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3-Equitable and Prime Labeling of Some Classes of Graphs

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Abstract Researchers have constructed a model to transform "word motion problems into an algorithmic form" in order to be processed by an intelligent tutoring system (ITS). This process has the following steps. Step 1: Categorizing the characteristics of motion problems, step 2: suggesting a model for the categories. "In order to solve all categories of problems, graph theory including backward and forward chaining techniques of artificial intelligence can be utilized". The adoption of graph theory into motion problems has evidence that the model solves almost all of motion problems. Graph labeling is sub field of graph theory which has become the area of interest due to its diversified applications. Formally, if the nodes are labeled under some constraint, the resulting labeling is known as vertex labeling and it will be an edge labeling if the labels are assigned to edges under some conditions. Graph labeling nowadays is one of the rapid growing areas in applied mathematics which has shown its presence in almost every field. The known applications are in Computer Science, Physics, Chemistry, Radar, Coding Theory, Connectomics, Socioloy, x-ray crystallography, Astronomy etc. "For a graph $G(V, E)$ and $k > 0$, give node labels from $\{0, 1, \ldots, k-1\}$ such that when the edge labels are induced by the absolute value of the difference of the node labels, the count of nodes labeled with i and the count of nodes labeled with j differ by at most one and the number of lines labeled with i and with j differ by at most 1. So G with such an allocation of labels is $k-$equitable and becomes 3-equitable labeling, when $k = 3$". In this paper, the existence and non-existence of 3-equitable labeling of certain graphs are established.

Keywords 3-equitable Graphs, Total Graph, Mycielski's Graph, Middle Graph, Central Graph, Degree Splitting Graph, Ladder Graph, Fan Graph, Friendship Graph, Lollipop Graph, Prime Labeling

AMS Subject Classification: 05C78

1 Introduction

The investigation of graph labeling techniques plays an important role in many areas of technology, especially in "intelligent systems, communication networks and network security". For $G(V, E)$ and $k > 0$, give node labels from $\{0, 1, \ldots, k-1\}$ such that when the line labels are induced by the absolute value of the difference of the node labels, the number of nodes labeled with i and with j differ by at most 1 and the number of lines labeled with i and with j differ by at most 1. A graph G with such an allocation of labels is known as $k-$ equitable [3].

1.1 Preliminaries

Definition 1.1 *[6] "A function $f : V \to \{0, 1, 2\}$ such that each edge uv receives $|f(v_i) - f(v_j)|$ where $v_i, v_j \in V$ is said to be a 3-equitable labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1 \; \forall 0 \leq i, j \leq 2$ where $v_f(i)$ is the count of nodes labeled with i, $v_f(j)$ is the count of nodes labeled with j, $e_f(i)$ is the count of lines labeled with i, and $e_f(j)$ is the count of lines labeled with j. A graph which admits 3-equitable labeling is called a 3-equitable graph".*

Definition 1.2 *[9] "Let G be with $V = S_1 \cup S_2 \cup \cdots \cup S_i \cup T$, where each S_i is a set of nodes having minimum 2 nodes of the same degree and $T = V \setminus S_i$. The degree splitting graph of G denoted by $DS(G)$ is derived from G by adding nodes $w_1, w_2, \ldots w_t$ and joining to each node of S_i $\forall 1 \leq i \leq t$".*

Definition 1.3 *[2] "The middle graph $M(G)$ of G whose node set is $V(G) \cup E(G)$ where two nodes are adjacent if and only if*
(i) They are adjacent lines of G or
(ii) One is a node and other is a line incident with it".

Definition 1.4 *[2] "The central graph $C(G)$ of G is obtained by subdividing each edge of G exactly once and joining all the non-adjacent nodes of G in $C(G)$".*

Definition 1.5 *[8] "The total graph $T(G)$ of G is a graph with node set $V(G) \cup E(G)$ and two nodes $x, y \in T(G)$ are adjacent if either*
(i) $x, y \in V(G)$ and x is adjacent to y in G or
(ii) $x, y \in E(G)$ and x, y are adjacent in G or
(iii) $x \in V(G)$, $y \in E(G)$ and x, y are incident in G".

Definition 1.6 *[10] "Let G with n nodes denoted by $v_1, v_2, \ldots, v_n$. The Mycielski graph, $\mu(G)$ is obtained by adding to each node v_i a new node u_i such that u_i is adjacent to the neighbors of v_i. Finally, add a new node w such that w is adjacent to each and every u_i".*

A significant amount of results on 3-equitable labeling are already available in the literature. In this paper, a few more new results are derived to enrich the discipline which may ultimately serve as a tool for complete characterization.

2 Main Results

Definition 2.1 *[5] "The lollipop graph $L_{3,n}$ is a special type of graph consisting of a complete graph K_3 and a path P_n, connected with a bridge".*

One can obtain the 3-equitable labeling of $C(L_{3,1})$ (See Figure 1). So, we consider $C(L_{3,n})$, for $n \geq 2$.

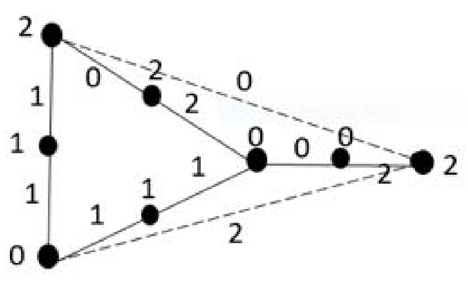

Figure 1. 3-equitable labeling of $C(L_{3,1})$

.

Theorem 2.1 *$C(L_{3,n})$ does not permit 3-equitable labeling $\forall n \geq 2$.*

Proof 1 *Consider $C(L_{3,n})$ on $n \geq 2$. We take $n = 2$ for the sake of discussion and so $|V(C(L_{3,2}))| = 10$ and $|E(C(L_{3,2}))| = 15$. Obtain of $C(L_{3,n})$ as given in Definition 1.4. We prove by the method of contradiction. Assume that $C(L_{3,2})$ has a 3-equitable labeling d with the property that "the number of nodes with labels i and j differ by at most 1 in the same way the number of lines with label i and j differ by at most 1, $0 \leq i, j \leq 2$, $i \neq j$". Also if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$, then one can observe that the number of lines labeled $0, 1,$ and 2 must be exactly 5 and that the number of nodes labeled $0, 1,$ and 2 must be at least 3 and at most 4 to satisfy the required 3-equitable labeling property $|e_d(i) - e_d(j)| \leq 1$ for all $0 \leq i, j \leq 2$. But in case of lines there are 6 lines with label 0, there are five lines with label 2, four lines with label 1, a contradiction (See Figure 2). Similar argument holds good for $C(L_{3,n})$ $n > 2$. Therefore, $C(L_{3,n})$ $n \geq 2$ does not admit 3-equitable labeling.*

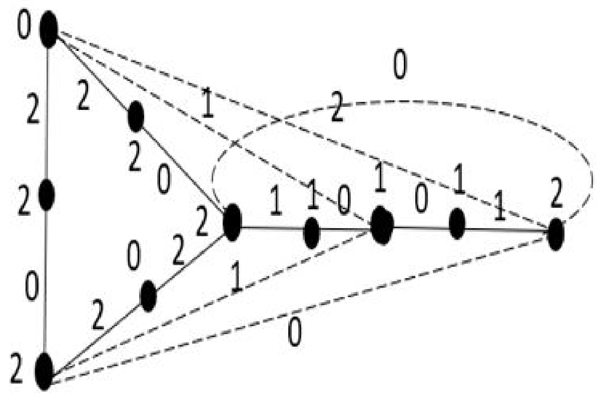

Figure 2. The non-existence of 3-equitable labeling for $C(L_{3,2})$

.

Definition 2.2 *[10] The fan $F_{1,n}$ $n \geq 2$ is defined as $F_{1,n} = P_n + K_1$.*

Theorem 2.2 *$T(F_{1,n})$ does not permit 3-equitable labeling $\forall n \geq 2$.*

Proof 2 *Consider $T(F_{1,n})$ on $n \geq 2$ nodes. We take $n = 2$ for the sake of discussion and so $|V(T(F_{1,2}))| = 6$ and $|E(T(F_{1,2}))| = 12$. Obtain of $T(F_{1,n})$ as given in Definition 3. We prove by the method of contradiction. Assume that $T(F_{1,n})$ has a 3-equitable labeling d with the property that "number of nodes with label i and j differ by at most 1 and in the same way the number of lines with label i and j differ by at most 1, $0 \leq i,j \leq 2, i \neq j$". Also if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i,j \leq 2$, then one can observe that the number of lines labeled $0, 1,$ and 2 must be exactly 4 and that the number of nodes labeled $0, 1,$ and 2 must be exactly 2 to satisfy the required 3-equitable labeling property $|e_d(i) - e_d(j)| \leq 1$ for all $0 \leq i,j \leq 2$. But in case of lines there are three lines with label 0, there are three lines with label 2, six lines with label 1, a contradiction (See Figure 3). Similar argument holds good for $T(F_{1,n})$ when $n \geq 2$.*

Therefore, $T(F_{1,n})$ for all $n \geq 2$, does not admit 3-equitable labeling.

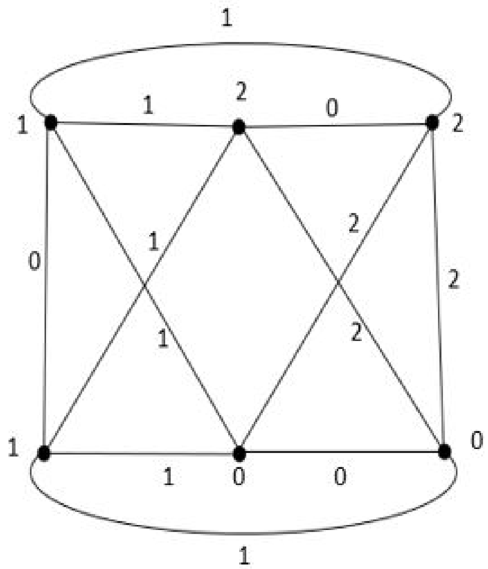

Figure 3. The non-existence of 3-equitable labeling for $T(F_{1,2})$

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Definition 2.3 *[7] The ladder L_n is defined as $L_n = P_n \times P_2$.*

One can obtain the 3-equitable labeling of $M(L_1)$,$M(L_2)$,$M(L_3)$ and consider $M(L_n)$, $\forall n \geq 4$.

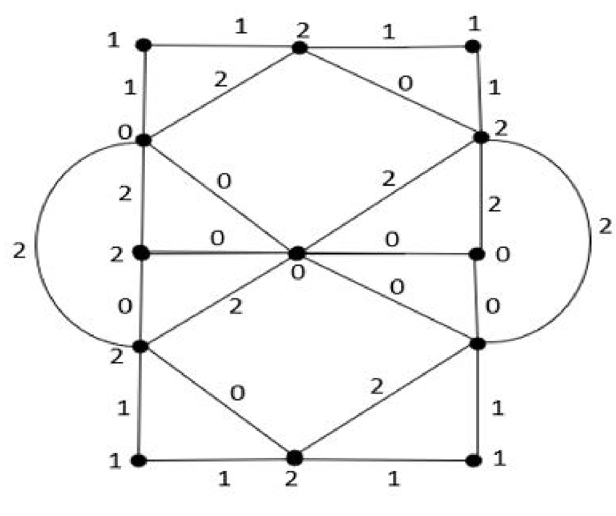

Figure 4. 3-equitable labeling for $M(L_3)$

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Theorem 2.3 *$M(L_n)$ does not permit 3-equitable labeling for all $n \geq 4$.*

Proof 3 *Let $M(L_n)$ on $n \geq 4$ nodes. We take $n = 4$ for the sake of discussion and so $|V(M(L_4))| = 18$ and*

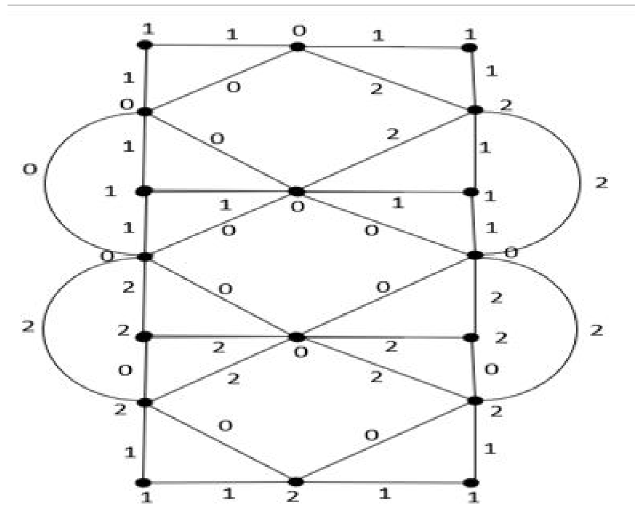

Figure 5. The non-existence of 3-equitable labeling for $M(L_4)$

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$|E(M(L_4))| = 36$. Obtain $M(L_4)$ as given in Definition1.3. We prove by the method of contradiction. Assume that $M(L_4)$ has a 3-equitable labeling d with the property that "number of nodes with label i and j differ by at most 1 and in the same way the number of lines with label i and j differ by at most 1, $0 \leq i, j \leq 2$, $i \neq j$". Also if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$, then one can observe that the number of lines labeled 0, 1, and 2 must be exactly 12 and that the number of nodes labeled $0, 1,$ and 2 must be exactly 6 to satisfy the required 3-equitable labeling property $|e_d(i) - e_d(j)| \leq 1 \; \forall \; i \leq 0, j \leq 2$. But in case of lines there are eleven lines with label 0, there are eleven lines with label 2, fourteen lines with label 1, a contradiction (See Figure 5). Similar argument holds good for $M(L_n)$ $n \geq 4$.

Definition 2.4 *[1, 4] "A friendship graph F_n is a graph which consists of n triangles with a common node".*

One can obtain the 3-equitable labeling of $DS(F_1)$, $DS(F_2)$ and consider $DS(F_n)$, $\forall \; n \geq 3$.

Theorem 2.4 *$DS(F_n)$ does not permit 3-equitable labeling $\forall$ $n \geq 3$.*

Proof 4 *Let $DS(F_n)$ be on $n \geq 3$ nodes. We take $n = 3$ for the sake of discussion and so $|V(DS(F_3))| = 8$ and $|E(DS(F_3))| = 15$. Obtain of $DS(F_3)$ as given in Definition 1.2. We prove by the method of contradiction. Assume that $DS(F_3)$ has a 3-equitable labeling d with the property that "number of nodes with label i and j differ by at most 1 and in the same way the number of lines with label i and j differ by at most 1, $i \leq 0$, $j \leq 2$, $i \neq j$". Also if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1 \; \forall \; 0 \leq i, j \leq 2$, then one can observe that the number of lines labeled $0, 1,$ and 2 must be exactly 5 and that the number of nodes labeled $0, 1,$ and 2 must be at least 2 and at most 3 to satisfy the required 3-equitable labeling property $|e_d(i) - e_d(j)| \leq 1 \; \forall 0 \leq i, j \leq 2$. But in case of lines there are four lines with label 0, there are five lines with label*

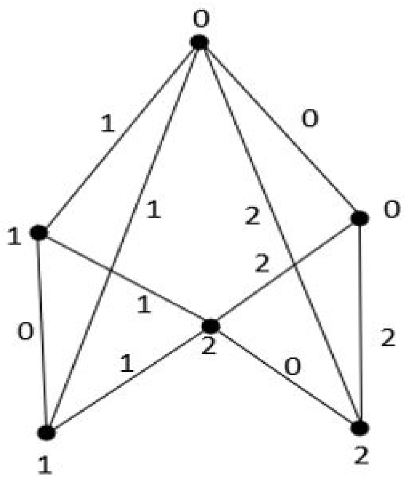

Figure 6. 3-equitable labeling for $DS(F_2)$

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2, *six lines with label* 1*, a contradiction (See Figure 7). Similar argument holds good for* $DS(F_n)$ $n > 4$*. Therefore,* $DS(F_n)$ $n \geq 4$*, does not admit 3-equitable labeling.*

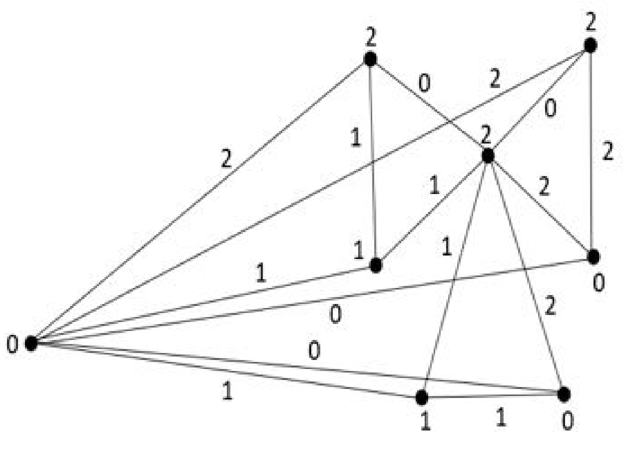

Figure 7. The non-existence of 3-equitable labeling for $DS(F_3)$

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Definition 2.5 *[4] "The path* P_n *is a tree with two nodes of node degree* 1*, and the other* $n-2$ *nodes of node degree* 2*".*

One can obtain the 3-equitable labeling of $\mu(P_2)$, $\mu(P_3)$, $\mu(P_4)$, $\mu(P_5)$, $\mu(P_6)$, $\mu(P_7)$. One such example is given in Figure 8. So, consider $\mu(P_n)$, $\forall\, n \geq 8$.

Theorem 2.5 $\mu(P_n)$ *does not permit 3-equitable labeling for all* $n \geq 8$.

Proof 5 *Let* $\mu(P_n)$ *be on* $n \geq 8$ *nodes. We take* $n = 8$ *for the sake of discussion and so* $|V(\mu(P_8))| = 17$ *and* $|E(\mu(P_8)| = 29$*. Obtain of* $\mu(P_8)$ *as given in Definition 1.6. We prove by the method of contradiction. Assume that* $\mu(P_8)$ *has a 3-equitable labeling* d *with the property that "number of nodes with label*

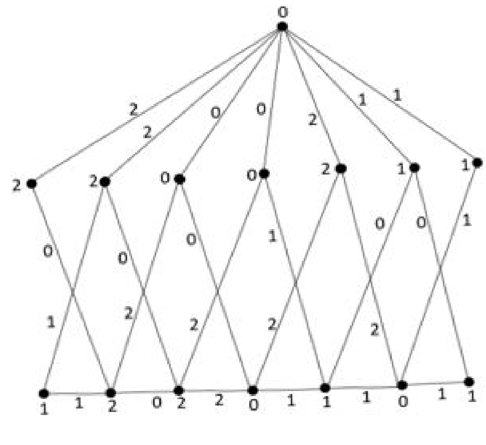

Figure 8. 3-equitable labeling of $\mu(P_7)$

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i *and* j *differ by at most* 1 *and in the same way the number of lines with label* i *and* j *differ by at most* 1*,* $i \leq 0$*,* $j \leq 2$*,* $i \neq j$*". Also if* $|v_f(i) - v_f(j)| \leq 1$ *and* $|e_f(i) - e_f(j)| \leq 1$ $\forall 0 \leq i, j \leq 2$*, then one can observe that the number of lines labeled* $0, 1,$ *and* 2 *must be at least* 9 *and at most* 10 *and that the number of nodes labeled* $0, 1,$ *and* 2 *must be at least* 5 *and at most* 6 *to satisfy the required 3-equitable labeling property* $|e_d(i) - e_d(j)| \leq 1 \;\forall 0 \leq i, j \leq 2$*. But in case of lines there are nine lines with label* 0*, there are eleven lines with label* 2*, nine lines with label* 1*, a contradiction (See Figure 9). Similar argument holds good for* $\mu(P_n)$ $n > 8$*. Therefore,* $\mu(P_n)$ $n > 8$*, does not admit 3-equitable labeling.*

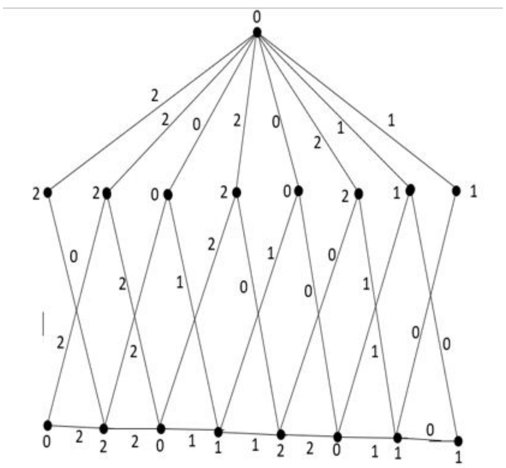

Figure 9. The non-existence of 3-equitable labeling of $\mu(P_8)$

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One can obtain the 3-equitable labeling of $\mu(L_1)$ (see Figure 10). So, consider $\mu(L_n)$, for $n \geq 2$.

Theorem 2.6 $\mu(L_n)$ *does not permit 3-equitable labeling for all* $n \geq 2$.

Proof 6 *Let* $\mu(L_n)$ *be on* $n \geq 2$ *nodes. Take* $n = 2$ *(say) and so* $|V(\mu(L_2))| = 9$ *and* $|E(\mu(L_2))| = 16$*. Obtain* $\mu(L_2)$ *as given in Definition 1.6. Assume that* $\mu(L_2)$ *has a 3-equitable*

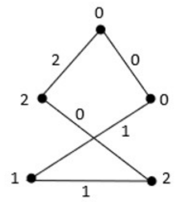

Figure 10. 3-equitable labeling of $\mu(L_1)$

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labeling d with the property that "number of nodes with label i and j differ by at most 1 and in the same way the number of lines with label i and j differ by at most 1, $0 \leq i, j \leq 2, i \neq j.$" Also if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$, then one can observe that the number of lines labeled 0, 1, and 2 must be at least 5 and at most 6 and that the number of nodes labeled 0, 1, and 2 must be exactly 3 to satisfy the required 3-equitable labeling property $|e_d(i) - e_d(j)| \leq 1$ $\forall 0 \leq i, j \leq 2$. But in case of lines there are six lines with label 0, there are three lines with label 2, seven lines with label 1, a contradiction (See Figure 11). Similar argument holds good for $\mu(L_n), n > 2$. Therefore, $\mu(L_n), n \geq 2$, does not admit 3-equitable labeling.

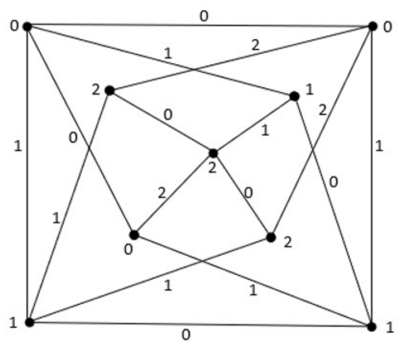

Figure 11. The non-existence of 3-equitable labeling of $\mu(L_2)$

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Definition 2.6 *"For any integers $m > 2, n > 1$, Umbrella graph $U_{(m,n)}$ is obtained by identifying the end node of P_n with a central node of F_m".*

One can obtain the 3-equitable labeling of $DS(U_{(3,k)})$, $2 \leq k \leq 11$ (See Figure 12). For $DS(U_{(3,n)}), n \geq 12$, the following theorem is derived.

Theorem 2.7 *$DS(U_{(3,n)})$ does not permit 3-equitable labeling for all $n \geq 12$.*

Proof 7 *Let $DS(U_{(3,n)})$ be on $n \geq 12$ nodes. Take $n = 12$ and so $|V(DS(U_{(3,12)})| = 17$ and $|E(DS(U_{(3,12)})| = 30$. Obtain $DS(U_{(3,12)})$ as given. Assume that $(DS(U_{(3,12)})$ has a 3-equitable labeling d with the property that "number of nodes with label i and j differ by at most 1 and in the same way the number of lines with label i and j differ by at most 1, $0 \leq i, j \leq 2, i \neq j$." Also if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$*

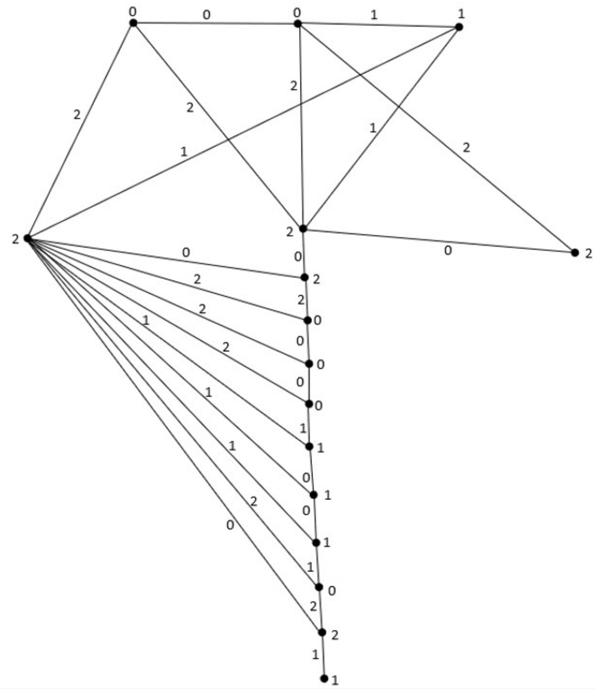

Figure 12. 3-equitable labeling of $DS(U_{(3,11)})$

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for all $0 \leq i, j \leq 2$, then one can observe that the number of lines labeled 0, 1, and 2 must be exactly 10 and that the number of nodes labeled 0, 1, and 2 must be at least 5 and at most 6 to satisfy the required 3-equitable labeling property $|e_d(i) - e_d(j)| \leq 1$ $\forall 0 \leq i, j \leq 2$. But in case of lines there are 11 lines with label 0, there are 10 lines with label 2, 9 lines with label 1, a contradiction (See Figure 13). Similar argument holds good for $DS(U_{(m,n)}), n > 12$. Therefore, $DS(U_{(m,n)}), n \geq 12$, does not admit 3-equitable labeling.

3 Prime Labeling

A graph $H' = (V(H'), E(H'))$ with p_1- nodes is said to admit prime labeling if there exists an injection $h_1 : V(H') \rightarrow \{1, 2, ..., p_1\}$ such that for each line $e = st$, $gcd\{h_1(s), h_1(t)\} = 1$. A graph that admits prime labeling is called prime graph. For more results on prime labeling, see [11, 12, 13]. It is of high interest to study and derive the relation between 3-equitable and prime labeling. One can also note that the graphs discussed in above the section do admit prime labeling and hence prime graphs (see Figure 14 and Figure 15). This interesting observation leads to the following list of open problems to the research community.

- Is every 3-equitable graph a prime?
- Is every prime graph a 3-equitable?
- What is the relation between 3-equitable and prime labeling? More specifically, what is the characterization theorem for these labeling?

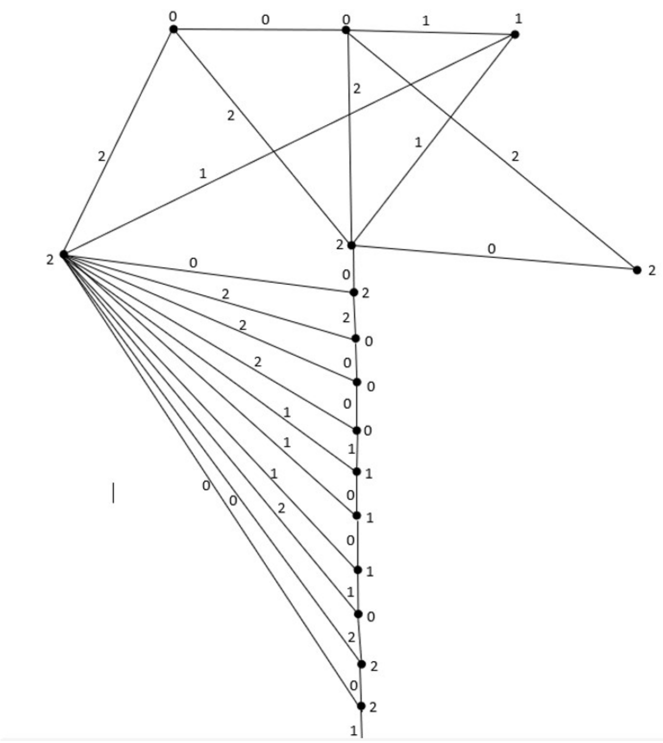

Figure 13. The non-existence of 3-equitable labeling for $DS(U_{(3,12)})$

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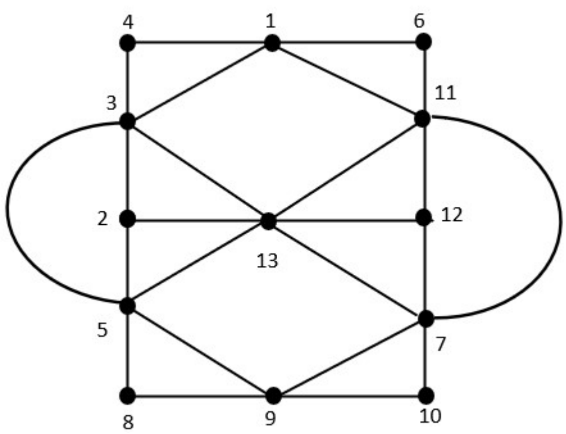

Figure 14. Prime labeling of $M(L_3)$

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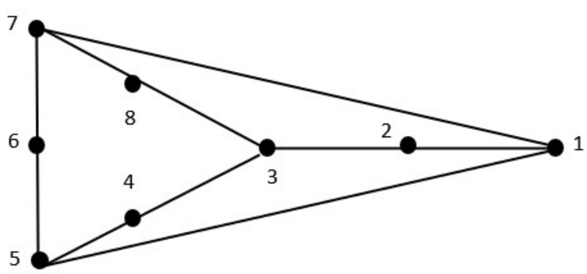

Figure 15. Prime labeling of $C(L_{3,1})$

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4 Conclusion

The existence and non-existence of 3-equitable labeling of the central graph of lollipop graph, total graph of fan graph, middle graph of ladder graph, degree splitting graph of friendship graph and Mycielskian graph of path are established. A complete characterization of 3-equitable labeling is still open and this is for future work. Authors also believe that the concept of 3-equitable labeling may find its applications in computing and intelligent systems.

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