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Approximation Method Using DP Ball Curves for Solving Ordinary Differential Equations

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Abstract Many researchers frequently developed numerical methods to explore the idea of solving ordinary differential equations (ODEs) approximately. Scholars started evolving approximation methods by developing algorithms to improve the accuracy in terms of error for the approximate solution. Polynomials, piece-wise polynomials in the form of Bézier curves, Bernstein polynomials, etc., are frequently used to represent the approximate solution of ODEs. To get the minimum error between the exact and approximate solutions of ODEs, the DP Ball curve (DPBC) using the least squares method (LSM) is proposed to improve the accuracy of the approximate solutions for the initial value problem IVPs. This paper explores the use of control points of the DPBC with error reduction by minimizing the residual function. The residual function is minimized by constructing the objective function by taking the sum of squares of the residue function for the least residual error. Then, by solving the constraint optimization problem, we obtained the best control points of DPBC. Two strategies are employed: investigating DPBC's control points through error reduction with LSM and computing the optimum control points through degree raising of DPBC for the best approximate solution of ODEs. Substituting the values of control points back into the DPBC allows for the best approximate solution to be obtained. Moreover, the convergence of the proposed method to the IVPs is successfully analyzed in this study. The error accuracy of the proposed method is also compared with the existing studies. Numerous numerical examples of first, second, and third orders are presented to illustrate the efficiency of the proposed method in terms of error. The results of the numerical examples are shown in which the error accuracy is considerably improved.

Keywords DP Ball Curve, Ordinary Differential Equations, Initial Value Problems, Residual Function, Least Squares Method

1. Introduction

Numerous problems in the different branches of computer science and engineering contain mathematical models which involve ordinary differential equations (ODEs). Solving ODEs can be done through three approaches such as analytical, approximate, and numerical ways. However, under the analytical approach, a satisfactory solution could not be found. To overcome this drawback, the approximate solution is considered where it can be found under numerical methods [2,30]. Several methods like shooting, finite element, segmentation, and series methods can produce the approximate solutions of ODEs where a solution is represented as polynomial functions [1,28]; [39-40]. For polynomial functions and data approximation solution presentations, Bézier curves are regularly employed because its control points have advantages in terms of error accuracy computation [2,3]. Furthermore, the Bézier curve's degree can be raised when the approximate solution is not satisfactory [6]. At the same time, while the degree of the Bézier curve is not necessary, it can be reduced to overcome the computational burden. Even with this drawback, the Bézier curve still becomes an

attractive method to perform computations based on the control points for solving ODEs [6,28]. Thus, in the 1970s, the popularity of Bézier curves led it to become a benchmark for researchers to compare with any curve method. New basis functions of generalized Ball curves were introduced and compared with the Bézier curve [7-11]. Researchers found that the properties of the shape or curve preservation are similar for both Bézier curves and generalized Ball curves. In terms of computation, the generalized Ball curves are faster and more efficient compared to the Bézier curves when used in the recursive algorithm and de Casteljau algorithm, respectively [10,12]. Furthermore, in terms of degree raising and reduction, the generalized Ball curves are well performed [13] due to the degree raising and reduction of the control point, becoming a good tool to find the best accurate shape of the curve [10]; [14-15]; [17]. Then, the control point's base becomes a significant geometrical instrument for the shape of the Ball curves [6,15, 17]. This was proven in the year 1974 when Ball presented a cubic polynomial Ball basis function and defined a cubic polynomial curve in the mathematical form while designing a renowned aircraft structure for the CONSURF [7-9]. Ball's work sparked a lot of researchers to study Ball curves, PBC, Said Ball curves (SBC), Wang Ball curves (WBC), and the accuracy in the shape of the curve through degree increment and decrement as well as through the theoretical calculation [5,10]; [13-18]. Till then, three different Ball curves, DPBC, SBC, and WBC, were proposed which were categorized as higher degree n polynomials generalizations [10-11]; [14]. The degree of these generalization curves can be obtained by overlying the control points of SBC, DPBC, and WBC [20]. During the development of the Ball curves, various works arose on numerical methods where many algorithms were developed such as a reduction method to obtain the highest accuracy for the approximate solutions of the higher order ODEs [22-25]. However, a lot of drawbacks are found in the reduction method like writing difficulties, needing a large amount of time to develop computer programs, and computational burden which affects the accuracy in terms of error [22-23]. Alternatively, the direct method was introduced to solve the higher-order ODEs as well as its system of simultaneous first-order ODEs [26]. This method is capable of obtaining the approximate solutions of higher order ODEs efficiently without a reduction to the first order as well as less time needed for computation, where the best approach is via Least Square [39]; [41-44]; [50-51]. Least squares methods (LSM) develop the highest level of error accuracy by using polynomial functions or piecewise polynomial functions and are used for solving the higher order ODEs approximately [30]; [50-51]. Furthermore, for better approximate solutions of higher ODEs, the least square objective function is established in LSM using the Bézier curves [27]. The literature review showed that the approximate solution of first and higher-order ODEs based on the representations of polynomial functions and the Bézier curves with LSM is satisfactory but not on the desired level of error accuracy [28,34]; [37-38]; [45]; [50-51]. Since the properties for the curve preservation of the generalized DPBC and Bézier curves are similar as well as DPBC being more efficient than the Bézier curves in terms of computations, then the representations of DPBC have yet to be examined through LSM to improve the level of error accuracy by solving ODEs approximately. We proposed the DPBC method, and this paper intends to investigate the control points of DPBC, and it will be employed for the best approximate solutions of ODEs using LSM.

The outline of this paper is as such: DPBC is reviewed briefly with its properties in Section 2. In Section 3, the proposed method, and representations of DPBC through minimizing the residual error are illustrated and its degree raising strategy is described. Meanwhile, the theory of the proposed method with convergence property is analyzed in Section 4. A few numerical examples are also demonstrated to validate the proposed method in Section 5. Lastly, in Section 6, the conclusions are provided.

2. Representations of DPBC

DPBC of degree m with $(m+1)$ control points $\{\boldsymbol{d_j}\}_{\boldsymbol{j=0}}^{\boldsymbol{m}}$ is as follows [14,15,17]:

$$D(x) = \sum_{j=0}^{m} d_j \;\; D_j^m(x), \;\; 0 \le x \le 1 \tag{2.1}$$

where the Polynomials $D_j^m(x)$ in DPBC is as follows:

$$D_j^m(x) = \begin{cases} (1-x)^m, & for \;\; j = 0, \\ x(1-x)^{m-j}, & for \;\; 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor - 1, \\ K_1^m(x) + K_2^m(x), & for \;\; j = \left\lfloor \frac{m}{2} \right\rfloor, \\ K_1^m(x) + K_3^m(x), & for \;\; j = \left\lceil \frac{m}{2} \right\rceil, \\ D_{m-j}^m(1-x), & for \;\; \left\lceil \frac{m}{2} \right\rceil + 1 \le j \le m \end{cases} \tag{2.2}$$

where

$$K_1^m(x) = \left(\frac{1}{2}\right)^{\left\lceil \frac{m}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor} \left(1 - x^{\left\lfloor \frac{m}{2} \right\rfloor + 1} - (1-x)^{\left\lfloor \frac{m}{2} \right\rfloor + 1}\right), \tag{2.3}$$

$$K_2^m(x) = \left(\left\lceil \frac{m}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor\right) x(1-x)^{\left\lfloor \frac{m}{2} \right\rfloor + 1}, \tag{2.4}$$

$$K_3^m(x) = \left(\left\lceil \frac{m}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor\right) \;\; (x)^{\left\lfloor \frac{m}{2} \right\rfloor + 1} \;\; (1-x). \tag{2.5}$$

Properties of DPBC basis functions:

i. Non-negativity of DPBC basis function.

$$D_j^m(x) \ge 0, \quad for \;\; all \;\; j = 0, 1, 2, \ldots \;\; , m \tag{2.6}$$

ii. Unity for the partition of DPBC basis function.

$$\sum_{j=0}^{m} D_j^m(x) = 1, \quad for \;\; all \;\; j = 0, 1, 2, \ldots \;\; , m \tag{2.7}$$

The properties in equations (2.6) and (2.7), the non-negativity of the basis function of the DPBC, and its partition unity are shown in the control point's convex combination. Hence, the justification of these properties implies the DPBC with its control points which lies in the convex hull [14,15].

3. Approximate Solutions of ODEs Using DP Ball Curves

Since then, the structural properties between the basis functions of Bézier and DPBC are the same in shape-preserving as well as the computational efficiency of DPBC is higher than the Bézier curve in terms of error accuracy. The approach using DPBC is proposed for the approximate solution of ODEs. The residual error is measured by building the objective function and minimizing by the sum of squares of the residual function $R(x) = Lg(x) - f(x)$. If this objective function equals zero by minimizing, then the quantity of residual error will equal zero as well. Hence, this indicates that the approximate and exact solution of ODEs is the same. The following IVPs are considered to find their approximate solution:

$$Lg(x) = \sum_{j=0}^{2n} \;\; \varphi_j(x) \;\; g^{(j)}(x) = f(x) \;\; , \quad 0 \le x \le 1 \tag{3.1}$$

with IVP

$$g(0) = g_0, \quad g^{(j)}(0) = g_j, \quad j = 1, 2, 3, \ldots, n-1 \tag{3.2}$$

where

$$L = \sum_{j=0}^{2n} \;\; \varphi_j(x) \;\; \frac{d^j}{dx^j}$$

is the differential operator of order $(2n)^{th}$ with polynomial coefficients $\varphi_j(x)$ and equals to $f(x)$ polynomial function in terms of x.

3.1. Algorithm using DP Ball Curves

Considering the IVPs of ODEs (3.1) with the initial conditions (3.2), the objective is to find the polynomial or piecewise

polynomial function as approximate solution ODEs such that it complies with the initial conditions (3.2) and minimizes the residual function $R(u) = Lg(x) - f(x)$ using DPBC with LSM. The algorithm of the proposed method for approximating the solutions of ODEs is illustrated step by step as follows:

Step 1.

Suppose that the approximate solution of ODEs in (3.1) with initial conditions (3.2) is $D(x)$ in the form of DPBC with degree m.

$$D(x) = \sum_{j=0}^{m} d_j \ D_j^m(x), \ \ 0 \le x \le 1 \tag{3.3}$$

where $\{d_j\}_{j=0}^{m} = d_0, d_1, d_2, \dots\dots\dots, d_m$ are $(m+1)$ control points that need to be computed.

Step 2.

By substituting $D(x)$ into ODEs (3.1) with initial condition (3.2), the residual function $R_D(x)$ is obtained which can be expressed as a polynomial of degree $\le k$ in the form of DPBC.

$$R_D(x) = Lg(x) - f(x) = \frac{d^n}{dx^n}\left(\sum_{j=0}^{m} d_j \ D_j^m(x) \ \right) - f\begin{pmatrix} \sum_{j=0}^{m} d_j \ D_j^m(x) \ , \frac{d}{dx}\left(\sum_{j=0}^{m} d_j \ D_j^m(x) \ \right), \\ \frac{d^2}{dx^2}\left(\sum_{j=0}^{m} d_j \ D_j^m(x) \ \right), \\ \dots, \frac{d^{n-1}}{dx^{n-1}}\left(\sum_{j=0}^{m} d_j \ D_j^m(x) \ \right) \end{pmatrix}$$

$$\Rightarrow R_D(x) = \sum_{j=0}^{k} \ c_{Dj} \ D_j^m(x) \tag{3.4}$$

With $k = max_j\{m - j + \deg(\varphi_j), \deg(f(x))\}$, control points c_{Dj} are the linear functions in d_j and $j = 0, 1, 2, 3, \dots, k$ and these linear functions can be found by degree raising and differentiation of DPBC as well as through other multiplication techniques.

Step 3.

By taking the sum of squares of the residual function, the objective function is developed as follows:

$$F_D = \sum_{j=0}^{k} c_{Dj}^2 \ \ d_j.$$

Step 4.

The values of d_j are obtained by applying any appropriate method or Lagrange Multiplier method.

$$\min F_D = \sum_{j=0}^{k} c_{Dj}^2 \ \ d_j \ ,$$

s.t $\ g^j(0) = g_j, \ j = 0, 1, 2, 3, \dots, m-1$ s)

Step 5.

By replacing the values of all control points d_j back into Equation (3.3), the approximate solutions of (3.1) are found.

3.2. The Degree Raising Strategy for DP Ball Curves

The DPBC is employed through the minimization of the residual error to compute the solutions of ODEs approximately. By substituting d_j, the coefficients of DPBC into the objective functions F_D, either the Euclidean norms of the DPBC's coefficients or through DPBS's convex hull property is obtained. Then, the upper bound on $R_D(x)$, from c_{Dj}'s values is shown below.

$$|R_D(x)| \le max_j\{|c_{Dj}|\}, \qquad 0 \le x \le 1 \tag{3.5}$$

If the values in Equation (3.5) are not small enough, then the approximate solution of ODEs will not be at a satisfactory

level. To improve the error accuracy for an acceptable level of approximation, the degree of DPBC is increased and run again. The next section demonstrates the theoretical development of the proposed method by conducting the convergence analysis.

4. Convergence Analysis

In this section, the convergence analysis of the DPBC method for IVPs is conducted. The convergence analysis employed to the IVPs of ODEs using the control point base of DPBC is as follows: Firstly, by employing the average squares of control points of DPBC, the square of L_2 norm polynomial can be estimated within [0, 1];

Lemma:

If DPBC is a polynomial as follows:

$$D(x) = \sum_{j=0}^{m} d_{j,m} \; D_j^m(x),$$

then

$$\frac{\sum_{j=0}^{m} d_{j,m}^2}{m+1} \geq \frac{\sum_{j=0}^{m+1} d_{j,m+1}^2}{m+2} \geq \frac{\sum_{j=0}^{m+2} d_{j,m+2}^2}{m+3} \geq \cdots \; \frac{\sum_{j=0}^{m+n} d_{j,m+n}^2}{m+n+1} \; \rightarrow \int_0^1 D^2(x) \; dx, \quad n \rightarrow +\infty.$$

By raising the degree to $(m + \mathrm{k})$, the $d_{j,m+k}$ are the coefficients of DPBC.

Proof: By degree raising

$$d_{j,m+1} = \frac{\mathrm{j} \; d_{j-1,m}}{m+1} + \frac{(\mathrm{m}+1-\mathrm{j}) \; d_{j,m}}{m+1},$$

therefore,

$$\begin{aligned}
&\frac{\sum_{j=0}^{m+1} d_{j,m+1}^2}{m+2} \\
&= \frac{\sum_{j=0}^{m+1} \left[\left(\frac{j}{m+1}\right)^2 d_{j-1,m}^2 + \left(\frac{m+1-j}{m+1}\right)^2 d_{j,m}^2 + 2 \; \frac{j(m+1-j)}{(m+1)^2} \; d_{j-1,m} \; d_{j,m} \right]}{m+2} \\
&\leq \frac{\sum_{j=1}^{m+1} \left(\frac{j}{m+1}\right)^2 d_{j-1,m}^2 + \sum_{j=0}^{m} \left(\frac{m+1-j}{m+1}\right)^2 d_{j,m}^2 + \sum_{j=1}^{m} \frac{j(m+1-j)}{(m+1)^2} \; \left(d_{j-1,m}^2 + d_{j,m}^2\right)}{m+2} \\
&= \frac{\sum_{j=0}^{m} d_{j,m}^2}{m+1}
\end{aligned}$$

⇒The sequence $\frac{\sum_{j=0}^{m+n} d_{j,m+n}^2}{m+n+1}$ is reducing monotonically and as the zero is its lower bound, it is convergent as $n \rightarrow +\infty$. Now,

$$\lim_{n \rightarrow +\infty} \frac{\sum_{j=0}^{m+n} d_{j,m+n}^2}{m+n+1} = \int_0^1 D^2(x) \; dx$$

requires proof.

Through calculus (fundamental theorem),

$$\int_0^1 D^2(x) \; dx = \lim_{n \rightarrow +\infty} \frac{\sum_{j=0}^{m+n} D^2\left(\frac{j}{m+n}\right)}{m+n+1}.$$

For the convergence, the property of degree elevation to DPBC having a small arbitrary quantity, ε_j, $j = 0, 1, 2, 3, \ldots, m+n$ is such that

$$D\left(\frac{j}{m+n}\right) = d_{j,m+n} + \varepsilon_j, j = 0, 1, 2, 3, \ldots, m+n, \text{ with } \lim_{n \rightarrow +\infty} max_j \left|\varepsilon_j\right| = 0.$$

Thus,

$$\frac{\sum_{j=0}^{m+n} D^2\left(\frac{j}{m+n}\right)}{m+n+1} = \frac{\sum_{j=0}^{m+n}\left(d_{j,n+m}^2 + 2d\ \varepsilon_j + \varepsilon_j^2\right)}{m+n+1}.$$

Let $\varepsilon_{m,n} = max_j\ \{|\varepsilon_j|\}_{j=0}^{m+n}$.

On the right-hand side, the last two terms can be approximated as

$$\left|\frac{\sum_{j=0}^{m+n}\left(\varepsilon_j^2\right)}{m+n+1}\right| \le \varepsilon_{m,n}^2$$

and

$$\left|\frac{\sum_{j=0}^{m+n}\left(d_{j,m+n}\ \varepsilon_j\right)}{m+n+1}\right| \le \frac{\varepsilon_{m,n}\sum_{j=0}^{m+n}\left|d_{j,m+n}\right|}{m+n+1} \le$$

$$\frac{\varepsilon_{m,n}\sum_{j=0}^{m+n}\left(\frac{j}{m+n}\left|d_{j-1,m+n-1}\right| + \frac{m+n-j}{m+n}\left|d_{j,m+n-1}\right|\right)}{m+n+1}$$

$$\le \frac{\varepsilon_{m,n}\sum_{j=0}^{m+n-1}\left|d_{j,m+n-1}\right|}{m+n} \le \cdots \le \frac{\varepsilon_{m,n}\sum_{j=0}^{m}\left|d_{j,m}\right|}{m+1}.$$

Consequently,

$$\lim_{n\to+\infty}\frac{\sum_{j=0}^{m+n}\left(2d_{j,m+n}\ \varepsilon_j + \varepsilon_j^2\right)}{m+n+1} = 0$$

$$\Rightarrow \lim_{n\to+\infty}\frac{\sum_{j=0}^{m+n} D^2\left(\frac{j}{m+n}\right)}{m+n+1} = \lim_{n\to+\infty}\frac{\sum_{j=0}^{m+n}\left(d_{j,n+m}^2\right)}{m+n+1}.$$

Thus, it is proved.

$$\lim_{n\to+\infty}\frac{\sum_{j=0}^{m+n}\left(d_{j,n+m}^2\right)}{m+n+1} = \int_0^1 D^2(x)\ \ dx$$

Theorem:

If the IVP (3.1) has a unique and C^2 continuous solution $\bar{g}(x)$, then the approximate solution $g(x)$ obtained from the control-points method converges to the exact solution as the degree of the approximate solution tends to infinity.

Proof:

Step 1.

There exists an arbitrary small quantity $\varepsilon > 0$, and by using the theorem from Weierstrass [47-49], we can obtain the polynomial $E_Z(x)$ with degree Z such that

$$\left\|\frac{d^i[E_Z(x)]}{dx^i} - \frac{d^i[\bar{g}(x)]}{dx^i}\right\|_\infty < \frac{\varepsilon}{2}, \qquad i = 0, 1, 2, \dots, n-1$$

where $\|\ .\|_\infty$ stands for the $L_\infty - norm$ over $0 \le x \le 1$. Generally, $E_Z(x)$ does not satisfy the conditions. Another polynomial $G_Z(x) = E_Z(x) + \mathrm{a}x + b$, with a small change such that $G_Z(x)$ validates the conditions $G_Z(0) = g_0$, ${G'}_Z(0) = g_1, {G''}_Z(0) = g_2, \dots, {G^{(\mathrm{n}-1)}}_Z(0) = g_{\mathrm{m}-1}$ can be found.

Thus, we have,

$$\left\|\frac{d^i[G_Z(x)]}{dx^i} - \frac{d^i[\bar{g}(x)]}{dx^i}\right\|_\infty < \varepsilon, \qquad i = 0, 1, 2, \dots, n-1.$$

Let $LG_Z(x) = L\left(x,\ g'(x),\ g''(x), \dots, g^{(n-1)}(x)\right)$ for every $x \in [0,1]$ Thus, the residual function can be approximated as shown below.

$$\|LG_Z(x) - f(x)\|_\infty = \|LG_Z(x) - \bar{g}(x)\|_\infty$$
$$\le \|\varphi_m(x)\|_\infty\|G_Z^n - \bar{g}^n\|_\infty + \cdots + \|\varphi_2(x)\|_\infty\|G_Z'' - \bar{g}''\|_\infty + \|\varphi_1(x)\|_\infty\|G_Z' - \bar{g}'\|_\infty$$
$$+ \|\varphi_0(x)\|_\infty\|G_Z - \bar{g}\|_\infty \le \left(\|\varphi_m(x)\|_\infty + \cdots + \|\varphi_2(x)\|_\infty + \|\varphi_1(x)\|_\infty + \|\varphi_0(x)\|_\infty\right)\ \varepsilon\ \le K_3\ \varepsilon$$

where $K_3 = \|\varphi_m(x)\|_\infty + \cdots + \|\varphi_2(x)\|_\infty + \|\varphi_1(x)\|_\infty + \|\varphi_0(x)\|_\infty$ is a constant.

Step 2.

Representation of the residual $R(G_Z)$ in terms of the DPBC is as shown below.

$$R(G_Z) = LG_Z(\mathrm{x}) - f(x) = \sum_{j=0}^{m} d_{j,m} \quad D_j^m(x)$$

From the Lemma, there exists an integer $N(\geq M)$ such that if $n > N$, then

$$\left| \frac{\sum_{j=0}^{m} d_{j,m}^2}{m+1} - \int_0^1 (R(G_Z))^2 \ dx \right| < \varepsilon$$

$$\Rightarrow \frac{\sum_{j=0}^{m} d_{j,m}^2}{m+1} < \varepsilon + \int_0^1 (R(G_Z)^2 \ dx \ \leq \varepsilon + [(\|\varphi_m(x)\|_\infty + \cdots + \|\varphi_2(x)\|_\infty + \|\varphi_1(x)\|_\infty + \|\varphi_0(x)\|_\infty)\varepsilon]^2$$
$$\leq \varepsilon + {K_3}^2 \ \varepsilon^2.$$

Step 3.

Let $D(x)$, the approximate solution of degree $n \geq N$, of (3.1) be obtained with DPBC's coefficients such that

$$R(x, D(x)) = L(x, \mathrm{D}(x)) - f(x) = \sum_{j=0}^{l} d_{j,l} \quad D_j^l(x)$$

with degree $l \geq n \geq N$.

The norm for the difference of exact solution $\bar{g}(x)$ and approximate solution $D(x)$ is as follows:

$$\|D(x) - \bar{g}(x)\| = \int_0^1 \sum_{j=0}^{2} \left| \frac{d^i[D(x)]}{dx^i} - \frac{d^i[\bar{g}(x)]}{dx^i} \right|^2 dx$$

$$\|D(x) - \bar{g}(x)\| \leq K \ \left(|D(0) - \bar{g}(0)| + \left\| R\big(x, D(x)\big) - \mathrm{R}(x, \bar{g}(x) \right\|_2^2 \right) = K \int_0^1 \left(\sum_{j=0}^{l} d_{j,l} \ \mathrm{D}_j^l(x) \right)^2 dx \leq \frac{K}{l+1} \sum_{j=0}^{l} c_{j,l}^2$$

where K is constant. Since then, using the control points DPBC, the average square of its residual is minimized, and the average of $D(x)$ is lesser than $G_Z(x)$. Therefore,

$$\|D(x) - \bar{g}(x)\| \leq \frac{K}{l+1} \sum_{j=0}^{l} c_{j,l}^2 \leq \frac{K}{l+1} \sum_{j=0}^{l} d_{j,l}^2 \leq \frac{K}{m+1} \sum_{j=0}^{m} d_{j,l}^2$$
$$\leq K \left(\varepsilon + \big((\|\varphi_m(x)\|_\infty + \cdots + \|\varphi_2(x)\|_\infty + \|\varphi_1(x)\|_\infty + \|\varphi_0(x)\|_\infty)\varepsilon \big)^2 \right) \leq K\big(\varepsilon + {K_3}^2 \ \varepsilon^2\big).$$

Thus, $D(x)$, the approximate solution converges to the exact solution if the degree of the approximate solution tendency is to infinity.

5. Numerical Examples

In this section, a few numerical examples are illustrated to demonstrate the proposed method in terms of error accuracy. We solved the initial value problem of ODEs.

Example 5.1 [50]

$$y'(x) = \tfrac{1}{2} \mathrm{e}^{\left(\frac{x}{2}\right)} y\left(\tfrac{x}{2}\right) + \tfrac{1}{2} y(x), \quad 0 \leq x \leq 1, \quad y(0) = 1 \tag{5.1}$$

The exact solution is given by $y = e^x$.

DPBC Degree 2

Firstly, suppose that degree 2 $(m = 2)$ DPBC in (2.1) is the approximate solution of (5.1).

$$y = D(x) = \sum_{j=0}^{2} d_j \ D_j^2(x), \qquad 0 \leq x \leq 1,$$

$$\text{i.e.} \quad y = D(x) = \ d_0 D_0^2(x) + \ d_1 D_1^2(x) + d_2 D_2^2(x).$$
$$\Rightarrow y = D(x) = \ d_0(1-x)^2 + \ d_1(1 - \ x^2 - (1-x)^2) + \ d_2 \ x^2.$$
$$\Rightarrow y = D(x) = (d_0 - 2d_1 + d_2) \ x^2 + (2d_1 - 2 \ d_0) \ x + d_0. \tag{5.2}$$

where d_0, d_1 and d_2 are the control points of DPBC which need to be obtained. By replacing $y = D(x)$ into (5.1), the residual function is obtained. i.e.

$$R_D(x) = \frac{d}{dx}\left(\sum_{j=0}^{2} d_j \ D_j^2(x)\right) - \frac{1}{2}\mathrm{e}^{\left(\frac{x}{2}\right)} \sum_{j=0}^{2} d_j \ D_j^2\left(\frac{x}{2}\right) - \frac{1}{2}\sum_{j=0}^{2} d_j \ D_j^2\left(\frac{x}{2}\right).$$

$$R_D(x) = 2d_1 - 2.5d_0 + \left(3d_0 - 5d_1 + 2d_2 + 0.5 \ e^{(0.5x)} \ (d_0 - d_1)\right)x \ - \ 0.5d_0 e^{(0.5x)}$$
$$- \left(0.5d_0 - d_1 + 0.5d_2 + 0.5 \ e^{(0.5x)}(0.25d_0 - 0.5d_1 + 0.25d_2)\right)x^2 = \sum_{j=0}^{2} \ c_{dj} \ D_j^2(x).$$

The residual error using least square is minimized by constructing the objective function and minimizing with $y(0) = d_0 = 1$.

$$F(d_1, d_2) = 1.795046917{d_1}^2 - 1.583067836d_1d_2 - 0.8597647316d_1 + 0.741522584{d_2}^2 - 1.744132412d_2 + 2.978687207$$

Thus, we have the following system of linear equations:

$$3.590093833d_1 - 1.583067836d_2 - 0.8597647316 = 0,$$
$$1.483045168d_2 - 1.583067836d_1 - 1.744132412 = 0.$$

The control points $d_1$and d_2 are obtained by solving the above system of linear equations through vpasolve in MATLAB. The values of control points are presented in Table 1 below.

Table 1. Values of control points of DPBC degree 2 for Equation (5.2)

Control Points	Values
d_0	1.000000000
d_1	1.432189464
d_2	2.704830287

By substituting the least values of the control points d_0, d_1 and d_2 back into (5.2), the approximate solution of (5.1) is obtained as follows:

$$y = D(x) = 0.8404513596x^2 + \ 0.864378927x \ + \ 1.$$

Comparisons of results and accuracy of the approximate solution of degree 2 for (5.1) with the exact solution and existing method are presented in Table 2.

Table 2. Comparison between the accuracy of approximate solutions by DPBC methods of degree 2 with the exact solutions and the existing method

x	Exact Solution	DPBC Methods	Error of DPBC Methods	Bézier [50]	Error of Bézier [50]
0.00	1.0000000000	1.0000000000	0.0000000000e+00	1.000000	0.00000000
0.10	1.1051709181	1.0948424063	1.0328511780e-02	1.0912373279	1.3933590143e-02
0.20	1.2214027582	1.2064938398	1.4908918376e-02	1.2003837347	2.1019023428e-02
0.30	1.3498588076	1.3349543005	1.4904507113e-02	1.3274392204	2.2419587179e-02
0.40	1.4918246976	1.4802237883	1.1600909307e-02	1.4724037849	1.9420912713e-02
0.50	1.6487212707	1.6423023034	6.4189673029e-03	1.6352774283	1.3443842375e-02
0.60	1.8221188004	1.8211898457	9.2895473896e-04	1.8160601506	6.0586498025e-03
0.70	2.0137527075	2.0168864151	3.1337076271e-03	2.0147519517	9.9924424652e-04
0.80	2.2255409285	2.2293920117	3.8510832428e-03	2.2313528317	5.8119032195e-03
0.90	2.4596031112	2.4587066356	8.9647559242e-04	2.4658627906	6.2596794161e-03
1.0	2.7182818285	2.7048302866	1.3451541874e-02	2.7182818283	1.5904522144e-10

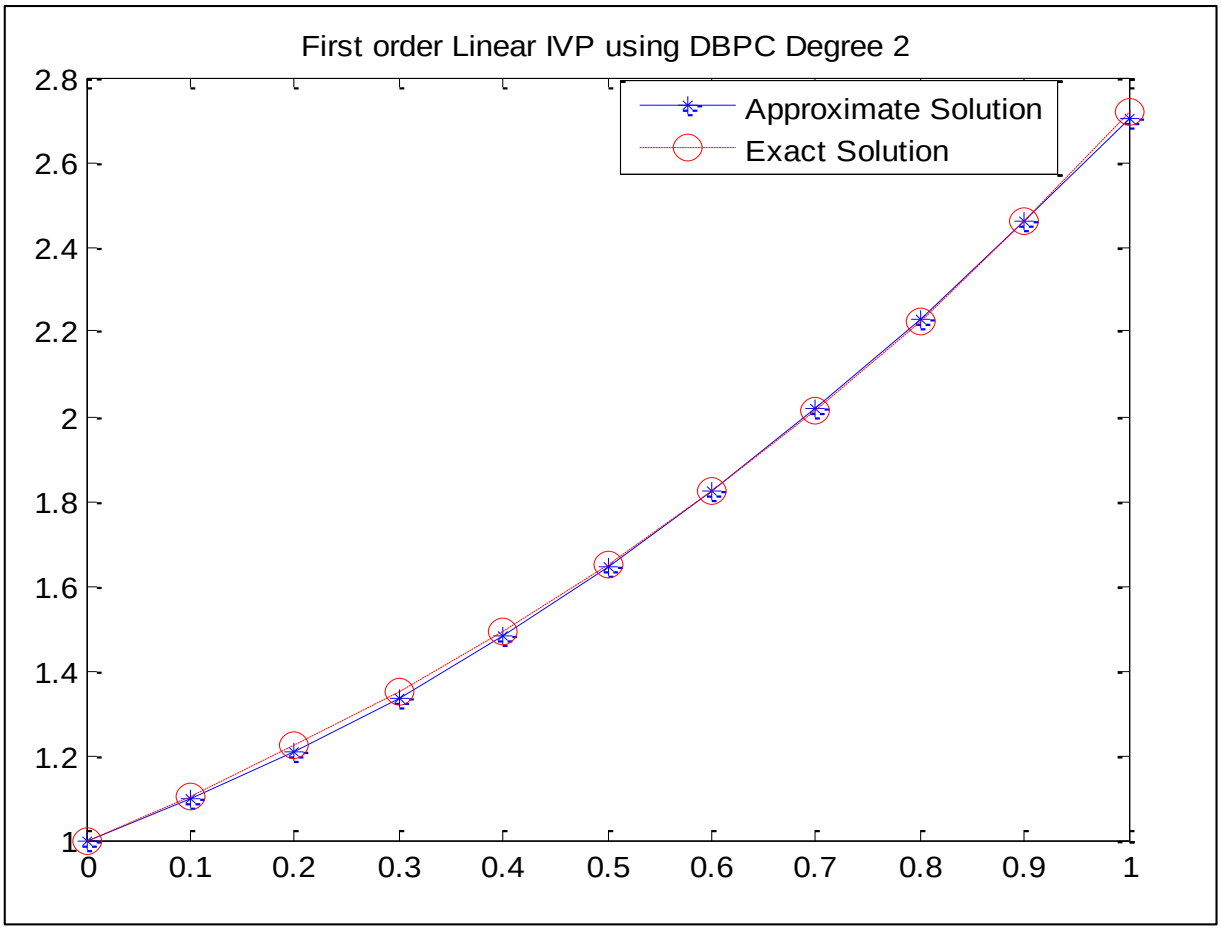

Figure 1. Approximate solution by DPBC degree 2 and exact solution of (5.1)

DPBC Degree 8

Assume that the DPBC in (2.1) of degree $m = 8$ is the approximate solution of (5.1).

$$y = D(x) = \sum_{j=0}^{8} d_j \; D_j^8(x), \qquad 0 \le x \le 1.$$

$$\begin{aligned} y = D(x) = {} & (d_0 - d_1 - d_7 + d_8)x^8 + (7d_1 - 8d_0 + d_2 - d_6 + d_7)x^7 \\ & + (28d_0 - 21d_1 - 6d_2 - d_3 - d_5 + d_6)x^6 + (35d_1 - 56d_0 + 15d_2 + 5d_3 + d_5)x^5 + \\ & (70d_0 - 35d_1 - 20d_2 - 10d_3 - 5d_4)x^4 + (21d_1 - 56d_0 + 15d_2 + 10d_3 + 10d_4)x^3 + \\ & (28d_0 - 7d_1 - 6d_2 - 5d_3 - 10d_4)x^2 + (d_1 - 8d_0 + d_2 + d_3 + 5d_4)x + d_0. \end{aligned} \tag{5.3}$$

where $d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7$ and d_8 are the control points or coefficients of DPBC that need to be found.

By substituting $D(x)$ into (5.1), the residual function is obtained, i.e.

$$R_D(x) = \frac{d}{dx}\left(\sum_{j=0}^{8} d_j \;\; D_j^8(x)\right) - \frac{1}{2}e^{\left(\frac{x}{2}\right)} \;\; \sum_{j=0}^{8} d_j \;\; D_j^8\left(\frac{x}{2}\right) - \frac{1}{2}\sum_{j=0}^{8} d_j \;\; D_j^8(x)$$

$$\begin{aligned}
&= d_1 - 8.5d_0 + d_2 + d_3 + 5d_4 \\
&- x^8\left(0.5d_0 - 0.5d_1 - 0.5d_7 + 0.5d_8\right. \\
&+ 0.5e^{0.5x}(0.00390625d_0 - 0.00390625d_1 - 0.00390625d_7 + 0.00390625d_8)\big) \\
&- x\big(14.5d_1 - 60.\,d_0 + 12.5d_2 + 10.5d_3 + 22.5d_4 \\
&+ 0.5e^{0.5x}(0.5d_1 - 4d_0 + 0.5d_2 + 0.5d_3 + 2.5d_4)\big) - 0.5d_0e^{0.5x} \\
&+ x^6\big(59.5d_1 - 70d_0 + 10d_2 + 0.5d_3 + 0.5d_5 - 7.5d_6 + 7d_7 \\
&+ 0.5e^{0.5x}(0.328125d_1 - 0.4375d_0 + 0.09375d_2 + 0.015625d_3 + 0.015625d_5 - 0.015625d_6)\big) \\
&+ x^2\big(66.5d_1 - 182d_0 + 48d_2 + 32.5d_3 + 35d_4 \\
&+ 0.5e^{0.5x}(1.75d_1 - 7d_0 + 1.5d_2 + 1.25d_3 + 2.5d_4)\big) \\
&- x^3\big(150.5d_1 - 308d_0 + 87.5d_2 + 45d_3 + 25d_4 \\
&+ 0.5e^{0.5x}(2.625d_1 - 7d_0 + 1.875d_2 + 1.25d_3 + 1.25d_4)\big) \\
&- x^5\big(143.5d_1 - 196d_0 + 43.5d_2 + 8.5d_3 + 6.5d_5 - 6d_6 \\
&+ 0.5e^{0.5x}(1.09375d_1 - 1.75d_0 + 0.46875d_2 + 0.15625d_3 + 0.03125d_5)\big) \\
&+ x^4\big(192.5d_1 - 315d_0 + 85d_2 + 30d_3 + 2.5d_4 + 5d_5 \\
&+ 0.5e^{0.5x}(2.1875d_1 - 4.375d_0 + 1.25d_2 + 0.625d_3 + 0.3125d_4)\big) \\
&- x^7\big(11.5d_1 - 12d_0 + 0.5d_2 - 0.5d_6 + 8.5d_7 - 8d_8 \\
&+ 0.5e^{0.5x}(0.0546875d_1 - 0.0625d_0 + 0.0078125d_2 - 0.0078125d_6 + 0.0078125d_7)\big) \\
&= \sum_{j=0}^{8} c_{dj} \;\; D_j^8(x).
\end{aligned}$$

The objective function is constructed by minimizing the residual error using least square and given the initial condition $y(0) = d_0 = 1$.

$$\begin{aligned}
&F(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8) \\
&= 0.03845072041{d_1}^2 + 0.08294480961d_1d_2 + 0.08868125052d_1d_3 + 0.3537277994d_1d_4 \\
&- 0.005642045809d_1d_5 - 0.003357859179d_1d_6 - 0.002105835749d_1d_7 \\
&- 0.01365608494d_1d_8 - 0.5180167109d_1 + 0.045597323{d_2}^2 + 0.0997333665d_2d_3 \\
&+ 0.445671021d_2d_4 - 0.008058871617d_2d_5 - 0.005059833207d_2d_6 - 0.003302969691d_2d_7 \\
&- 0.02459963634d_2d_8 - 0.6012732354d_2 + 0.05602702244{d_3}^2 + 0.5799589071d_3d_4 \\
&- 0.01166936786d_3d_5 - 0.007817572636d_3d_6 - 0.005367854185d_3d_7 \\
&- 0.04584929173d_3d_8 - 0.706255942d_3 + 7.192456{d_4}^2 + 0.5441973621d_4d_5 \\
&+ 0.4258997169d_4d_6 + 0.3450397827d_4d_7 - 7.985868262d_4d_8 - 6.811213546d_4 \\
&+ 0.05209259979{d_5}^2 + 0.09307727621d_5d_6 + 0.08316282589d_5d_7 - 0.7306854703d_5d_8 \\
&- 0.01616842021d_5 + 0.04267903402{d_6}^2 + 0.07794563618d_6d_7 - 0.6201846941d_6d_8 \\
&- 0.008062529413d_6 + 0.0362500475{d_7}^2 - 0.5345952552d_7d_8 - 0.004495883883d_7 \\
&+ 3.77842941{d_8}^2 - 0.01927796127d_8 + 5.558854951.
\end{aligned}$$

Then, by minimizing the objective function, thus we have an 8×8 system of linear equations.

$$\begin{aligned}
&0.07690144082d_1 + 0.08294480961d_2 + 0.08868125052d_3 + 0.3537277994d_4 - 0.005642045809d_5 \\
&- 0.003357859179d_6 - 0.002105835749d_7 - 0.01365608494d_8 - 0.5180167109 = 0.
\end{aligned}$$

$$\begin{aligned}
&0.08294480961d_1 + 0.09119464601d_2 + 0.0997333665d_3 + 0.445671021d_4 - 0.008058871617d_5 \\
&- 0.005059833207d_6 - 0.003302969691d_7 - 0.02459963634d_8 - 0.6012732354 = 0.
\end{aligned}$$

$$\begin{aligned}
&0.08868125052d_1 + 0.0997333665d_2 + 0.1120540449d_3 + 0.5799589071d_4 - 0.01166936786d_5 \\
&- 0.007817572636d_6 - 0.005367854185d_7 - 0.04584929173d_8 \;\; - 0.706255942 = 0.
\end{aligned}$$

$$\begin{aligned}
&0.3537277994d_1 + 0.445671021d_2 + 0.5799589071d_3 + 14.384912d_4 + 0.5441973621d_5 + 0.4258997169d_6 \\
&+ 0.3450397827d_7 - 7.985868262d_8 - 6.811213546 = 0.
\end{aligned}$$

$$\begin{aligned}
&0.5441973621d_4 - 0.008058871617d_2 - 0.01166936786d_3 - 0.005642045809d_1 + 0.1041851996d_5 \\
&+ 0.09307727621d_6 + 0.08316282589d_7 - 0.7306854703d_8 - 0.01616842021 = 0.
\end{aligned}$$

$$0.4258997169d_4 - 0.005059833207d_2 - 0.007817572636d_3 - 0.003357859179d_1 + 0.09307727621d_5 + 0.08535806805d_6 + 0.07794563618d_7 - 0.6201846941d_8 - 0.008062529413 = 0.$$

$$0.3450397827d_4 - 0.003302969691d_2 - 0.005367854185d_3 - 0.002105835749d_1 + 0.08316282589d_5 + 0.07794563618d_6 + 0.07250009501d_7 - 0.5345952552d_8 - 0.004495883883 = 0.$$

$$7.55685882d_8 - 0.02459963634d_2 - 0.04584929173d_3 - 7.985868262d_4 - 0.7306854703d_5 - 0.6201846941d_6 - 0.5345952552d_7 - 0.01365608494d_1 - 0.01927796127 = 0.$$

The remaining control points are obtained by vpasolve in MATLAB and given in Table 3 as shown below.

Table 3. Control point values of DPBC of degree 8

Control Points	Values
d_0	1.0000000000
d_1	-3.020835162
d_2	12.83631354
d_3	-8.913692816
d_4	1.619642887
d_5	13.76130447
d_6	-9.570621902
d_7	6.739075801
d_8	2.718281828

We obtained the approximate solution of (5.1) by substituting the values of all the control points back into (5.3).

$$y = D(x) = 0.00004118943341x^8 + 0.0001651086821x^7 + 0.001423615194x^6 + 0.00831279643x^5 + 0.0416736239x^4 + 0.1666653873x^3 + 0.5000001107x^2 + 0.9999999969x + 1.$$

Comparisons of results and accuracy of the approximate solution of degree 8 for (5.1) with the exact solution and existing method are presented in Table 4.

Table 4. Comparison between the accuracy of approximate solutions by DPBC methods of degree 8 with the exact solutions and the existing method.

x	Exact Solution	DPBC Methods	Error of DPBC Method	Bézier [50]	Error of Bézier [50]
0.00	1.0000000000	1.0000000000	0.0000000000e+00	1.0000000000	0.0000000000e+00
0.10	1.1051709181	1.1051709181	3.8512304457e-11	1.1056199126	4.4899447785e-04
0.20	1.2214027582	1.2214027581	3.3204550220e-11	1.2222861275	8.8336935985e-04
0.30	1.3498588076	1.3498588076	1.5205836590e-11	1.3511555123	1.2967047349e-03
0.40	1.4918246976	1.4918246977	5.1722404137e-11	1.4934970059	1.6723082274e-03
0.50	1.6487212707	1.6487212707	5.2624571367e-14	1.6507015142	1.9802435108e-03
0.60	1.8221188004	1.8221188003	5.4615645340e-11	1.8242924231	2.1736226845e-03
0.70	2.0137527075	2.0137527075	1.2270184868e-11	2.0159368626	2.1841550884e-03
0.80	2.2255409285	2.2255409285	3.5617286898e-11	2.2274578704	1.9169418791e-03
0.90	2.4596031112	2.4596031111	3.7406522324e-11	2.4608476120	1.2445008665e-03
1.0	2.7182818285	2.7182818285	5.5289106626e-12	2.7182818280	4.5904569035e-10

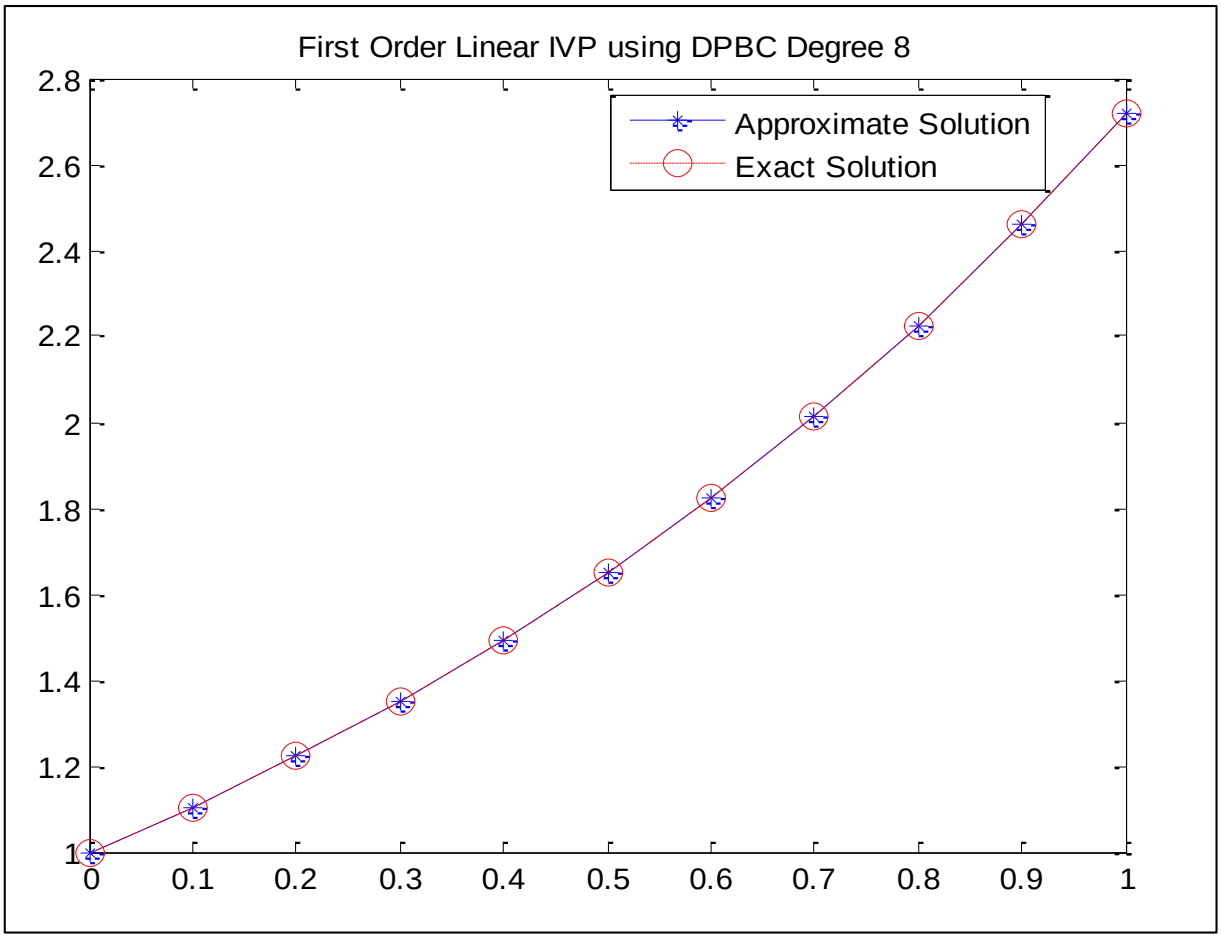

Figure 2. Approximate solution by DPBC degree 8 and exact solution of (5.1)

Example 5.2 [51]

$$\left(\frac{1}{1+\sqrt{3}}\right)\frac{dy}{dx} + xy(x) = 0, y(0) = 1 \tag{5.4}$$

The exact solution is given by $y = e^{-\frac{1}{2}(1+\sqrt{3})x^2}$.

DPBC Degree 3

Suppose that the approximate solution of (5.4) is the DPBC in (2.1) of degree $m = 3$.

$$y = D(x) = \sum_{j=0}^{3} d_j \; D_j^3(x), \qquad 0 \le x \le 1.$$

$$\Rightarrow y = D(x) = (d_1 - d_0 - d_2 + d_3)\; x^3 + (3d_0 - 3d_1)\; x^2 + (2d_1 - 3\; d_0 + d_2)\; x + d_0. \tag{5.5}$$

where $d_0, d_1, d_2,$ and d_3 are the control points of DPBC that need to be obtained. By substituting $D(x)$ into (5.4), we obtained the residual function. i.e.

$$R_D(x) = \left(\frac{1}{1+\sqrt{3}}\right)\frac{d}{dx}\left(\sum_{j=0}^{3} d_j \; D_j^3(x)\right) + x\sum_{j=0}^{3} d_j \; D_j^3(x)$$
$$= (d_1 - d_0 - d_2 + d_3)\; x^4 \; + (3d_0 - 3d_1)\; x^3$$
$$+ \; (3.098076211d_1 - 4.098076211d_0 - 0.09807621135d_2 + 1.098076211d_3)x^2$$
$$+ (3.196152423d_0 - 2.196152423d_1)x \; - 1.098076211d_0 + 0.7320508076d_1$$
$$+ 0.3660254038d_2 = \sum_{j=0}^{4} \; c_{dj} \; D_j^4(x).$$

The residual error is minimized using least square by constructing the objective function and using $y(0) = d_0 = 1$.

$$F(d_1, d_2, d_3) = 0.09754599541d_1^2 + 0.1755956053d_1d_2 - 0.1752872471d_1d_3 - 0.2450797176d_1 + 0.1046888526d_2^2 - 0.1926987652d_2d_3 - 0.2086205804d_2 + 0.6660014447d_3^2 - 0.0816454381d_3 + 0.1928331838$$

Then, by minimizing the objective function, thus we have a 3×3 system of linear equations.

$$0.1950919908d_1 + 0.1755956053d_2 - 0.1752872471d_3 - 0.2450797176 = 0,$$

$$0.1755956053d_1 + 0.2093777051d_2 - 0.1926987652d_3 - 0.2086205804 = 0,$$

$$1.332002889d_3 - 0.1926987652d_2 - 0.1752872471d_1 - 0.0816454381 = 0.$$

By using vpasolve in MATLAB, the remaining control points are obtained and presented in Table 5 as shown below.

Table 5. Control points values of DPBC of degree 3

Control Points	Values
d_0	1.000000000
d_1	1.539069857
d_2	-0.05946688022
d_3	0.2552288473

We obtained the approximate solution of (5.4) by substituting the values of all the control points back into (5.5).

$$y = D(x) = 0.8537655843x^3 - 1.61720957x^2 + 0.01867283341x + 1$$

Results and accuracy of the approximate solution of degree 3 for (5.4) are presented in Table 6 and then compared with the exact solution and existing method.

Table 6. Comparison between the accuracy of approximate solutions by DPBC methods of degree 3 with the exact solutions and the existing method DLSM

x	Exact Solution	DPBC Methods	Error of DPBC Methods	Method [51]	Error of Method [51]
0.00	1.0000000000	1.0000000000	0.0000000000e+00	1.000000	0.00000000
0.10	0.9864326238	0.9865489532	1.1632938094e-04	0.984892	1.54029e-03
0.20	0.9468249818	0.9458763085	9.4867329041e-04	0.942929	3.89601e-03
0.30	0.8843146981	0.8831046595	1.2100386673e-03	0.879149	5.16528e-03
0.40	0.8036719909	0.8033565995	3.1539137979e-04	0.798593	5.07886e-03
0.50	0.7106989540	0.7117547221	1.0557681344e-03	0.706300	4.39934e-03
0.60	0.6115435272	0.6134216209	1.8780936943e-03	0.607308	4.23521e-03
0.70	0.5120400431	0.5134798893	1.4398461966e-03	0.506659	5.38129e-03
0.80	0.4171721726	0.4170521208	1.2005181854e-04	0.409390	7.7818e-03
0.90	0.3307208619	0.3292609090	1.4599528969e-03	0.320543	1.01782e-02
1.0	0.2551189419	0.2552288473	1.0990536306e-04	0.245155	9.9638e-03

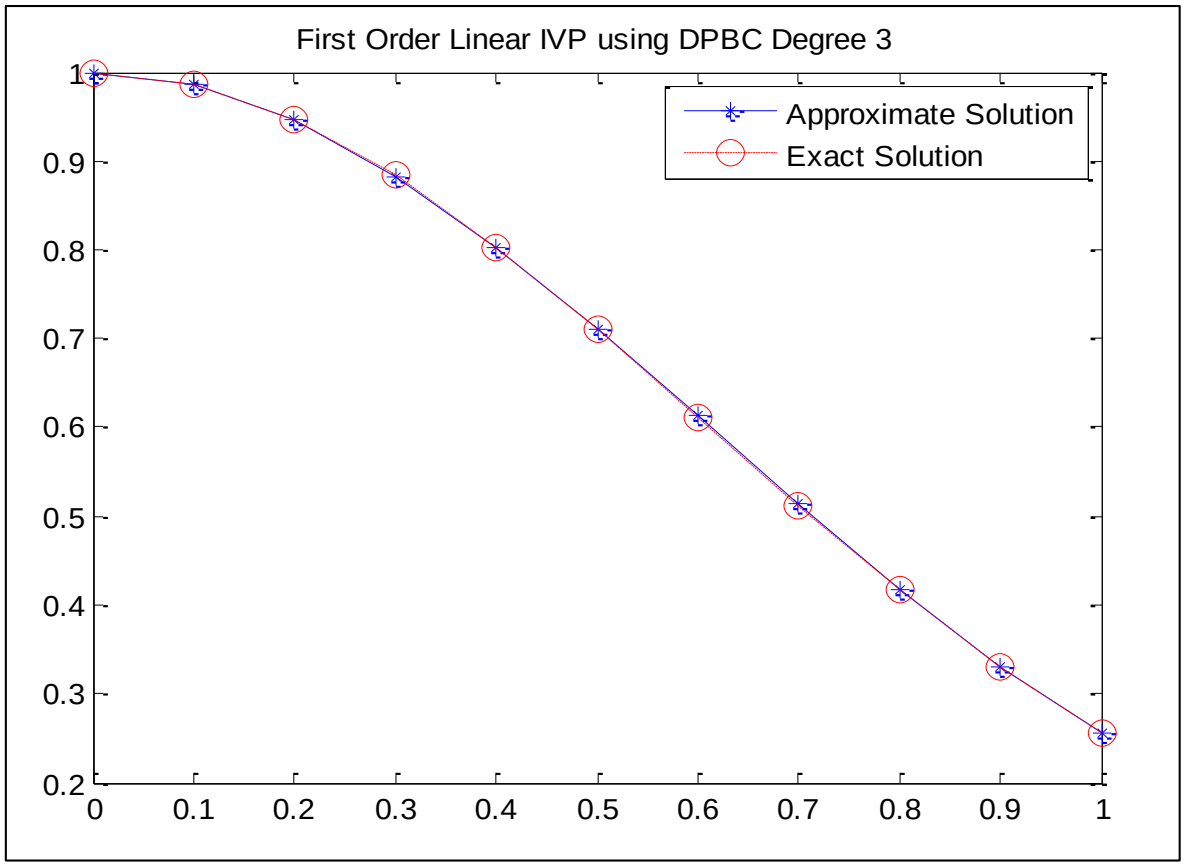

Figure 3. Approximate solution by DPBC degree 3 and exact solution of (5.5)

DPBC Degree 6

Suppose that the approximate solution of (5.4) is the DPBC in (2.1) of degree $m = 6$.

$$y = D(x) = \sum_{j=0}^{6} d_j \; D_j^6(x), \qquad 0 \le x \le 1.$$

$$\begin{aligned} y = D(x) = {} & (d_0 - d_1 - d_5 + d_6)x^6 + (5d_1 - 6d_0 + d_2 - d_4 + d_5)x^5 \\ & + (15d_0 - 10d_1 \; - 4d_2 - 2d_3 + d_4)x^4 + (10d_1 - 20d_0 + 6d_2 + 4d_3)x^3 \\ & + (15d_0 - 5d_1 - 4d_2 - 6d_3)x^2 + (d_1 - 6d_0 + d_2 + 4d_3)x + d_0. \end{aligned} \tag{5.6}$$

where $d_0, d_1, d_2, d_3, d_4, d_5$, and d_6 are the control points or coefficients of DPBC that need to be found. By substituting $y = D(x)$ into (5.4), the residual function is obtained. i.e.

$$R_D(x) = \left(\frac{1}{1+\sqrt{3}}\right)\frac{d}{dx}\left(\sum_{j=0}^{6} d_j \; D_j^6(x)\right) + x\sum_{j=0}^{6} d_j \; D_j^6(x)$$

$$\begin{aligned} & = (d_0 - d_1 - d_5 + d_6)x^7 + (5d_1 - 6d_0 + d_2 - d_4 + d_5)x^6 \\ & + (17.19615242d_0 - 12.19615242d_1 - 4d_2 - 2d_3 + d_4 - 2.196152423d_5 \\ & + 2.196152423d_6)x^5 \\ & + (19.15063509d_1 - 30.98076211d_0 + 7.830127019d_2 + 4d_3 \; - 1.830127019d_4 \\ & + 1.830127019d_5)x^4 \\ & + (36.96152423d_0 - 19.64101615d_1 - 9.856406461d_2 - 8.92820323d_3 + 1.464101615d_4)x^3 \\ & + (11.98076211d_1 - 27.96152423d_0 + 7.588457268d_2 + 8.392304845d_3)x^2 \\ & + (11.98076211d_0 - 3.660254038d_1 - 2.92820323d_2 - 4.392304845d_3)x - 2.196152423d_0 \\ & + 0.3660254038d_1 + 0.3660254038d_2 + 1.464101615d_3 = \sum_{j=0}^{7} \; c_{dj} \; D_j^7(x). \end{aligned}$$

The residual error is minimized using least square by constructing the objective function and minimizing with $y(0) = d_0 = 1$.

$$
\begin{aligned}
&F(d_1, d_2, d_3, d_4, d_5, d_6)\\
&= 0.00640639078d_1^2 + \ 0.01397669856d_1d_2 + 0.04360195829d_1d_3 - 0.002175374606d_1d_4\\
&- 0.001369821996d_1d_5 - 0.002904383474d_1d_6 - 0.0774033855d_1 + 0.007922278464\ d_2^2\\
&+ 0.05884212352d_2d_3 - 0.003047050648d_2d_4 - 0.002147624578d_2d_5\\
&- 0.006364803792d_2d_6 - 0.09436273643d_2 + 0.5484930097d_3^2\ + 0.07341643809d_3d_4\\
&+ 0.05529026998d_3d_5 - 0.6714181347d_3d_6 - 0.7340443973d_3 + 0.008932379475d_4^2\\
&+ 0.01560840019d_4d_5 - 0.08387322594d_4d_6 - 0.006481353909d_4 + 0.007072391446d_5^2\\
&- 0.06824587634d_5d_6 - 0.002932133502d_5 + 0.842998585d_6^2 - 0.003496402888d_6\\
&+ 0.4110391153
\end{aligned}
$$

Then, by minimizing the objective function, we have a 6×6 system of linear equations.

$$
\begin{aligned}
&0.01281278156d_1 + 0.01397669856d_2 + 0.04360195829d_3 - 0.002175374606d_4 - 0.001369821996d_5\\
&- 0.002904383474d_6 - 0.0774033855 = 0,\\
&0.01397669856d_1 + 0.01584455693d_2 + 0.05884212352d_3 - 0.003047050648d_4 - 0.002147624578d_5\\
&- 0.006364803792d_6 - 0.09436273643 = 0,\\
&0.04360195829d_1 + 0.05884212352d_2 + 1.096986019d_3 + 0.07341643809d_4 + 0.05529026998d_5\\
&- 0.6714181347d_6 - 0.7340443973 = 0,\\
&0.07341643809d_3 - 0.003047050648d_2 - 0.002175374606d_1 + 0.01786475895d_4 + 0.01560840019d_5\\
&- 0.08387322594d_6 - 0.006481353909 = 0,\\
&0.05529026998d_3 - 0.002147624578d_2 - 0.001369821996d_1 + 0.01560840019d_4 + 0.01414478289d_5\\
&- 0.06824587634d_6 - 0.002932133502 = 0,\\
&1.68599717d_6 - 0.006364803792d_2 - 0.6714181347d_3 - 0.08387322594d_4 - 0.06824587634d_5\\
&- 0.002904383474d_1 - 0.003496402888 = 0.
\end{aligned}
$$

Thus, by solving the above system of linear equations using vpasolve in MATLAB, the remaining control points are found. The values of all the control points are shown in Table 7.

Table 7. Control point values of DPBC of degree 6

Control Points	Values
d_0	1.0000000000
d_1	-0.4332998493
d_2	3.54293521
d_3	0.7222775777
d_4	-2.209834267
d_5	1.549359451
d_6	0.255118904

We obtained the approximate solution of (5.4) by substituting the values of all the control points back into (5.6).

$$
\begin{aligned}
y = D(x) &= 0.139059302x^6 - 0.864370318x^5 + 1.50686823x^4 - 0.1862769218x^3 - 1.33890706x^2\\
&- 0.001254328288x \ + \ 1
\end{aligned}
$$

Numerical results and accuracy of the approximate solution of degree 6 for the (5.4) are presented in Table 8 and then compared with the exact solution and existing method.

Table 8. Comparison between the accuracy of approximate solutions by DPBC methods of degree 6 with the exact solutions and the existing method.

x	Exact Solution	DPBC Methods	Error of DPBC Methods	Method [51]	Error of Method [51]
0.00	1.0000000000	1.0000000000	0.0000000000e+00	1.000000	0.00000e+00
0.10	0.9864326238	0.9864414018	8.7779876961e-06	0.972952	1.34802e-02
0.20	0.9468249818	0.9468459270	2.0945196460e-05	0.931079	1.57463e-02
0.30	0.8843146981	0.8842991762	1.5521888722e-05	0.870665	1.36498e-02
0.40	0.8036719909	0.8036456776	2.6313259297e-05	0.79209	1.15819e-02
0.50	0.7106989540	0.7107019491	2.9951476732e-06	0.699549	1.11498e-02
0.60	0.6115435272	0.6115696838	2.6156573916e-05	0.600776	1.07671e-02
0.70	0.5120400431	0.5120490571	9.0140171854e-06	0.506769	5.27143e-03
0.80	0.4171721726	0.4171521579	2.0014747922e-05	0.431508	1.43362e-02
0.90	0.3307208619	0.3307165411	4.3207480930e-06	0.391687	6.09662e-02
1.0	0.2551189419	0.2551189040	3.7928547414e-08	0.406429	1.5131e-01

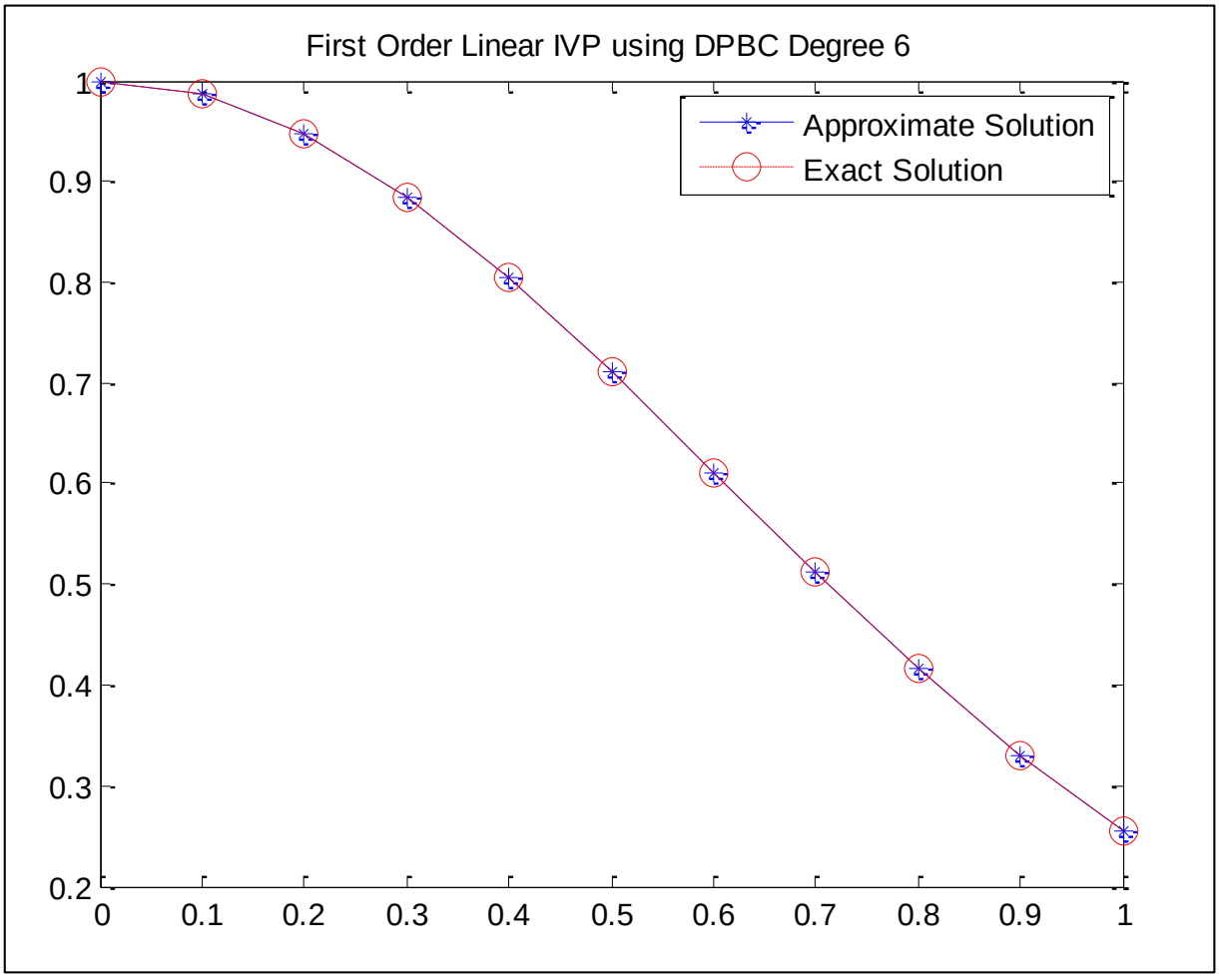

Figure 4. Approximate solution by DPBC degree 6 and exact solution of (5.4)

Example 5.3 [50]

$$y''(x) = \frac{3}{4}y(x) + y\left(\frac{x}{2}\right) - x^2 + 2, \quad 0 \le x \le 1, \quad y(0) = 0, \quad y'(0) = 0 \tag{5.7}$$

The exact solution is given by $y = x^2$.

DPBC Degree 8

Suppose that DPBC in (2.1) of degree $m = 8$ is the approximate solution of (5.7).

$$y = D(x) = \sum_{j=0}^{8} d_j \; D_j^8(x), \qquad 0 \le x \le 1.$$

$$\begin{aligned} y = D(x) = {} & (d_0 - d_1 - d_7 + d_8)x^8 + (7d_1 - 8d_0 + d_2 - d_6 + d_7)x^7 \\ & + (28d_0 - 21d_1 - 6d_2 - d_3 - d_5 + d_6)x^6 \\ & + (35d_1 - 56d_0 + 15d_2 + 5d_3 + d_5)x^5 \\ & + (70d_0 - 35d_1 - 20d_2 - 10d_3 - 5d_4)x^4 \\ & + (21d_1 - 56d_0 + 15d_2 + 10d_3 + 10d_4)x^3 \\ & + (28d_0 - 7d_1 - 6d_2 - 5d_3 - 10d_4)x^2 + (d_1 - 8d_0 + d_2 + d_3 + 5d_4)x + d_0. \end{aligned} \tag{5.8}$$

where $d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7$ and d_8 are the control points of DPBC that need to be found. By substituting $y = D(x)$ into (5.8), the residual function is obtained. i.e.

$$\begin{aligned} R_D(x) = {} & \frac{d^2}{dx^2}\left(\sum_{j=0}^{8} d_j \; D_j^8(x)\right) - \frac{3}{4}\sum_{j=0}^{8} d_j \; D_j^8(x) + \sum_{j=0}^{8} d_j \; D_j^8\left(\frac{x}{2}\right) - x^2 + 2 \\ & = (0.75390625d_1 - 0.75390625d_0 + 0.75390625d_7 - 0.75390625d_8)x^8 \\ & + (6.0625d_0 - 5.3046875d_1 - 0.7578125d_2 + 0.7578125d_6 \\ & - 0.7578125d_7)x^7 \\ & + (34.5625d_0 - 39.921875d_1 + 4.59375d_2 + 0.765625d_3 + 0.765625d_5 \\ & - 0.765625d_6 - 56d_7 + 56d_8)x^6 \\ & + (266.65625d_1 - 292.25d_0 + 30.28125d_2 - 3.90625d_3 - 0.78125d_5 \\ & - 42d_6 + 42d_7)x^5 \\ & + (783.125d_0 - 601.5625d_1 - 163.75d_2 - 21.875d_3 + 4.0625d_4 - 30d_5 \\ & + 30d_6)x^4 \\ & + (681.625d_1 - 1071d_0 + 286.875d_2 + 91.25d_3 - 8.75d_4 + 20d_5)x^3 \\ & + (812d_0 - 413d_1 - 234d_2 - 115d_3 - 50d_4 + 1)x^2 \\ & + (124.75d_1 - 326d_0 + 88.75d_2 + 58.75d_3 + 53.75d_4)x + 54.25d_0 \\ & - 14d_1 - 12d_2 - 10d_3 - 20d_4 - 2 = \sum_{j=0}^{8} c_{dj} \; D_j^8(x). \end{aligned} \tag{5.9}$$

By using the initial conditions, $y(0) = 0$ and $y'(0) = 0$ in (5.10), we obtained $d_0 = 0$ and $d_1 = -2.5$. After minimizing the residual error using least square by constructing the objective function, we have the following system of linear equations.

$$0.2837660867d_2 + 0.5918929949d_3 + 6.098910652d_4 - 0.01992480176d_5 + 0.0151214111d_6 + 0.03012114023d_7 + 0.1803017313d_8 + 0.05099657287 = 0.$$

$$0.5918929949d_2 + 1.253008844d_3 + 13.46949354d_4 - 0.1016324686d_5 - 0.005630067784d_6 + 0.04440415988d_7 + 0.6100678266d_8 + 0.1242976641 = 0.$$

$$6.098910652d_2 + 13.46949354d_3 + 391.1534524d_4 + 36.49843696d_5 + 37.68333036d_6 + 38.59638663d_7 - 244.5558564d_8 + 15.48098733 = 0.$$

$$36.49843696d_4 - 0.1016324686d_3 - 0.01992480176d_2 + 14.49077041d_5 + 15.51007094d_6 + 16.27804249d_7 - 81.17633174d_8 + 2.153397817 = 0.$$

$$0.0151214111d_2 - 0.005630067784d_3 + 37.68333036d_4 + 15.51007094d_5 + 16.87958671d_6 + 17.95356599d_7 - 86.02969188d_8 + 2.111445933 = 0.$$

$$0.03012114023d_2 + 0.04440415988d_3 + 38.59638663d_4 + 16.27804249d_5 + 17.95356599d_6 + 19.30969197d_7 - 89.78459528d_8 + 2.084903725 = 0.$$

$$0.1803017313d_2 + 0.6100678266d_3 - 244.5558564d_4 - 81.17633174d_5 - 86.02969188d_6 - 89.78459528d_7 + 471.2700727d_8 - 19.35755997 = 0.$$

The remaining control points are obtained through vpasolve in MATLAB. The values of control points are given in Table 9 as follows:

Table 9. Control point values of DPBC of degree 8.

Control Points	Values
d_0	0.0000000000
d_1	-2.5000000000
d_2	7.214285714
d_3	-5.785714286
d_4	0.2142857143
d_5	8.214285714
d_6	-6.785714286
d_7	3.5000000000
d_8	1.0000000000

We obtained the approximate solution of (5.8) by substituting the values of all the control points back into (5.9).

$$y = D(x) = 8.881784197 \times 10^{-16}x^7 + 8.881784197 \times 10^{-16}x^6 + 8.881784197 \times 10^{-15}x^5 + 5.107025913 \times 10^{-15}\ x^4 + 1.110223025 \times 10^{-14}x^3 + x^2 + 2.220446049 \times 10^{-16}x.$$

Comparisons of results and accuracy of the approximate solution of degree 8 for the (5.7) with the exact solution and existing method are presented in Table 10.

Table 10. Comparison between the accuracy of approximate solutions by DPBC methods of degree 8 with the exact solutions and the existing method.

x	Exact Solution	DPBC Methods	Error of DPBC Methods	Bézier [50]	Error of Bézier [50]
0.00	1.0000000000	1.0000000000	0.0000000000e+00	1.0000000000	0.0000000000e+00
0.10	0.0100000000	0.0100000000	0.0000000000e+00	0.0100000000	1.7347234760e-18
0.20	0.0400000000	0.0400000000	0.0000000000e+00	0.0400000000	1.3877787808e-17
0.30	0.0900000000	0.0900000000	0.0000000000e+00	0.0900000000	2.7755575616e-17
0.40	0.1600000000	0.1600000000	0.0000000000e+00	0.1600000000	2.7755575616e-17
0.50	0.2500000000	0.2500000000	0.0000000000e+00	0.2500000000	0.0000000000e+00
0.60	0.3600000000	0.3600000000	0.0000000000e+00	0.3600000000	5.5511151231e-17
0.70	0.4900000000	0.4900000000	0.0000000000e+00	0.4900000000	0.0000000000e+00
0.80	0.6400000000	0.6400000000	0.0000000000e+00	0.6400000000	1.1102230246e-16
0.90	0.8100000000	0.8100000000	0.0000000000e+00	0.8100000000	0.0000000000e+00
1.0	1.0000000000	1.0000000000	0.0000000000e+00	1.0000000000	0.0000000000e+00

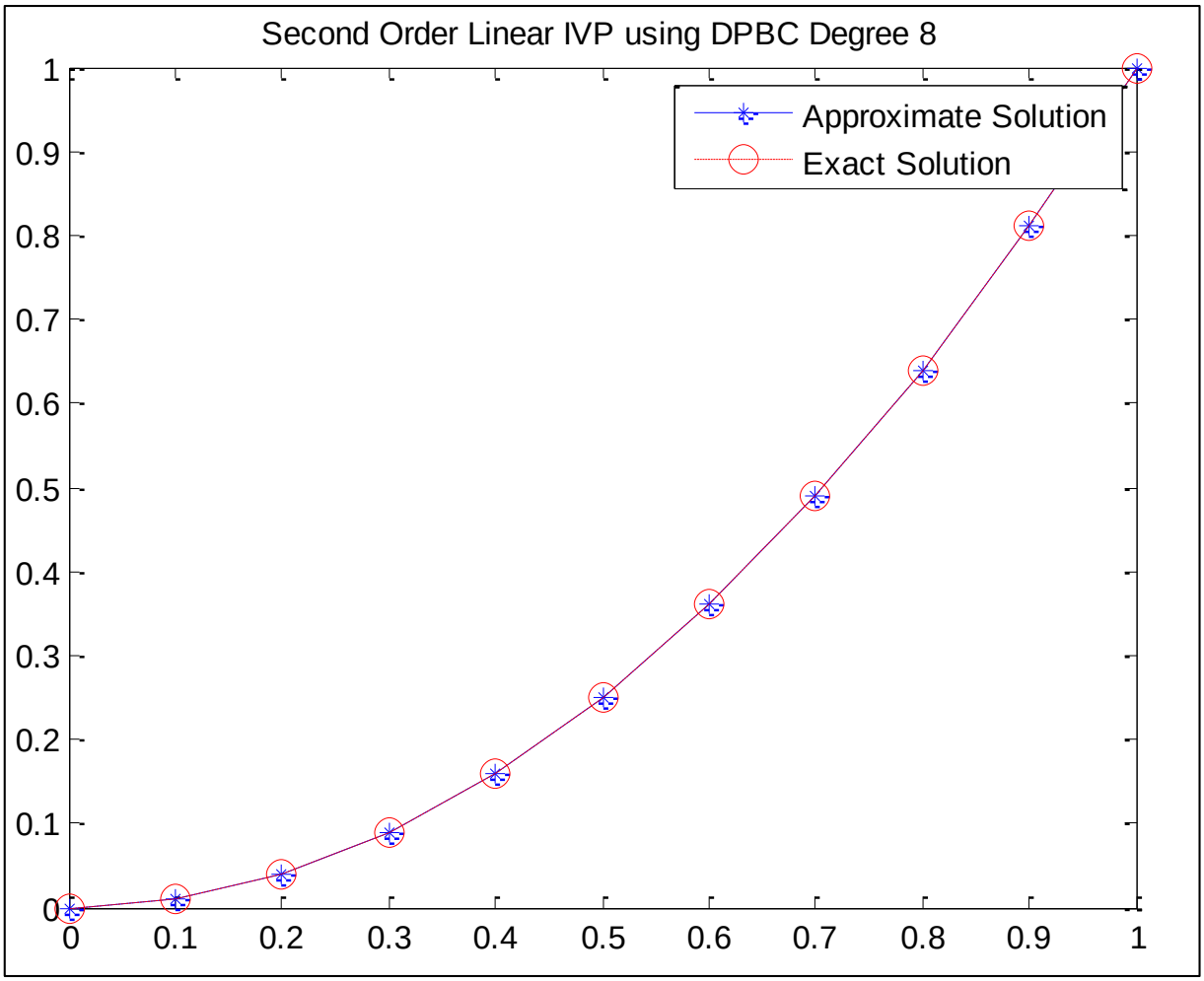

Figure 5. Approximate solution by DPBC degree 8 and exact solution of (5.7)

Example 5.4 [53]

$$y''(x) = \frac{2}{x}y'(x) - (4x^2 + 6)\ y(x), \quad 0 \le x \le 1, \quad y(0) = 1, \quad y'(0) = 0 \tag{5.9}$$

The exact solution is given by $y = e^{x^2}$.

DPBC Degree 20

Suppose that DPBC in (2.1) of degree $m = 20$ is the approximate solution of (5.9).

$$y = D(x) = \sum_{j=0}^{20} d_j\ D_j^{20}(x), \quad 0 \le x \le 1.$$

$$
\begin{aligned}
y = D(x) = {} & (d_0 - d_1 - d_{19} + d_{20})x^{20} + (19d_1 - 20d_0 + d_2 - d_{18} + d_{19})x^{19} \\
& + (190d_0 - 171d_1 - 18d_2 - d_3 - d_{17} + d_{18})x^{18} \\
& + (969d_1 - 1140d_0 + 153d_2 + 17d_3 + d_4 - d_{16} + d_{17})x^{17} \\
& + (4845d_0 - 3876d_1 - 816d_2 - 136d_3 - 16d_4 - d_5 - d_{15} + d_{16})x^{16} \\
& + (11628d_1 - 15504d_0 + 3060d_2 + 680d_3 + 120d_4 + 15d_5 + d_6 - d_{14} \\
& + d_{15})x^{15} \\
& + (38760d_0 - 27132d_1 - 8568d_2 - 2380d_3 - 560d_4 - 105d_5 - 14d_6 - d_7 \\
& - d_{13} + d_{14})x^{14} \\
& + (50388d_1 - 77520d_0 + 18564d_2 + 6188d_3 + 1820d_4 + 455d_5 + 91d_6 \\
& + 13d_7 + d_8 - d_{12} + d_{13})x^{13} \\
& + (125970d_0 - 75582d_1 - 31824d_2 - 12376d_3 - 4368d_4 - 1365d_5 - 364d_6 \\
& - 78d_7 - 12d_8 - d_9 - d_{11} + d_{12})x^{12} \\
& + (92378d_1 - 167960d_0 + 43758d_2 + 19448d_3 + 8008d_4 + 3003d_5 \\
& + 1001d_6 + 286d_7 + 66d_8 + 11d_9 + d_{11})x^{11} \\
& + (184756d_0 - 92378d_1 - 48620d_2 - 24310d_3 - 11440d_4 - 5005d_5 \\
& - 2002d_6 - 715d_7 - 220d_8 - 55d_9 - 11d_{10})x^{10} \\
& + (75582d_1 - 167960d_0 + 43758d_2 + 24310d_3 + 12870d_4 + 6435d_5 \\
& + 3003d_6 + 1287d_7 + 495d_8 + 165d_9 + 55d_{10})x^{9} \\
& + (125970d_0 - 50388d_1 - 31824d_2 - 19448d_3 - 11440d_4 - 6435d_5 \\
& - 3432d_6 - 1716d_7 - 792d_8 - 330d_9 - 165d_{10})x^{8} \\
& + (27132d_1 - 77520d_0 + 18564d_2 + 12376d_3 + 8008d_4 + 5005d_5 + 3003d_6 \\
& + 1716d_7 + 924d_8 + 462d_9 + 330d_{10})x^{7} \\
& + (38760d_0 - 11628d_1 - 8568d_2 - 6188d_3 - 4368d_4 - 3003d_5 - 2002d_6 \\
& - 1287d_7 - 792d_8 - 462d_9 - 462d_{10})x^{6} \\
& + (3876d_1 - 15504d_0 + 3060d_2 + 2380d_3 + 1820d_4 + 1365d_5 + 1001d_6 \\
& + 715d_7 + 495d_8 + 330d_9 + 462d_{10})x^{5} \\
& + (4845d_0 - 969d_1 - 816d_2 - 680d_3 - 560d_4 - 455d_5 - 364d_6 - 286d_7 \\
& - 220d_8 - 165d_9 - 330d_{10})x^{4} \\
& + (171d_1 - 1140d_0 + 153d_2 + 136d_3 + 120d_4 + 105d_5 + 91d_6 + 78d_7 \\
& + 66d_8 + 55d_9 + 165d_{10})x^{3} \\
& + (190d_0 - 19d_1 - 18d_2 - 17d_3 - 16d_4 - 15d_5 - 14d_6 - 13d_7 - 12d_8 \\
& - 11d_9 - 55d_{10})x^{2} \\
& + (d_1 - 20d_0 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 + d_9 + 11d_{10})x + d_0.
\end{aligned} \tag{5.10}
$$

where $d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}, d_{12}, d_{13}, d_{14}, d_{15}, d_{16}, d_{17}, d_{18}, d_{19}$ and d_{20} are the control points of DPBC that need to be found. By substituting $y = D(x)$ into (5.8), the residual function is obtained. i.e.

$$
\begin{aligned}
R_D(x) & \\
& = x\frac{d^2}{dx^2}\left(\sum_{j=0}^{20} d_j \; D_j^{20}(x)\right) - 2\frac{d}{dx}\left(\sum_{j=0}^{20} d_j \; D_j^{20}(x)\right) + (4x^3 + 6x)\sum_{j=0}^{20} d_j \; D_j^{20}(x) \\
& = \sum_{j=0}^{23} c_{dj} \; D_j^{23}(x).
\end{aligned} \tag{5.11}
$$

By using the initial conditions, $y(0) = 1, y'(0) = 0$ and minimizing the above residual error in (5.11) through least square by constructing the objective function, thus we have an (18×18) system of linear equations. We obtained the control points through vpasolve in MATLAB. The values of control points are given in the following table:

Table 11. Control point values of DPBC of degree 20.

Control Points	Values
d_0	1.0000000000
d_1	-2967.170834
d_2	26579.04409
d_3	-104834.1226
d_4	238133.7402
d_5	-341083.0203
d_6	315891.3465
d_7	-185059.8203
d_8	62849.8977
d_9	-9503.748145
d_{10}	1.259431854
d_{11}	11110.00185
d_{12}	-70834.80924
d_{13}	202847.8061
d_{14}	-338556.9436
d_{15}	358843.1229
d_{16}	-246639.9081
d_{17}	107141.229
d_{18}	-26846.31262
d_{19}	2970.889111
d_{20}	2.718281828

We obtained the approximate solution of (5.9) by substituting the values of all the control points back into (5.10).

$$\begin{aligned} y = D(x) = {} & 0.000005231662521x^{20} - 0.00003927245062\ x^{19} + 0.000156084061x^{18} - 0.0003808379321x^{17} \\ & + 0.000689990639x^{16} - 0.0008603464957x^{15} + 0.001047141437x^{14} - 0.0006496690568x^{13} \\ & + 0.001778279332x^{12} - 0.000183303125x^{11} + 0.008400969223x^{10} - 0.0000194339242x^{9} \\ & + 0.04167097005x^{8} - 0.0000007266284628x^{7} + 0.1666667609x^{6} \\ & - 0.00000001041760387x^{5} + 0.5000000013x^{4} - 0.0000000002210208444x^{3} + x^{2} \\ & - 2.106132542e-12x + 1 \end{aligned}$$

Comparisons of results and accuracy of the approximate solution of degree 8 for the (5.9) with the exact solution and existing method are presented in Table 12.

Table 12. Comparison between the accuracy of approximate solutions by DPBC methods of degree 8 with the exact solutions and the existing method.

x	Exact Solution	DPBC Methods	Error of DPBC Methods	Error in [53]
0.00	1.0000000000	1.0000000000	0.00000e+00	0.00000e+00
0.10	1.0100501671	1.0100501671	4.72955e-14	8.3156e−013
0.20	1.0408107742	1.0408107742	1.77636e-14	9.3925e−013
0.50	1.2840254167	1.2840254167	2.22045e-16	6.9234e−013
0.70	1.6323162200	1.6323162200	2.70894e-14	4.1545e−013
0.80	1.8964808793	1.8964808793	1.64757e-13	2.8710e−013
0.90	2.2479079867	2.2479079867	5.34239e-13	1.8963e−013
1.0	2.7182818285	2.7182818285	8.88178e-16	6.5414e−013

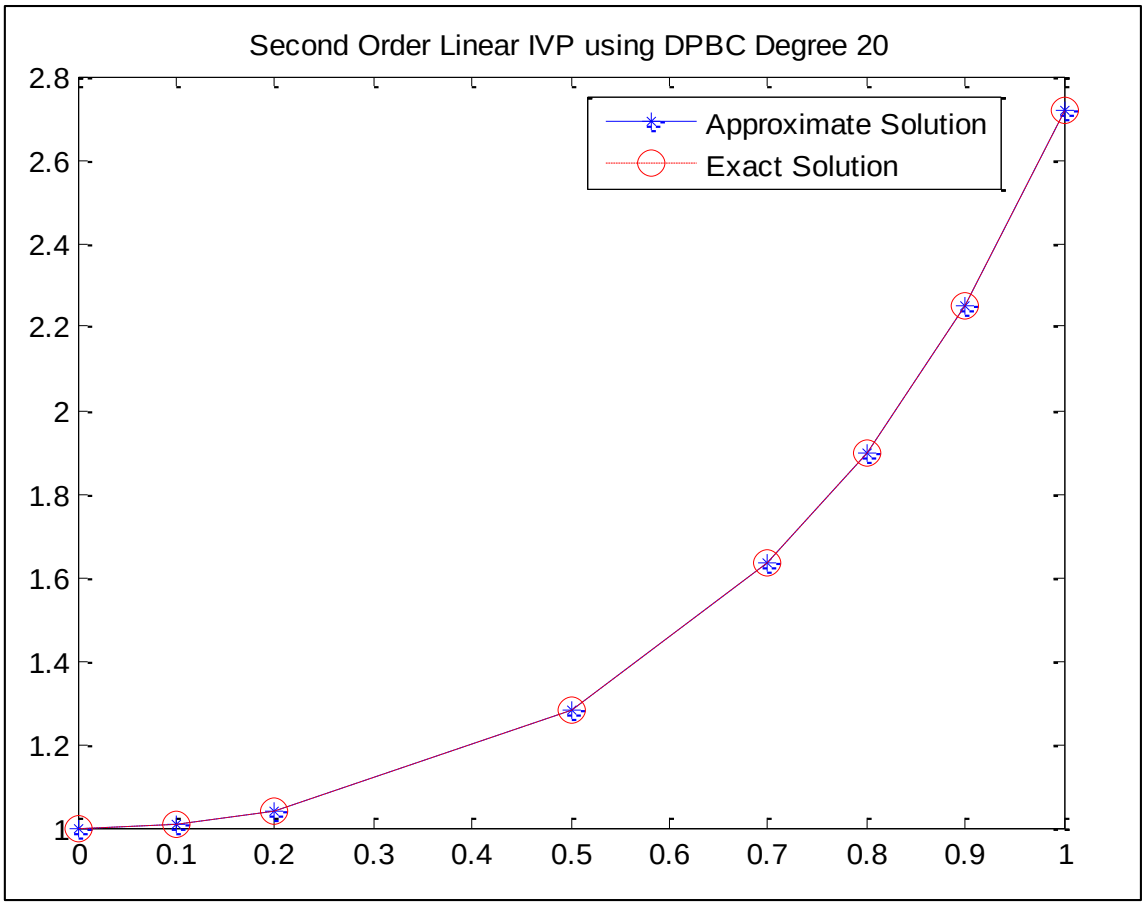

Figure 6. Approximate solution by DPBC degree 8 and exact solution of (5.7)

Example 5.5 [52]

$$y''' = 3\ sin(x), \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2 \tag{5.12}$$

with the given exact solution $y(x) = 3\,cos(x) + \frac{x^2}{2} - 2$.

DP Ball Curves Degree 14

Suppose that the DP Ball curves in (2.1) are the approximate solution of degree $m = 14$ for third order linear ODE in (5.12) as follows:

$$y = D(x) = \sum_{j=0}^{14} d_j\ D_j^{14}(x), \qquad 0 \le x \le 1.$$

$$\begin{aligned} y = D(x) = {} & (d_0 - d_1 - d_{13} + d_{14})\ x^{14} + (13d_1 - 14d_0 + d_2 - d_{12} + d_{13})\ x^{13} \\ & + (91d_0 - 78d_1 - 12d_2 - d_3 - d_{11} + d_{12})\ x^{12} \\ & + (286d_1 - 364d_0 + 66d_2 + 11d_3 + d_4 - d_{10} + d_{11})\ x^{11} \\ & + (1001d_0 - 715d_1 - 220d_2 - 55d_3 - 10d_4 - d_5 - d_9 + d_{10})\ x^{10} \\ & + (1287d_1 - 2002d_0 + 495d_2 + 165d_3 + 45d_4 + 9d_5 + d_6 - d_8 + d_9)\ x^9 \\ & + (3003d_0 - 1716d_1 - 792d_2 - 330d_3 - 120d_4 - 36d_5 - 8d_6 - 2d_7 \\ & + d_8)\ x^8 \\ & + (1716d_1 - 3432d_0 + 924d_2 + 462d_3 + 210d_4 + 84d_5 + 28d_6 + 8d_7)\ x^7 \\ & + (3003d_0 - 1287d_1 - 792d_2 - 462d_3 - 252d_4 - 126d_5 - 56d_6 - 28d_7)\ x^6 \\ & + (715d_1 - 2002d_0 + 495d_2 + 330d_3 + 210d_4 + 126d_5 + 70d_6 + 56d_7)\ x^5 \\ & + (1001d_0 - 286d_1 - 220d_2 - 165d_3 - 120d_4 - 84d_5 - 56d_6 - 70d_7)\ x^4 \\ & + (78d_1 - 364d_0 + 66d_2 + 55d_3 + 45d_4 + 36d_5 + 28d_6 + 56d_7)\ x^3 \\ & + (91d_0 - 13d_1 - 12d_2 - 11d_3 - 10d_4 - 9d_5 - 8d_6 - 28d_7)\ x^2 \\ & + (d_1 - 14d_0 + d_2 + d_3 + d_4 + d_5 + d_6 + 8d_7)\ x + d_0. \end{aligned} \tag{5.13}$$

where $d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}, d_{12}, d_{13}$ and d_{14} are the control points or coefficients of DP Ball curves that need to be found.

By substituting $y = D(x)$ into (5.11), the residual function is obtained. i.e.

$$R_D(x) = \frac{d^3}{dx^3}\left(\sum_{j=0}^{14} d_j \; D_j^{14}(x)\right) - 3\,sin(x). \tag{5.13}$$

The above residual in (5.66) can be expressed as a polynomial of degree 14 as follows:

$$R_D(x) = \sum_{j=0}^{14} \; c_{dj} \; D_j^{14}(x) \tag{5.14}$$

where c_{dj} are the linear functions in terms of $d_j,\ j = 0, 1, 2, 3, \ldots, 14$.

By using the initial conditions, $y(0) = 1,\ y'(0) = 0$ and $y''(0) = -2$ in (5.14), we obtained $d_0 = 1,\ d_1 = -62.0125$ and $d_2 = 376.3778846$ and then by constructing and minimizing the objective function, thus we have a 12×12 system of linear equations. The remaining control points are obtained by solving the system of linear equations through vpasolve in MATLAB. The values of control points are presented in Table 13.

Table 13. Control point values of DPBC of degree 14.

Control Points	Values
d_0	1.0000000000
d_1	-62.0125
d_2	376.3778846
d_3	-909.2262821
d_4	1128.705536
d_5	-717.3319639
d_6	191.2985917
d_7	0.7735916861
d_8	-217.1888339
d_9	783.1222772
d_{10}	-1191.402724
d_{11}	940.0154582
d_{12}	-380.6512085
d_{13}	63.13340692
d_{14}	0.1209069176

We obtained the approximate solution of (5.11) by substituting the values of all the control points back into (5.12).

$$\begin{aligned} y = D(x) = &- \; 3.00489301e-11 \; x^{14} - 1.937880436e-11 \; x^{13} + 0.000000006305824304 \; x^{12} \\ &- 5.855202782e-11 \; x^{11} - 0.000000826666001 \; x^{10} - 3.378967431e-11 \; x^{9} \\ &+ 0.00007440477667 \; x^{8} - 4.407911784e-12 \; x^{7} - 0.004166666666 \; x^{6} - 1.050396755e \\ &- 13 \; x^{5} + 0.125 \; x^{4} - 1.773971773e-16 \; x^{3} - x^{2} + 1.880790961e-37 \; x + 1. \end{aligned}$$

Numerical results and accuracy of the approximate solution of degree 14 for (5.64) are presented in Table 14 and then compared with the exact solution and existing method.

Table 14. Comparison between the accuracy of approximate solutions by DPBC methods of degree 14 with the exact solutions and the existing method.

x	Exact Solution	DPBC Methods	Error of DPBC Methods	Bernstein [52]	Error of Bernstein [52]
0.1	0.990012495834077	0.990012495834077	2.22e-16	0.990012495834077	1.05e-16
0.2	0.960199733523725	0.960199733523725	2.22e-16	0.960199733523724	5.02e-16
0.3	0.911009467376818	0.911009467376818	0.00e+00	0.911009467376816	1.20e-15
0.4	0.843182982008655	0.843182982008655	1.11e-16	0.938448064449894	2.18e-15
0.5	0.757747685671118	0.757747685671118	1.11e-16	0.843182982008653	3.43e-15
0.6	0.656006844729035	0.656006844729035	3.33e-16	0.656006844729029	4.89e-15
0.7	0.539526561853465	0.539526561853465	0.00e+00	0.539526561853458	6.42e-15
0.8	0.410120128041497	0.410120128041496	4.44e-16	0.410120128041488	7.78e-15
0.9	0.269829904811993	0.269829904811993	2.22e-16	0.269829904811984	8.63e-15
1.0	0.120906917604419	0.120906917604419	2.22e-16	0.120906917604410	8.67e-15

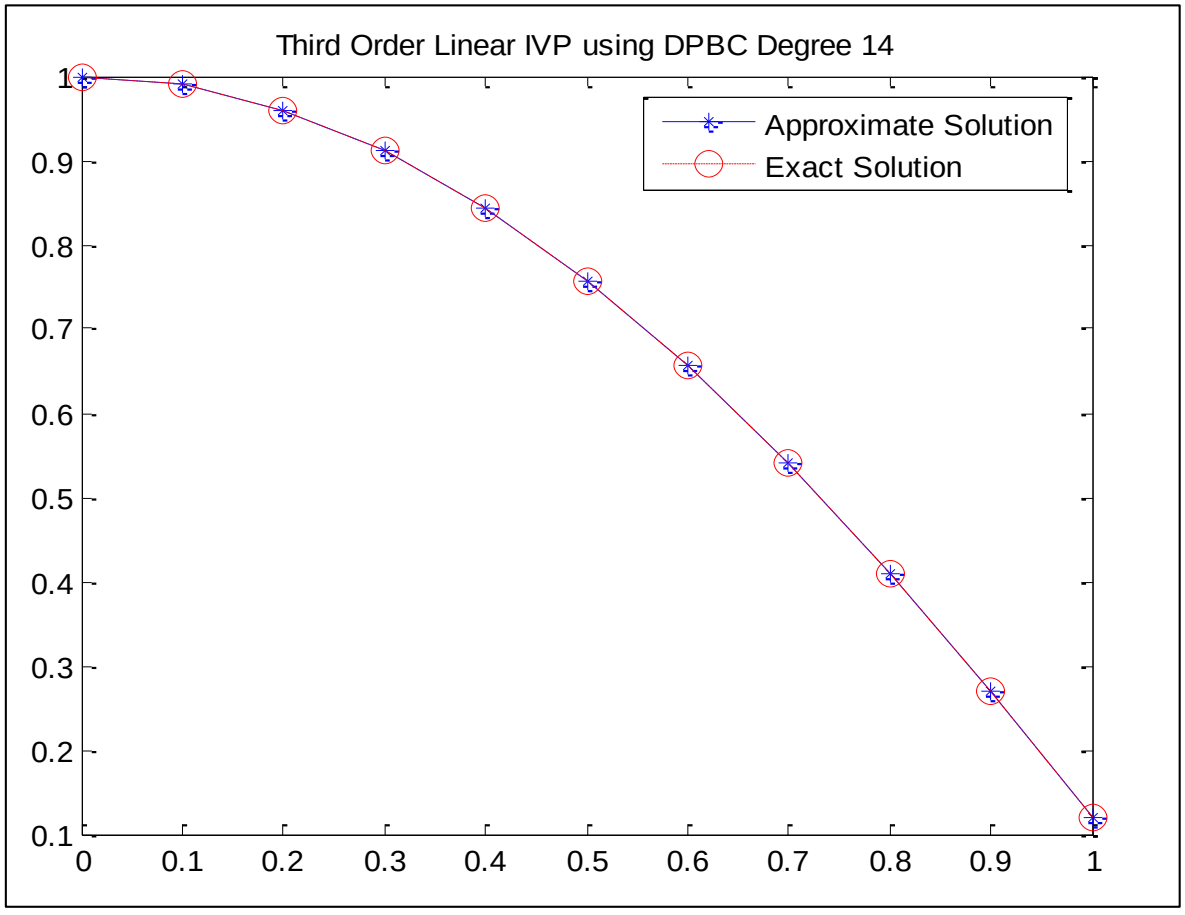

Figure 7. Approximate solution by DPBC degree 14 and exact solution of (5.11)

6. Conclusions

The DP Ball curves are represented to approximate the solution of first, second and third order IVP of ODEs through investigation of the control points of DPBC by reducing the residual function with LSM. The best values of control points are computed through the minimization of the residual function by decreasing the residual error. The DPBC is represented as the approximate solution of ODEs. By comparing the Bézier curve's method with Bernstein's basis, the results of the numerical examples show that the representations of DPBC performed more satisfactory approximate solutions in terms of error accuracy. In addition, we solved the linear ODE with a polynomial coefficient based on the control points of DPBC using LSM. The convergence analysis is successfully conducted for the

proposed method of DPBC to the IVPs. The computation in the proposed method is very simple and the idea of DPBC is quite general. The DPBC method can be applied to solve the different types of higher-order differential equations approximately.

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