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Historical Review of Existing Sequences and the Representation of the Wing Sequence

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Abstract A sequence is simply an ordered list of numbers. Sequences exist in mathematics very often. The Fibonacci, Lucas, Perrin, Catalan, and Motzkin sequences are a few that have drawn academics' attention over the years. These sequences have arisen from different perspectives. By investigating the construction of each sequence, these sequences can be classified into three groups, i.e., those that arise from nature, are constructed from other existing sequences, or are generated from geometric representation. This outcome may assist the researchers in adding a new number sequence to the family of sequences. Our observation of the geometric representation of the Motzkin sequence shows that a new sequence can be constructed, namely the Wing sequence. Therefore, we demonstrate the iterations of the Wing sequence for $3 \leq n \leq 5$. The wings are constructed by classifying them into $(n-1)$ classes and determining the first and second points. It will then provide $(n-2)$ wings in each class. This technique will construct $(n-1)(n-2)$ wings for each n. The iterations may provide a basic technique for researchers to construct a sequence using the technique of geometric representation. The observation of geometric representations can develop people's thinking skills and increase their visual abilities. Hence, the study of geometric representation may lead to new lines of research that go beyond only sequences.

Keywords Motzkin, Wing Sequence, Geometric Representation

1. Introduction

Number sequences arise in mathematics very often and in a perfectly natural manner. A sequence is simply an ordered list of numbers. The sequence is finite when it comes to a stop at some point; otherwise, it is infinite [1].

Some examples of infinite sequences are the sequence of squares: 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, …, the sequence of powers of 2: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, …, the sequence of primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, …, and the sequence of factorial numbers where the nth number is the product $1 \times 2 \times 3 \times \cdots \times (n-1) \times n$: $1, 2, 6, 24, 120, 720, 5040, 40320, \ldots$. Further examples of sequences may be found in [2] and [3].

The nth term of a sequence may be denoted by symbols such as a_n, b_n or c_n with "n" shown as a subscript. Each number in the sequence is called a term (or an element). For example, in the sequence of squares provided above, the first term is 0, the second term is 1, the third term is 4, the fourth term is 9, and the list goes on. However, there are sequences that start with subscript 0. For instance, the Fibonacci sequence starts the first term with subscript 0, i.e., $F_0, F_1, F_2, F_3, \cdots, F_n$ [4].

A sequence may also be denoted by any alphabet or symbol depending on its elements [3]. For example, a sequence of natural numbers can be denoted by the symbol '$\mathbb{N}$'. The brackets '⟨' and '⟩' or '(' and ')' are sometimes used to enclose the sequence [3]. For example, we may write a sequence of odd natural numbers as $\mathbb{N} = (1, 3, 5, 7, 9, 11, 13, \ldots)$.

When the terms of a sequence are defined using the preceding terms, then the sequence is said to be defined recursively, or it is called a recursive sequence. Otherwise, it is a non-recursive sequence [3]. For example, consider a sequence of powers of 2 written as $= 1, 2, 4, 8, 16, \ldots$. We can see that each number in the sequence is twice the preceding one. Therefore, the nth term of the powers-of-2 sequence can be written recursively as $P_n = 2P_{n-1}$ for all $n > 1$. Further details on sequences can be found in [5].

Since number sequences often arise as a result of some arithmetical procedures, many examples can be given since the sequences grow with tremendous rapidity. Hence, problems that have to do with counting frequently produce incredibly intriguing sequences [3]. For instance, the Fibonacci sequence [4], [6], Catalan sequence [7], Lucas sequence [4], Perrin sequence [8], Padovan sequence [8], Motzkin sequence [9], Narayana’s cow sequence [10], [11], Horadam sequence [12], Jacobsthal sequence [13], Narayana sequence [14], [15], Riordan sequence [16], Delannoy sequence [17], [18] and Schroeder sequence [19], [20]. We discussed these sequences in the following section.

2. Historical Review of Existing Sequence

Our research shows that the aforementioned sequences have their own history in terms of how mathematicians came to discover them. The Fibonacci sequence, denoted by F_n, was introduced by Leonardo of Pisa, an Italian mathematician, known as Fibonacci. Fibonacci sequence has been arisen by the rabbit problem as shown in Figure 1 and was first discussed in the Latin manuscript, Liber Abaci, in 1202. The manuscript has been translated into a modern language and published as Fibonacci's Liber Abaci [6]. It is assumed that the rabbit will continue to live and breed each month. Each pair of rabbits will reproduce another pair of rabbits, assuming a female and a male every month. This situation yields a sequence known as the Fibonacci sequence. From the rabbit problem above, the Fibonacci sequence is formed by adding the previous two terms in the sequence using the formula

$$F_n = F_{n-1} + F_{n-2} \quad (1)$$

for $n \geq 2$ with the initial values of $F_0 = 0$ and $F_1 = 1$. The first 10 terms of the Fibonacci sequence are $F_n = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34$ [6].

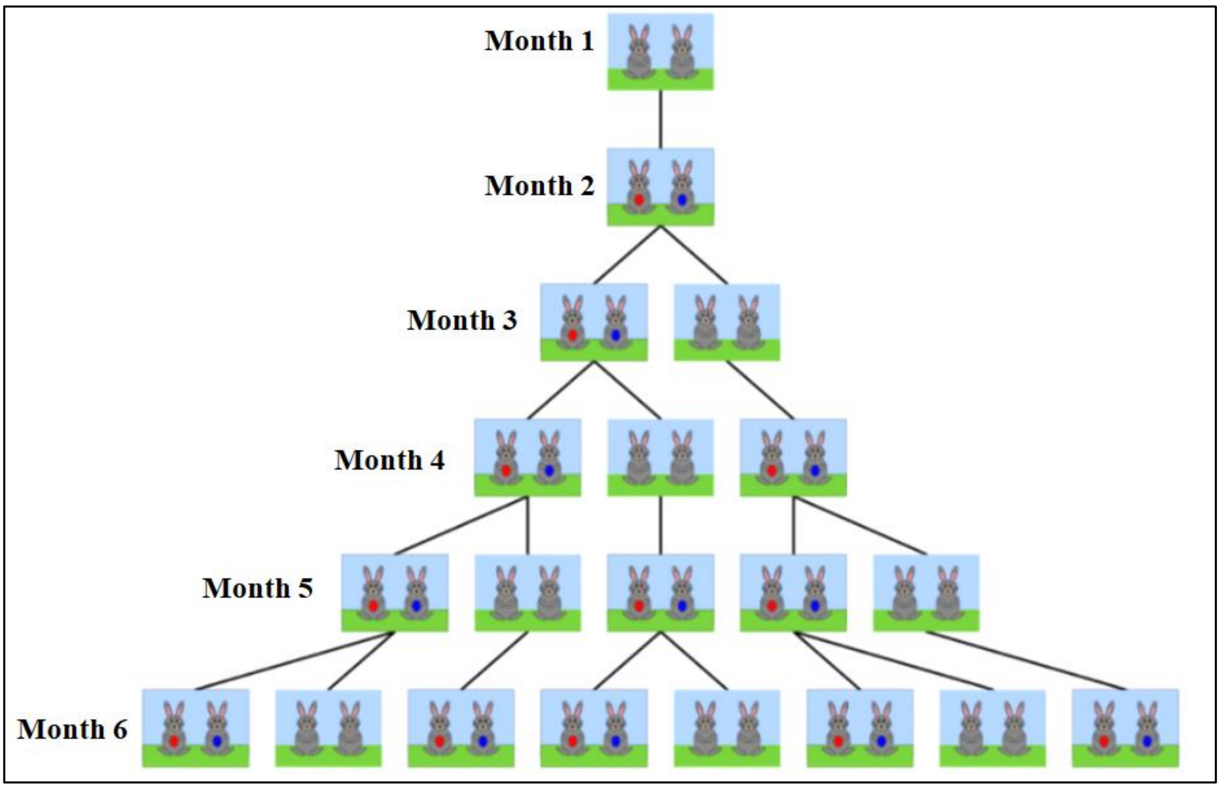


Figure 1. Fibonacci’s rabbit problem

The Lucas sequence, denoted by L_n, was introduced by the French mathematician, Edouard Lucas (1842-1891). According to [4], Edouard Lucas was the one who found that Leonardo of Pisa was the discoverer of the Fibonacci sequence. The Lucas sequence was then introduced, using the same formula as the Fibonacci sequence but with different initial conditions. The Lucas sequence is also formed by adding the previous two terms in the sequence using the formula

$$L_n = L_{n-1} + L_{n-2} \quad (2)$$

for $n \geq 2$ with initial conditions $L_0 = 2$ and $L_1 = 1$. The first 10 terms of the Lucas sequence are $L_n = 2, 1, 3, 4, 7, 11, 18, 29, 47, 76$ [4].

The Perrin sequence, denoted by P_n, was initially discovered in 1876 by Edouard Lucas (1842-1891). The Perrin sequence was named after a mathematician, R. Perrin, who continued the study about the sequence in 1899 [8]. As with the Fibonacci and Lucas sequences, the Perrin sequence can also be formed by adding two previous terms to the sequence. The formula is given by

$$P_n = P_{n-2} + P_{n-3} \quad (3)$$

for $n \geq 3$ with initial conditions $P_0 = 3,\ P_1 = 0,\ P_2 = 2$. The first 10 terms of the Perrin sequence can be written as $P_n = 3, 0, 2, 3, 2, 5, 5, 7, 10, 12$ [8].

The Padovan sequence, also denoted by P_n, was discovered in 1994 and named after a mathematician, Richard Padovan. The Padovan sequence is attributed to Richard Padovan while mentioning the works of an architect, Hans van der Laan, which finally contributed to the plastic number [8]. The Padovan sequence can be formed using the formula

$$P_n = P_{n-2} + P_{n-3} \quad (4)$$

when $n \geq 3$ and $P_0 = P_1 = P_2 = 1$. The first 10 terms of the sequence can be written as $P_n = 1, 1, 1, 2, 2, 3, 4, 5, 7, 9$ [21].

The Catalan sequence, denoted by C_n, was named after a French mathematician, Eugene Charles Catalan (1814-1894). It was originally identified by Leonhard Euler in 1751 when the Catalan sequence described the number of triangulations of $(n+2)$-gons. For example, when $n = 3$, there are 5 ways of triangulating the 5-gon as shown in Figure 2. Take note that the first term of the Catalan sequence is denoted by C_0.

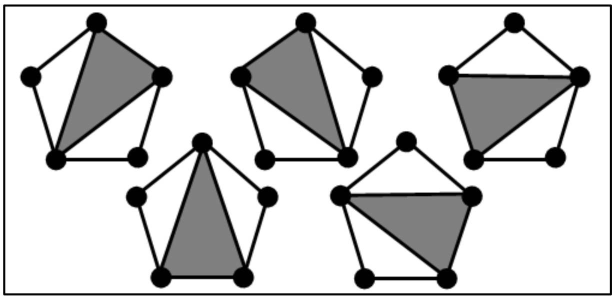


Figure 2. The 4th term of Catalan sequence, $C_3 = 5$.

The standard formula of the Catalan sequence was obtained by Eugene Charles Catalan in 1838 and is written as

$$C_n = \frac{(2n)!}{(n+1)!\,n!} \quad (5)$$

where $n \geq 0$. The first 10 terms of the Catalan sequence are $C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 486$ [22], [23].

The Motzkin sequence, denoted by M_n, was discovered by Theodore Motzkin in 1948. The Motzkin sequence can be formed using the formula

$$M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i} = \frac{2n+1}{n+2} M_{n-1} + \frac{3n-3}{n+2} M_{n-2}. \quad (6)$$

The Motzkin sequence represents the number of ways of connecting a subset of n points on a circle by non-intersecting chords [9]. For example, when $n = 4$, there are 9 ways of connecting the subsets as presented in Figure 3 below. The first 10 terms of the Motzkin sequence can be written as $M_n = 1, 1, 2, 4, 9, 21, 51, 127, 323, 835$.

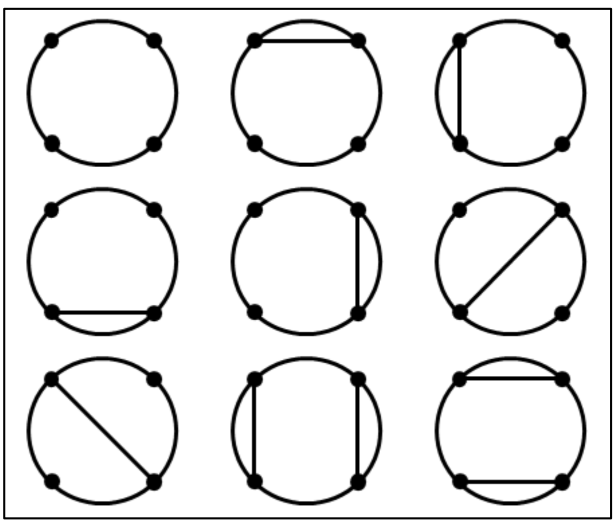


Figure 3. 9 ways of connecting a subset of 4 points on a circle by non-intersecting chords ($M_4 = 9$)

During the period from 1325 to 1400, a mathematician named Narayana Panditha from India produced Narayana's cow sequence that arouse from the cow problem. His idea was motivated by the Fibonacci sequence. The Narayana cow's sequence described the number of cows presents each year as equal to the sum of the cows from the year before and the three years prior. The sequence can be written as

$$G_n = G_{n-1} + G_{n-3} \tag{7}$$

for $n \geq 3$ with initial conditions $G_0 = 0$, $G_1 = 1$, $G_2 = 1$ and $G_3 = 1$. The first 10 terms of Narayana cow's sequence are $G_n = 1, 1, 1, 2, 3, 4, 6, 9, 13, 19$ [10].

Later in 1961, A. F. Horadam modified the first two terms of the Fibonacci sequence to obtain the Horadam sequence, which generalized the Fibonacci sequence. The Horadam sequence can be written as

$$H_n = rH_{n-2} + sH_{n-1} \tag{8}$$

for $n \geq 2$ with the initial conditions $H_0 = 0$, $H_1 = 1$, $r = 1$ and $s = 1$. The first 10 terms of the Horadam sequence are $H_n = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34$ [12], [24].

A. F. Horadam also modified the Lucas sequence and produced another new sequence, that is, the Jacobsthal sequence. The Jacobsthal sequence was named after the German mathematician, Ernst Jacobsthal. The Jacobsthal sequence is the numbers that can be found in the Lucas sequence using a similar formula to the Fibonacci sequence. The Lucas sequence can be written as

$$J_n = J_{n-1} + 2J_{n-2} \tag{9}$$

for $n \geq 0$ with initial conditions $J_0 = 0$ and $J_1 = 1$. The first 10 terms of the Jacobsthal sequence are $J_n = 0, 1, 1, 3, 5, 11, 21, 43, 85, 171$ [12].

Furthermore, a Canadian mathematician, T. V. Narayana, produced the Narayana sequence motivated by the Catalan sequence. The Narayana sequence describes the numbers in a triangular array, where the sum of each number in n^{th} row gives the n^{th} term of the Catalan sequence. The Narayana sequence can be written as

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \tag{10}$$

for $n \in \mathbb{N}^+$ and $1 \leq k \leq n$. The first 10 terms of the Narayana sequence are $N(n,1) = 1, 1, 1, 1, 3, 1, 1, 6, 6, 1$ [14], [15].

Moreover, the Riordan sequence has been produced and motivated by the Motzkin sequence. Each number in the Riordan sequence describes the number of Motzkin paths without horizontal steps of height zero. The Riordan sequence can be written as

$$R_n = \frac{(n-1)(2R_{n-1} + 3R_{n-2})}{n+1} \tag{11}$$

for $n \geq 2$ with initial conditions $R_0 = 1$ and $R_1 = 0$. The first 10 terms of the Riordan sequence are $R_n = 1, 0, 1, 1, 3, 6, 15, 36, 91, 232$ [25].

Furthermore, the Delannoy sequence has been produced and motivated by the lattice path enumeration. Each number in the Delannoy sequence described the number of paths on a rectangular grid from $(0,0)$ to (m,n) with jumps $(0,1)$, $(1,1)$ or $(1,0)$. The Delannoy sequence can be written as

$$D(m,n) = \sum_{j=0}^{n} \binom{m+n-j}{n}\binom{n}{j} \tag{12}$$

for $m \geq 0$ and $n \geq 0$. The first 7 terms of the Delannoy sequence are $D(3,n) = 1, 7, 25, 63, 129, 231, 377$ [17].

Besides, the Schroeder sequence has been produced and motivated by the Delannoy sequence. It consists of the number of lattice paths on a square grid from $(0,0)$ to (n,n) with jumps $(0,1)$, $(1,1)$ and $(1,0)$. The Schroeder sequence can be written as

$$S_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\binom{n+k}{k} \tag{13}$$

for $n \geq 0$. The first 10 terms of the Schroeder sequence are $S_n = 1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098$ [19], [26].

There are many advantages when investigating number sequences. For example, the Fibonacci and Lucas sequences can utilize the symmetrical keys to create the encryption or decryption methods in the application of symmetric cryptosystems [27]. In addition, the Catalan sequence could improve the existing data hiding technique [28], [29], encoding of data [30] as well as steganography image [31]. Generally, the decomposition of number sequences can be implemented in a programming language such as Java [32].

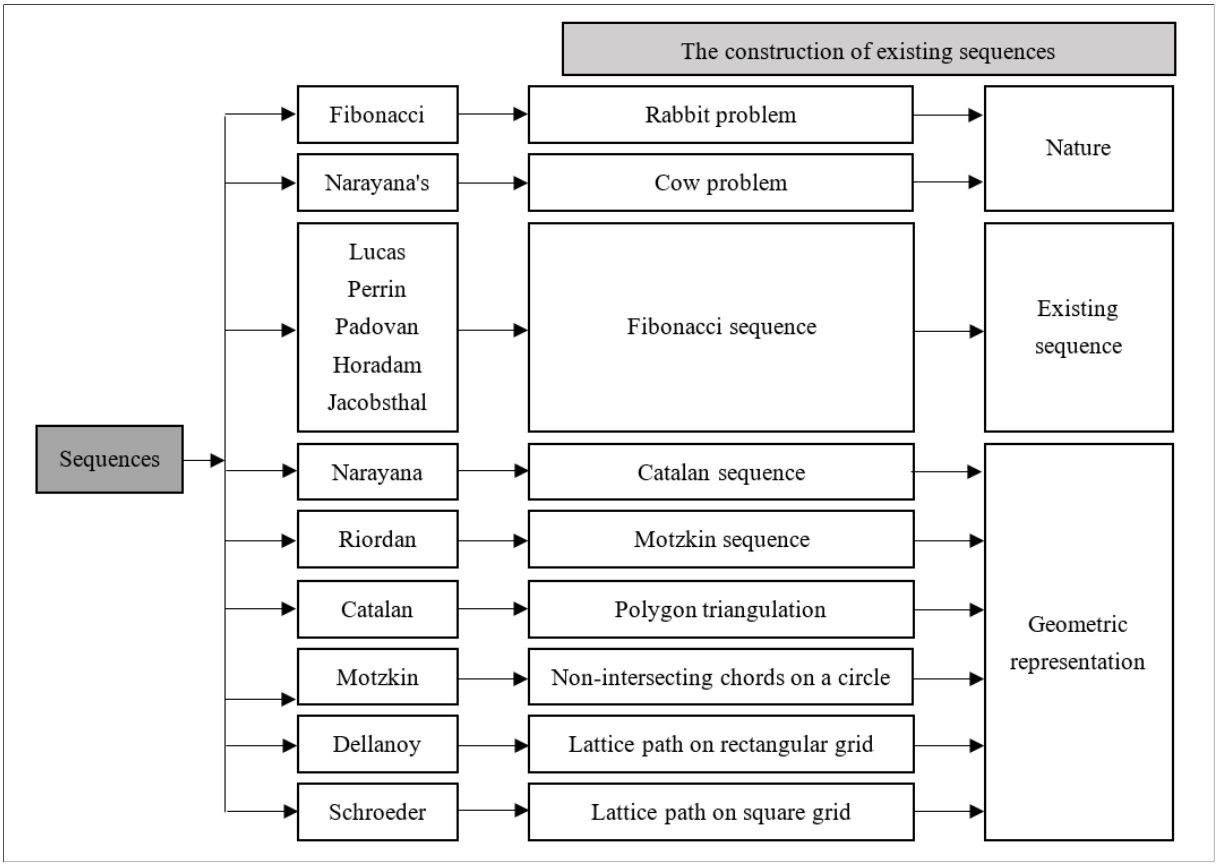


Figure 4. The construction of existing sequences

Indeed, these sequences offer a lot of benefits, especially to computer scientists. For example, when they have to run an infinite formula into the system, the program will be fast and smooth.

Based on our exploration of the Fibonacci, Lucas, Perrin, Padovan, Catalan, Motzkin, Narayana cow's, Horadam, Jacobsthal, Narayana, Riordan, Delannoy, and Schroeder sequences, these sequences have been produced based on the nature and existing sequences through geometric enumerations.

For example, the Fibonacci and Narayana cow's sequences have been discovered from nature; and Lucas, Perrin, Padovan, Horadam and Jacobsthal have been constructed from the existing sequences. Furthermore, Catalan sequence has been obtained from the geometric representation of polygon triangulation [7], Motzkin sequence from the geometric representation of non-intersecting chords on a circle [9], and Dellanoy and Schroeder sequences from the geometric representation of lattice paths on a grid [19], [17], [20]. We summarized our reviews regarding the construction of the existing sequences in Figure 4.

Geometric representation has various advantages for researchers, especially in the family of sequences. For instance, the geometric representation obviously can generate a new sequence, such as Catalan, Motzkin, Narayana, Riordan, Dellanoy and Schroeder. The construction of such sequences shows that geometric representation can develop people's thinking skills and encourage their visual abilities [33], [34], [35]. Geometric representation also plays a significant role in solving mathematical problems for visualization purposes [36], [37], [38]. It may reduce the work while exploring huge data [39], [40], [41], [42], [43], and doing data comparison and interpretation in a regression model [44]. Furthermore, it could solve fingerprint classification, which is mainly used in criminal investigation [45]. In addition, geometric representation can solve various problems in our daily lives such as file hierarchy on a computer system, browsing history and document management [46].

Based on our observation, obviously these sequences have been constructed mainly on the basis of a pattern. For example, the pattern of rabbit production in a year had generated the Fibonacci sequence, and the idea had motivated other researchers to develop the Lucas, Perrin, Padovan, Horadam and Jacobsthal sequences. Furthermore, the pattern of cow production had produced the Narayana cow's sequence, the pattern of polygon triangulation had generated the Catalan sequence, the pattern of non-intersecting chords on a circle had developed the Motzkin sequence, the pattern of lattice path on rectangular grids had yielded the Delannoy sequence, and the pattern of lattice path on square grids had constructed the Schroeder sequence. Moreover, the constructions of the

Catalan and Motzkin sequences had developed the Narayana and Riordan sequences respectively.

The aforementioned situations lead to the emergence of number sequences from nature, geometric enumeration, or existing sequences. In conclusion, the geometric representation of non-intersecting chords on a circle that had yielded the Motzkin sequence motivated us to develop a new number sequence. We discuss and present the sequence in the following paragraph.

3. The Wing Sequence

The idea for constructing the Wing sequence has been inspired by the geometric representation of the Motzkin sequence. The Motzkin sequence considered the non-intersecting chords on a circle. Whereas, the construction of the Wing sequence has considered the unique directed paths on n horizontal points which are inspired by the Half Butterfly Method (HBM). The HBM has been introduced to decompose a complete graph K_n into Hamiltonian circuits [47], [48]. The initial concept of HBM was motivated by [49] in decomposing a bipartite graph into smaller graphs using the shift-rotate and Butterfly strategy. The benefit of applying the HBM to decompose a complete graph is that the Hamiltonian circuits can be represented in a graphical visualization [50]. The current application of HBM is listing $n!$ permutations [51].

The construction of the Wing sequence has several advantages. In terms of education, as highlighted by [33], the construction of the Wing sequence may enhance people's thinking skills in developing their visual abilities. Indeed, the Wing sequence is unique since it is not only a sequence but can also be presented in geometric representation.

The Wing sequence might be seen as a normal sequence; however, it contributes to developing a novel geometrical representation since not all existing sequences can be represented geometrically. As a result, it offers a new perspective on the Wing sequence.

The construction of the Wing sequence can open up new research by observing the geometric representation instead of the normal sequence representation. New theorems might also be developed, and the geometric representation can play a significant role in solving the theorems. Therefore, we proposed a new sequence for the family of sequences, namely the Wing sequence. The construction of the Wing sequence is discussed below.

3.1. The Construction of the Wing Sequence

Scientists and mathematicians have to make guesses based on their observations of the number pattern. A "guess" of this sort is called a *hypothesis* (or theory), where it is put forward to explain some observed phenomenon [3]. Therefore, we present our hypothesis on the Wing sequence.

The proposed Wing sequence, denoted by W_n, is inspired by the Half Butterfly Method (HBM) that was introduced in 2015 [47]. The general HBM is presented in Figure 5.

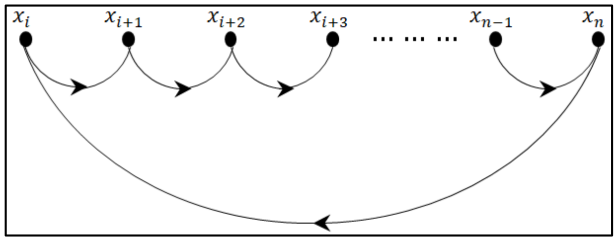


Figure 5. The general representation of HBM.

By investigating a few enumerations on the HBM, we were able to construct the Wing sequence in response to the observed pattern. We were inspired to ask this question after observing the aforementioned HBM: "How many different directed paths can be formed among n points?" We believe that there is a way to modify the HBM to produce a sequence. Then, we thought back to the Motzkin sequence. Since the Motzkin sequence has considered the non-intersecting chords on a circle, at this point, another question arose: "How many different directed paths can be formed among n horizontal points such that each point is passed by at least once?" By referring to Figure 5, the HBM could be seen as the movement of arrows from one point to another. The HBM also starts and ends at the same point since it has been proposed to decompose a complete graph K_n into circuits.

From our observation of the HBM, we have tried the iterations to construct W_1, W_2, W_3, W_4 and W_5, but we modified the directed path to start at point 1 and end at point n. This modification has been implemented after some investigation of the existing sequences. For example, as discussed earlier, Riordan considered the paths, Motzkin considered the intersecting chords, and Delannoy and Schroeder considered the lattice paths. Whereas, the Wing sequence describes the number of unique directed paths that can be constructed among n horizontal points. Related definitions are presented below.

Definition 1. For any $n \geq 1$, STP is defined as the starting point such that $STP = 1$.

Definition 2. For any $n \geq 2$, SCP is defined as the second point such that $2 \leq SCP \leq n$.

Definition 3. For any $n \geq 3$, MDP is defined as the middle point such that $2 \leq MDP \leq n$.

Definition 4. For any $n \geq 4$, ENP is defined as the end point such that $n-2 \leq ENP \leq n$.

Definition 5. A *Wing sequence* denoted by W_n consists of elements (or terms) where each term is the number of unique ways of drawing directed paths among n horizontal points such that each point is necessarily passed by at least once and that satisfies the following conditions:

i) All n horizontal points are fixed to be numbered with 1 until n (from left to right).
ii) Each directed path among n horizontal points will have $n-1$ arrows.
iii) Each n will have $n-1$ *class*.
iv) In each *class*, the STP and SCP will be fixed. The third point onward will be chosen from all the remaining points in ascending order, consecutively.
v) Each *class* will have $n-2$ directed paths.
vi) Each n will have $(n-1)(n-2)$ total directed paths from all *class*.

Definition 6. $Class\ i[SSP(1,i+1)]$ for $1 \le i \le n-1$, where SSP represents the "Starting and Second Point" describes the set of STP and SCP of the directed paths in each class.

We present the construction of the first five terms of the Wing sequence in Table 1.

Table 1. The construction of the first five terms of the Wing sequence

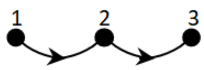
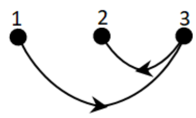
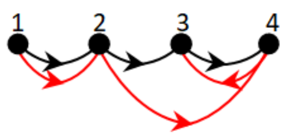
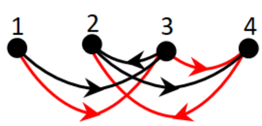
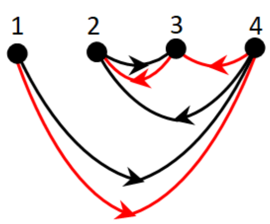
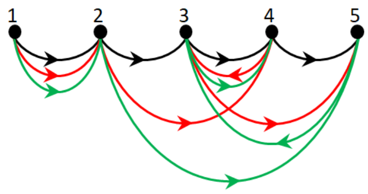
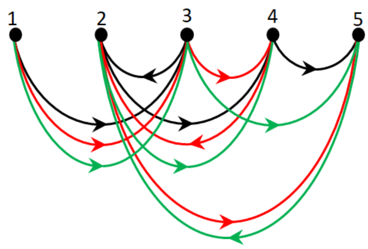
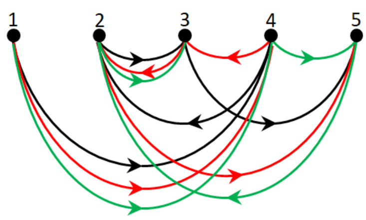
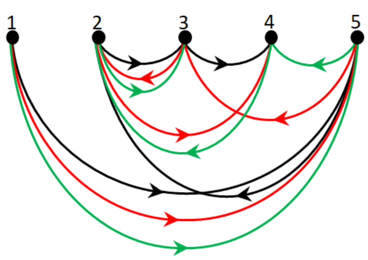
W_n	The construction of W_n	Explanation
The first term of Wing sequence, $W_1 = 0$.	Trivial solution.	For W_1, there is only one point to be considered. This is a trivial solution since it is obvious that no directed path can be constructed with only one point.
The second term of Wing sequence, $W_2 = 1$.		• Trivial solution, $Class\ 1[SSP(1,2)]$ contains one directed path. • The directed path starts at STP 1, then moves to SCP 2. • The movement of the path is: $(1,2)$.
The third term of Wing sequence, $W_3 = 2$.		$Class\ 1[SSP(1,2)]$ contains one directed path. • The path starts at STP 1, then passed by at SCP 2, and ends at ENP 3. • The movement of the path is: $(1,2,3)$.
		$Class\ 2[SSP(1,3)]$ contains one directed path. • The path starts at STP 1, then passed by at SCP 3, and ends at ENP 2. • The movement of the path is: $(1,3,2)$.
The fourth term of Wing sequence, $W_4 = 6$.		$Class\ 1[SSP(1,2)]$ contains two directed paths. • The first path, identified in black colour, starts at STP 1, then passed by at SCP 2, then MDP 3, and ends at ENP 4. • The second path, identified in red colour, starts at STP 1, then passed by at SCP 2, then MDP 4 and ends at ENP 3. • The movements of the path are: $(1,2,3,4)$ and $(1,2,4,3)$.
		$Class\ 2[SSP(1,3)]$ contains two directed paths. • The first path, identified in black colour, starts at STP 1, then passed by at SCP 3, then MDP 2, and ends at ENP 4. • The second path, identified in red colour, starts at STP 1, then passed by at SCP 3, then MDP 4 and ends at ENP 2. • The movements of the path are: $(1,3,2,4)$ and $(1,3,4,2)$.
		$Class\ 3[SSP(1,4)]$ contains two directed paths. • The first path, identified in black colour, starts at STP 1, then passed by at SCP 4, then MDP 2, and ends at ENP 3. • The second path, identified in red colour, starts at STP 1, then passed by at SCP 4, then MDP 3 and ends at ENP 2. • The movements of the path are: $(1,4,2,3)$ and $(1,4,3,2)$.

Table 1 continued.

The fifth term of Wing sequence, $W_5 = 12$.		$Class\ 1[SSP(1,2)]$ contains three directed paths. • The first path, identified in black colour, starts at STP 1, then passed by at SCP 2, then MDP 3, MDP 4, and ends at ENP 5. • The second path, identified in red colour, starts at STP 1, then passed by at SCP 2, then MDP 4, MDP 3, and ends at ENP 5. • The third path, identified in green colour, starts at STP 1, then passed by at SCP 2, then MDP 5, MDP 3, and ends at ENP 4. • The movements of the path are: $(1,5,2,3,4)$, $(1,5,3,2,4)$ and $(1,5,4,2,3)$.
		$Class\ 2[SSP(1,3)]$ contains three directed paths. • The first path, identified in black colour, starts at STP 1, then passed by at SCP 3, then MDP 2, MDP 4, and ends at ENP 5. • The second path, identified in red colour, starts at STP 1, then passed by at SCP 3, then MDP 4, MDP 2, and ends at ENP 5. • The third path, identified in green colour, starts at STP 1, then passed by at SCP 3, then MDP 5, MDP 2, and ends at ENP 4. • The movements of the path are: $(1,3,2,4,5)$, $(1,3,4,2,5)$ and $(1,3,5,2,4)$.
		$Class\ 3[SSP(1,4)]$ contains three directed paths. • The first path, identified in black colour, starts at STP 1, then passed by at SCP 4, then MDP 2, MDP 3, and ends at ENP 5. • The second path, identified in red colour, starts at STP 1, then passed by at SCP 4, then MDP 3, MDP 2, and ends at ENP 5. • The third path, identified in green colour, starts at STP 1, then passed by at SCP 4, then MDP 5, MDP 2, and ends at ENP 3. • The movements of the path are: $(1,4,2,3,5)$, $(1,4,3,2,5)$ and $(1,4,5,2,3)$.
		$Class\ 4[SSP(1,5)]$ contains three directed paths. • The first path, identified in black colour, starts at STP 1, then passed by at SCP 5, then MDP 2, MDP 3, and ends at ENP 4. • The second path, identified in red colour, starts at STP 1, then passed by at SCP 5, then MDP 3, MDP 2, and ends at ENP 4. • The third path, identified in green colour, starts at STP 1, then passed by at SCP 5, then MDP 4, MDP 2, and ends at ENP 3. • The movements of the path are: $(1,5,2,3,4)$, $(1,5,3,2,4)$ and $(1,5,4,2,3)$.

4. Conclusion and Future Research

This paper provided a historical review of sequences. The constructions, techniques, and related methods are discussed such as Fibonacci, Catalan, Motzkin and HBM. Our research on sequences has found that the HBM that has been used in graph decomposition can yield a new sequence, namely the Wing sequence.

The Wing sequence has been constructed from the modification of HBM by using the idea of geometric representation from the Motzkin sequence. Hence, in this paper, we have demonstrated several iterations as an initial idea of constructing the Wing sequence as presented in Table 1.

From Table 1, the construction of W_1 and W_2 is trivial. For W_3, there are two directed paths that can be constructed among three horizontal points, i.e., 2 *class*, 1 path in each class. For W_4, there are six directed paths that can be constructed among four horizontal points, i.e., 3 *class*, 2 paths in each class. For W_5, there are twelve directed paths that can be constructed among five horizontal points, i.e., 4 *class*, paths in each class. This construction yields the following conjecture.

Conjecture 1. For any $n \geq 3$, there exists a Wing sequence W_n such that $W_n = (n-1)(n-2)$.

We believe that some further research can be conducted to prove the general formula of the Wing sequence from the presented conjecture. Hence, some properties of each term of the Wing sequence can be formulated based on the general formula.

Motivated by the modified HBM and the existing sequence (i.e., the Motzkin sequence), the Wing sequence may have numerous applications, especially in combinatorics. Moreover, the Wing sequence has been extracted from the geometric representation of the HBM and not directly produced from the existing number sequences. Hence, it may contribute to many new studies among mathematicians.

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