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Some Convergence Results for the Strong Versions of Order-integrals in Lattice Spaces

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Abstract Integration in Riesz spaces has received significant attention in recent papers. The existing body of literature provides comprehensive analyses of the concepts related to order-type integrals for functions that are defined in ordered vector spaces and Banach lattices, as indicated by the studies covered in [3], [4], [5], [7], [8], [9], and [10]. In our work on strongly order-McShane (Henstock-Kurzweil) equi-integration, we have drawn upon the earlier works of Candeloro and Sambucini [6], as well as Boccuto et al. [1-2], who have conducted investigations in the field of order-type integrals. We have expanded upon their research to develop our own findings.

This paper focuses on studying the (o)-McShane integral in ordered spaces, where we emphasize the important fact that investigating the (o)-McShane integral is essential in addition to the (o)-Henstock integral. We highlight that the (o)-McShane integration in Banach lattices has richer properties and is more convenient compared to the (o)-Henstock integral. The properties of (o)-convergence exhibited by ordered McShane integrals are prominently featured in our study.

By using (o)-convergence, we have obtained valuable results related to the (o)-McShane integral. We arrive at the same results in Banach lattices as on McShane (Henstock-Kurzweil) norm-integrals, and we demonstrate that the (o)-McShane integral opens up a wide field of study where similar results with Henstock integration can be obtained. The outcomes demonstrate the benefits of utilizing this integration technique in ordered spaces, with potentially significant implications for diverse areas of mathematics and related fields.

Keywords Banach Lattice, Strongly Order-McShane (Henstock-Kurzweil) Integration, Strongly Order-McShane (Henstock-Kurzweil) Equi-integration

1. Introduction

In the last few years, several publications have concentrated on integration in Riesz spaces, exploring the notions of order-type integrals for functions that map to ordered vector spaces and Banach lattices. These include [3], [4], [5], [7], [8], [9], and [10]. Among these works, noteworthy contributions are made by Candeloro and Sambucini [6] and Boccuto et al. [1-2]. This paper defines the notion of strongly (oM)-equi-integral (strongly (oH)-equi-integral) and proves convergence theorems for order-equi-integrals with values in Banach lattices, with a particular focus on the strong version of order-equi-integrals.

In the subsequent sections, we will adopt the notations S to denote a compact metric space, and μ to denote a regular, nonatomic, and σ-additive measure that maps from the σ-algebra $\mathfrak{B}$ of Borel sets of S, such that $\mu : \mathfrak{B} \to \mathbb{R}^+$.

The sequence $(a_n)_n$ is deemed (o)-convergent, or order-convergent, to a if a sequence $(p_a)_a \in R$ exists such that $q_n \downarrow 0$ and $|a_n - a| \le q_n, \forall n \in \mathbb{N}$ ([4],[9]). This is denoted as $(o) - \lim_n a_n = a$.

A gauge is a function β that maps from a compact metric space S to the set of non-negative real numbers $\mathbb{R}^+$. On the other hand, a partition of S is a finite collection of pairs $(F_j, l_j)_{j=1}^{s}$, where the sets F_j are non-overlapping and $\cup_{j=1}^{s} F_j = S$, and the points l_j are called tags. A partition is said to be of Henstock type if each tag l_j belongs to its corresponding set F_j; otherwise, it is either a free or McShane partition.

A gauge is considered β-fine if the distance between any point $\lambda \in F$ and its corresponding tag l_j is less than $\beta(l_j)$ for all $j = 1, \ldots, s$. An alternative definition of a gauge is a function that associates each point $l_j \in S$ covering the set F_j and with an open ball centered at l_j.

The paper discusses fundamental concepts of integrals for Banach lattices with an order-continuous norm, assuming Y to be such a lattice.

Definition 1.1.

A collection $\mathcal{G}$ of functions $g : S \to Y$ is (o)-McShane ((o)-Henstock)-equi-integrable under two conditions, if each $g \in \mathcal{G}$ is (o)-McShane ((o)-Henstock) integrable, and for every (o)-sequence $(a_n)_n$ there exists a corresponding sequence $(\beta_n)_n$ of gauges, such that for any $g \in \mathcal{G}$, the inequality

$$\left|\sum_{j=1}^{s} g(l_j)\mu(F_j) - (oM)\int_S g\right| \leq a_n$$

$$\left(\left|\sum_{j=1}^{s} g(l_j)\mu(F_j) - (oH)\int_S g\right| \leq a_n\right)$$

holds on the condition that $\{(F_j, l_j), j = 1, ..., s\}$ is (β_n)-fine M-partition (H-partition) of S.

Theorem 1.2. [6]

If $g : S \to Y$ is a mapping, it is (o)-McShane ((o)-Henstock) integrable if and only if there exists a sequence of gauges $(\beta_n)_n$ and a corresponding (o)-sequence $(a_n)_n$ such that for every n, whenever $\prod^{''}$ and $\prod^{'}$ are two β_n-fine McShane (Henstock) partitions, the following condition is satisfied:

$$\left|\sigma(g, \prod^{''}) - \sigma(g, \prod^{'}\right| \leq a_n$$

Preposition 1.3. [6]

Let any (o)-Henstock integrable function $g : S \to Y$, there exist (o)-sequences $(a_n)_n$ and $(\beta_n)_n$ of gauges such that, for every n and every β_n-fine Henstock partition $\prod$, the following condition holds

$$\sum_{F \in \prod} Ob_n(g, F) \leq a_n,$$

where

$$Ob_n(F) = \overset{sup}{\prod}\left\{\left|\sum_{E' \in \prod''} g(t_{E'})\mu(E^{''}) - \sum_{E' \in \prod'} g(H_{E'})\mu(E^{'})\right|\right\}$$

Lemma 1.4. (Saks-Henstock)

Suppose that there exists a collection $\mathcal{G}$ of functions $g : S \to Y$, all of which are (o)-McShane ((o)-Henstock)-equi-integrable. Additionally, assume that for an (o)-sequence $(a_n)_n$, there is a corresponding sequence $(\beta_n)_n$ of gauges on S satisfying the following condition:

$$\left|\sum_{j=1}^{m} g(l_j)\mu(F_j) - (oM)\int_S g\right| \leq a_n$$

$$\left|\sum_{j=1}^{m} g(l_j)\mu(F_j) - (oH)\int_S g\right| \leq a_n$$

for every n and every β_n-fine McShane (Henstock)-partition $\prod = \{(F_j, l_j), j = 1, ..., m\}$ of S. Then, if $\{(K_r, \sigma_r), r = 1, ..., v\}$ is a random β_n-fine McShane (Henstock)-system, we have

$$\left|\sum_{r=1}^{v} g(\sigma_r)\mu(K_r) - (oM)\int_{K_r} g\right| \leq a_n$$

$$\left|\sum_{r=1}^{v} g(\sigma_r)\mu(K_r) - (oH)\int_{K_r} g\right| \leq a_n$$

for any $g \in \mathcal{G}$.

2. Materials and Methods

In this section, we focus on a convergence result for the sum integrals as defined in Definition 1.1. We can see that defining an integral requires a particular process of taking limits, and convergence theorems for integrals address the issue of whether we can swap the limit and the integral. The possibility of swapping the limit and the integral depends on whether one of the limit processes is uniform with respect to the other.

Theorem 2.1.

Suppose that, if $\mathcal{G} = \{g_v : S \to Y; v \in \mathbb{N}\}$ is (oM)-equi-integrable sequence, such that,

$$(o) - \lim_{v \to \infty} g_v(l) = g(l) \qquad l \in S$$

then $g : S \to Y$ is (o)-McShane integrable and the following equation

$$(o) - \lim_{v \to \infty} (oM)\int_S g_v = (oM)\int_S g.$$

holds.

Proof. If $(\beta_n)_n$ is the sequence of gauges from Definition 1.1 of the sequence (g_v) corresponding to (o)-sequence $(a_n)_n$, then, for any $v \in \mathbb{N}$, we have

$$\left|\sum_{j=1}^{s} g_v(l_j)\mu(F_j) - (oM)\int_S g_v\right| \leq a_n \qquad (1)$$

for any value of n and any McShane-partition $(F_j, l_j), j = 1, ..., s$ of S that is (β_n)-fine. Assuming that the partition (F_j, l_j) is constant, the convergence of g_v to g at every point implies

$$(o)-\lim_{v\to\infty}\sum_{j=i}^{s} g_v(l_j)\mu(F_j)=\sum_{j=1}^{s} g(l_j)\mu(F_j).$$

Choose $v_0\in\mathbb{N}$ such that for $v>v_0$ holds the inequality

$$\left|\sum_{j=1}^{s} g_v(l_j)\mu(F_j)-\sum_{j=1}^{s} g(l_j)\mu(F_j)\right|\leq a_n.$$

Then, we have

$$\left|\sum_{j=1}^{s} g(l_i)\mu(F_i)-(oM)\int_S g_v\right|\leq\left|\sum_{j=1}^{s}[g(l_j)\mu(F_j)-g_v(l_j)\mu(F_j)]\right|$$

$$+\left|\sum_{j=1}^{s}[g_v(l_j)\mu(F_j)-(oM)\int_S g_v\right|<2a_n$$

for $v>v_o$. For every $v,t>v_o$, the following inequality is given

$$\left|(oM)\int_S g_v-(oM)\int_S g_t\right|\leq 4a_n,$$

which shows that the sequence $(oM)\int_S g_v, v\in\mathbb{N}$ of elements of Y is a Cauchy sequence, and consequently,

$$(o)-\lim_{v\to\infty}\int_S g_v=M\in Y \qquad (2)$$

exists. By (1) and (2), choose such $P\in\mathbb{N}$ that

$$\left|(oM)\int_S g_v-M\right|<a_n$$

for all $v\geq P$. Assume that $\prod=\{(F_j,l_j), j=1,...,s\}$ is β_n-fine McShane-partition of S for every n. As g_v converges pointwise to g, the following inequality holds for some $v_1\geq P$,

$$\left|\sum_{j=1}^{s} g_{v_1}(l_i)\mu(F_i)-\sum_{j=1}^{s} g(l_j)\mu(F_i)\right|\leq a_n.$$

Therefore,

$$\left|\sum_{j=1}^{s} g(l_j)\mu(F_i)-M\right|\leq\left|\sum_{j=1}^{s} g(l_j)\mu(F_j)-\sum_{j=1}^{s} g_{v_1}(l_j)\mu(F_j)\right|$$

$$+\left|\sum_{j=i}^{s} g_{v_j}(l_j)\mu(F_j)-(oM)\int_S g_{v_1}\right|+\left|(oM)\int_S g_{v_1}-M\right|\leq 3a_n$$

and, as a result, g is (o)-McShane integrable on S, and

$$(o)-\lim_{v\to\infty}(oM)\int_S g_v=M=(oM)\int_S g.$$

□

Therefore, the (o)-Henstock-Kurzweil version of the theorem can be demonstrated similarly.

Definition 2.2.

A collection $\mathcal{G}$ of functions $g:S\to Y$ is considered strongly (o)-McShane ((o)-Henstock)-equi-integrable if each $g\in\mathcal{G}$ is integrable under the respective McShane ((o)-Henstock) variation, and if for any (o)-sequence $(a_n)_n$, there exists a corresponding sequence $(\beta_n)_n$ of gauges, provided that for any n and any $g\in\mathcal{G}$, the following

$$\left|\sum_{j=1}^{s} g(l_j)\mu(F_j)-E(F_j)\right|\leq a_n$$

holds if $\prod=(F_j,l_j), j=1,...,s$ is a β_n-fine McShane (Henstock)-partition of S, and E is an interval function with values in Y that is additive and corresponds to $g\in\mathcal{G}$.

Theorem 2.3.

Suppose that $\mathcal{G}=\{g_v:S\to Y; v\in\mathbb{N})$ is a strongly (oM)-equi-integrable sequence

$$(o)-\lim_{v\to\infty} g_v(l)=g(l) \qquad l\in S,$$

it follows that $g:S\to Y$ is strongly (o)-McShane integrable as well as

$$(o)-\lim_{v\to\infty} E_v(S)=E(S)$$

holds. The functions E_v and E are interval functions with values in Y that are additive, and they correspond to g_v and g, respectively.

Proof. It can be noted that the definition of strong (oM)-equi-integrability given in Definition 2.2 implies the (oM)-equi-integrability described in Definition 1.1. The implication of Theorem 2.1 is that g satisfies (o)-McShane integrability, and the following relation holds for every interval $F\subset S$:

$$(o)-\lim_{v\to\infty} E_V(F)=E(F)$$

Let (o)-sequence $(a_n)_n$ be given and let $(\beta_n)_n$ be the corresponding sequences of gauges from Definition 2.2. Assume that $\prod=\{(F_j,l_j), j=1,...,s\}$ is an arbitrary β_n-fine McShane-partition of S. As such, examine the sum

$$\sum_{j=1}^{s}\left|g(l_j)\mu(F_j)-E(F_j)\right|\leq\sum_{j=1}^{s}\left|g(l_j)\mu(F_j)-g_v(l_j)\mu(F_j)\right|$$

$$+\sum_{j=1}^{s}\left|g_v(l_j)\mu(F_j)-E_v(F_j)\right|+\sum_{i=1}^{s}\left|E_v(F_j)-E(F_j)\right|$$

$$\leq a_n+a_n+a_n.$$

If we choose a sufficiently large $v\in\mathbb{N}$, we can obtain the inequality

$$\sum_{j=1}^{s}\left|g(l_j)\mu(F_j)-E(F_j)\right|\leq 3a_n.$$

□

Similarly, the (o)-Henstock-Kurzweil version of the convergence theorem can be demonstrated in the same way.

Lemma 2.4.

Assume that $g_v : S \to Y, v \in \mathbb{N}$ are (o)-McShane integrable functions, provided that,

1. $g_v(l) \to g(l)$ for $l \in S$, and
2. $\{g_v; v \in \mathbb{N}\}$ establishes an (oM)-equi-integrable sequence.

For all (o)-sequence $(a_n)_n$, there is an $\alpha > 0$, such that for any finite collection $G = \{R_k, k = 1, ..., m\}$ of disjoint intervals of S with $\sum_{k=1}^{m} \mu(R_k) < \alpha$, we have

$$\left|\sum_{k=1}^{m}(oM)\int_{R_k} g_v\right| < a_n$$

for every $v \in N$.

Proof. Let (o)-sequence $(a_n)_n$ be given. Since g_v is (oM)-equi-integrable on S, there exists a sequence $(\beta_n)_n$ of gauges for every n on S, where

$$\left|\sum_{j=1}^{s} g_v(l_j)\mu(F_j) - (oM)\int_S g_v\right| \le a_n$$

whenever $\prod = \{(F_j, l_j), j = 1, ..., s\}$ is an arbitrary β_n-fine McShane-partition of S and $v \in \mathbb{N}$. Fixing a β_n-fine McShane-partition of S, $\prod = \{(F_j, l_j), j = 1, ..., s\}$, let $v_0 \in \mathbb{N}$ be such that

$$\left|g_v(l_j) - g(l_j)\right| < a_n$$

for $v > v_0$, as well as $A = max\{|g(l_i)||g_v(l_j)|, j = 1, ..., s, v \le v_0\}$, and set $\alpha = \frac{a_n}{A+1}$. Suppose that $P = \{R_k, k = 1, ..., m\}$ is a finite collection of disjoint intervals in S such that

$$\sum_{j=1}^{m} \mu(R_k) < \alpha.$$

We may assume that for each $k = 1, ..., m$ and for some j, $R_k \subset F_j$ exists. For all $j \in \mathbb{N}$, $j = 1, ..., s$, let $D_j = \{k, k = 1, ..., m; R_k \subset F_j\}$ and let

$$\gamma = \{(R_k, l_j), k \in D_j, j = 1, ..., s\}.$$

Observe that γ is an McShane-system in S that is β_n-fine. Based on Lemma 1.4, we get

$$\left|\sum_{k=1}^{m}(oM)\int_{R_k} g_v\right|$$

$$\le \left|\sum_{k=1}^{m}(oM)\int_{R_k} g_v - g_v(l_j)\mu(R_k)\right|$$

$$+\sum_{k=1}^{m}\left|g_v(l_j)\right|\mu(R_k)$$

$$\le a_n + (A + a_n)\sum_{k=1}^{m}\mu(R_k)$$

$$< a_n + (A + a_n)\alpha < a_n(2 + \frac{a_n}{A+1})$$

for every $v \in \mathbb{N}$. □

Lemma 2.5.

If $g : S \to Y$ is (o)-McShane integrable on S, for all (o)-sequence $(a_n)_n$ there is an $\alpha > 0$ such that for any finite collection $G = \{R_k, k = 1, ..., m\}$ of disjoint intervals with S with

$$\sum_{k=1}^{m}\mu(R_k) < \alpha,$$

we have

$$\left|\sum_{k=1}^{m}(oM)\int_{R_k} f\right| < a_n.$$

Lemma 2.6.

If $g : S \to Y$ is (o)-McShane integrable function on S, then for any sequence $\pi = \{F_j; j \in \mathbb{N}\}$ of non-overlapping intervals of S, the limit

$$(o) - \lim_{p\to\infty}\sum_{j=1}^{p}(oM)\int_{F_j} g = \sum_{j=1}^{\infty}(oM)\int_{F_j} g = L \in Y$$

exists.

Proof. By Lemma 2.5, for every (o)-sequence $(a_n)_n$, there is an $\alpha > 0$, such that, for any finite collection $\pi = \{F_j, j = 1, ..., m\}$ of disjoint intervals of S with

$$\sum_{j=1}^{m}\mu(F_j) < \alpha,$$

$$\sum_{j=1}^{\infty}\mu(F_j \le \mu(S) < \infty,$$

there exists a $K \in \mathbb{N}$ in a way that for all $m > K$, we see that

$$\sum_{j=m}^{\infty}\mu(F_j) < \alpha.$$

Assume $p, m \in \mathbb{N} : K < p < m$. Then, by Lemma 2.5, we have

$$\left|\sum_{j=1}^{m}(oM)\int_{F_j} g - \sum_{j=1}^{p}(oM)\int_{F_j} g\right|$$

$$= \left|\sum_{j=p+1}^{m}(oM)\int_{F_j} g\right| < a_n,$$

because

$$\sum_{j=p+1}^{m}\mu(F_j) \le \sum_{j=p+1}^{\infty}\mu(F_j) < \alpha.$$

Thus, it follows that

$$\sum_{j=1}^{p}(oM)\int_{F_j} g$$

is a Cauchy sequence in Y and

$$(o)-\lim_{p\to\infty}\sum_{j=1}^{p}(oM)\int_{F_j} g=\sum_{j=1}^{\infty}(oM)\int_{F_j} g = L\in Y$$

exists. □

Lemma 2.7.

If $g : S \to Y$ is (o)-McShane integrable function on S, then for every (o)-sequence $(a_n)_n$ there is an $\alpha > 0$, such that, if the sequence $\pi = \{F_j, j \in \mathbb{N}\}$ of non-overlapping intervals $F_j \subset S$, which satisfies

$$\sum_{j=1}^{\infty}\mu(F_j) < \alpha,$$

then

$$\left|\sum_{j=1}^{\infty}(oM)\int_{F_j} g\right| \le a_n.$$

Proof. If $\sum_{j=1}^{\infty}\mu(F_j) < \alpha$, then

$$\sum_{j=1}^{m}\mu(F_j) \le \sum_{j=1}^{\infty}\mu(F_j < \alpha$$

for every $m \in \mathbb{N}$, and, therefore,

$$\left|\sum_{j=1}^{m}(oM)\int_{F_j} g\right| \le a_n$$

by Lemma 2.5. Since by Lemma 2.6, the series $\sum_{j=1}^{\infty}(oM)\int_{F_j} g$ converges in Y, we obtain

$$\left|\sum_{j=1}^{\infty}(oM)\int_{F_j} g\right| =$$

$$\left|(o)-\lim_{m\to\infty}\sum_{j=1}^{m}(oM)\int_{F_j} g\right| \le a_n.$$

□

Theorem 2.8.

A collection of $\mathcal{G}$ of functions $g : S \to Y$ represents (oM)-equi-integrable ((oH)-equi-integrable) if and only if for all (o)-sequence $(a_n)_n$, there is a sequence $(\beta_n)_n$ of gauges $\beta_n(l) : S \to]0, +\infty[$ on S, such that,

$$\left|\sum_{j=1}^{s} g(l_j)\mu(F_j) - \sum_{k=1}^{q} g(v_k)\mu(T_k)\right| < a_n$$

holds for every n and β_n-fine M-partition (H-partition) $\{(F_j, l_j), j = 1, ..., s\}$ and $\{(T_k, v_k), k = 1, ..., q\}$ of S, and all $g \in \mathcal{G}$.

Proof. It is observed that if $\mathcal{G}$ satisfies (o)-equi-integrability, then the condition is also satisfied by a sequence of gauges $(\beta_n)_n$ corresponding to $\frac{1}{2}a_n$ in the definition of (o)-equi-integrability. If the theorem's condition is met, then every function g in $\mathcal{G}$ is (oM)-equi-integrable ((oH)-equi-integrable) individually, using the same corresponding sequence $(\beta_n)_n$ of gauges for a given (o)-sequence $(a_n)_n$, regardless of which $g \in \mathcal{G}$ is selected. □

3. Conclusions

The paper explores (o)-McShane integration in ordered spaces as an important complement to the study of (o)-Henstock integration. Our focus is on examining the concept of equi-integrability of functions that map to a Banach lattice equipped with an order-continuous norm, with a particular emphasis on the order-McShane and Henstock-Kurzweil integrability. Our research establishes several convergence theorems for strongly order-McShane (Henstock-Kurzweil) equi-integrals on Banach lattices, yielding the same results as those for McShane (Henstock-Kurzweil) norm-integrals. These findings illustrate the benefits of using (o)-McShane integration in ordered spaces and open up avenues for further research, such as the study of Walsh series and the application of (o)-McShane integration to intermediate value problems.

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