for concrete of a rectangular form of a compressed cross-sectional area. As evidenced by the results of the work of many researchers, in particular the works [5, 9], when the rectangular form of the compressed area is transformed into a trapezoidal or triangular one the values of the ultimate fiber strain are concentrated at the level of the most compressed edge of the member, that is, they grow. Such situation is observed in conditions of biaxial bending.

The strain values for biaxial deformation were obtained analytically on the basis of the extreme strength criterion for the case of the tensile reinforcement reaching the yield point [9]. As for the use of the proposed criterion (8) – (9) in the case of other forms of the compressed area, provided that the stress of the tensile reinforcement are not a constant value, then such studies should be continued. Their completion will make it possible to reveal the full range of changes in the ultimate compressive strains of concrete for rectangular and other forms of the compressed concrete area. That is, in the end, a diagram of the ultimate concrete compressive strains depending on the form of the cross-section of the members will be obtained. As a result, it will be possible to design reinforced concrete members according to the method described in this article not only in plane bending but also in complex loading (in biaxial bending, biaxial bending and axial load, bending with torsion and other loads).

## 4. Conclusions

A criterion has been formulated for calculating the ultimate strains of concrete of the compressed area as part of reinforced concrete members, in which the stress of the reinforcement at the moment of failure is not a constant value. This criterion, together with the extreme concrete strength criterion, may be used as the basis of the methodology for calculating the strength of reinforced concrete structures. Its use is possible for flexural members with high-strength reinforcement and members with over-reinforced cross-sections. The derived dependencies allow calculating the ultimate values of fiber strains of concrete. The resulting diagram of values may be used to normalize the ultimate values of concrete strains for different classes of concrete. Based on the strain model, the formulas for calculating the strength of reinforced concrete members using the calculated ultimate strains of concrete are derived. The calculation is based on the equations of the mechanics of a deformed solid body and a complete stress-strain diagram of concrete and reinforcement.

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