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The Form of σ –Algebra on Probability Hilbert Space

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Abstract Measure theory is used as the basis for probability theory. One of the most useful measure theories for statistics and probability theory is the concept of distance. The concept of distance introduced in the inner product space is closely related to the order relation in each sequence of elements. In statistics, random variables can be seen as a sequence that can be an object to study, including the partial ordering relation, expectation value, convergence, also infimum and supremum. This study aims to obtain the properties of a partial ordering relation which is useful for forming probability Hilbert spaces, more specifically the σ-algebra. If in ordinary sets, σ-algebra uses the concept of intersection and combination of sets, in probability Hilbert space, σ-algebra uses the concept of partial relation ordering, lattice, and indicator lattice. This research is quantitative research with a method of proof to generalize the concept of the order of elements. The novelty of this research is to find the associative properties of lattice in Hilbert probability space as described in Corollary 1. Furthermore, based on the definition of absolute value in Hilbert probability space, we derive the properties of addition and subtraction of absolute values and find their relationship with the lattice stated in Proposition 1. In the Hilbert probability space, the convergence property of random variables also applies which results in the lattice convergence stated in Proposition 2. Finally, it can be shown that the set of indicators in the Hilbert probability space form the algebra σ which is stated in Proposition 3. This study also gave use of the dataset shares of 42 energy companies in Indonesia in 2022. The results of plotting the data using the probability density function of the Normal distribution, Log-Normal distribution, and Cauchy distribution.

Keywords Hilbert Space, Lattice, Partial Ordered Relation, σ-Algebra

1. Introduction

Probability theory has been a prestigious branch of mathematics since the early 1930s. Previously, the discipline was viewed with skepticism by some mathematicians because it dealt with the concepts of random variables and independence that were not properly and formally defined [1]. This condition has been improved thanks to the efforts of Andrei Kolmogorov and Norbert Wiener who introduced probability theory into measure theory [2]. Although, at least some thirty years earlier Henry Lebesgue's research had led to the discovery of the Lebesgue measure, not in probability but in harmonic analysis problems. It seems odd that it took more than 30 years for the merger of probability and measure theory to occur [3]. After that, probability theory and measurement theory were so well intertwined that many mathematicians discussed these two topics. It also sounds odd that the basic concepts and theory of Lebesgue, which naturally arose out of practical questions about probability, could have been achieved without probability theory as their primary source of inspiration [4].

We will be interested in the measure theory because it is used as the basis for probability theory. Suppose that in a space $\mathcal{V}$ is defined size ϑ which is a nonnegative function defined in the collection of subsets $\mathcal{V}$ which is a measurable set. If $\mathcal{A}$ is a subset of $\mathcal{V}$ of a measurable set then the non-negative number $\vartheta(\mathcal{A})$ is called the measure of $\mathcal{A}$. Let $\mathcal{X}$ be illustrated as a variable that describes the state of the world, and the "measure" of a set is the probability of an event in this set. However, measure theory is much more general than that [5]. For example, if we think about intervals on the real number line, the natural measure is the length of the interval i.e., for $[a, b]$, the size is $b - a$. The measure resulting from this proposition is called the Lebesque measure, which is one way to make formal concepts such as integration [6][7].

In this paper, we introduce a partially ordered set to generalize the concept of the order of elements or elements [8]. In a partially ordered set, there is a relationship between pairs of elements, for example, "less than" and "divides", so that the elements can be compared. Partial ordered set theory is to identify an element from a series of alternatives. In general, it is assumed that the object is chosen as "most prefer" [9]. This intuitive idea would lead to a Hilbert-space of probability densities as σ −algebra of probability Hilbert space [10].

We adopt this latter possibility although no difficulty arises using the alternative null extension [11]. In applied statistics, statistics is defined as the science of making inferences about the population characteristics. Getting population information, of course, is done by observation. If the population is too large and it is not possible to do so, then observations are made on a representative sample. Sometimes, it is possible for scientists to generate datasets that follow a certain probability distribution. In general, the population characteristics of concern are the mean, variance, standard deviation, and proportion [12].

In its application, the expectation value is identical to the mean which is defined as the sum of the values of each sample observation divided by the number of samples. Intuitively, we can use the relative frequency method to calculate the population mean for the edge occurrences for the experiment of tossing two coins a finite number of times. This value is often called the mean of the random variable $\mathcal{X}$ or theoretically called the expected value of the random variable X. The expected value of a continuous random variable is obtained by approximation. approximate a discrete random variable using a Riemann sum. Therefore, the expected values for continuous random variables are determined by integration [13].

Wider, the concept of estimating function can be considered as an element of a Hilbert space. The expected value of random variable $\boldsymbol{x}$ is a continuous linear functional on Hilbert space [14]. The set of all finite variance random variables over a given probability space is a Hilbert space and the conditional expectation properties necessary to derive the usual formulas of the theory [15]. Research on embeddings on probability measures using the Hilbert space concept was conducted by Sriperumbudur in 2010. Previously, the expansion of expectation value in a discrete Hilbert space was deepened by [16]. In 2020, [17] introduces Hilbert's spatial averaging on the construction of typical states and on the aforementioned conditions for dynamic quantum typicality.

The expectation value approach in the Hilbert space is also carried out by [18]. More specifically, his research yields analytic properties of linear conditional expectations in infinite-dimensional Hilbert spaces. Inspired by previous research on expectation value in the Hilbert space, this study discusses the definition of expectation value in the Hilbert space with unitary elements. This study also gives examples of algebraic propositions regarding the properties that apply to the expectation value in the Hilbert space.

2. Results and Discussions

Probabilities are traditionally defined as measurements over a sample space, [14] define Hilbert space with unitary element as follows:

Definition 1. Given a Hilbert space, namely $\mathcal{H}$, and $\mathbf{1}$ be an element of $\mathcal{H}$ then the pair $(\mathcal{H}, \mathbf{1})$ is called a Hilbert space with unitary element if $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. In other words, the scalar multiples of $\mathbf{1}$ as constant elements of $\mathcal{H}$.

These are examples of Hilbert space with unitary element.

Example 1. Let $\mathcal{H} = \mathbb{R}^3$ be vector space with an inner product is defined by

$$\langle x, y \rangle = \sum_{i=1}^{3} x_i y_i$$

and the metric is

$$d(x, y) = \left(\sum_{i=1}^{3} (x_i - y_i)^2\right)^{\frac{1}{2}} .$$

As we know, a Hilbert space is a complete metric that generated by the inner product [19], first we will show that every sequence in $\mathbb{R}^3$ converges in $\mathbb{R}^3$.

Given (x_k) is a Cauchy sequence in $\mathbb{R}^3$, where

$$x_k = (x_{k,1}, x_{k,2}, x_{k,3}).$$

From induced metric we have,

$$d(x_k, x_m)^2 = \sum_{i=1}^{3} \left(x_{k,i} - x_{m,i}\right)^2 \to 0 \text{ as } k, m \to \infty.$$

For each coordinate position i, we have

$$\left(x_{k,i} - x_{m,i}\right)^2 \leq d(x_k, x_m)^2.$$

In other words, $\left(x_{k,i} - x_{m,i}\right)^2 \to 0$ as $k, m \to \infty$ and the sequence $\left(x_{k,i}\right)_{k=1,2,\cdots}$ of i-th coordinates is a Cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is complete, we have

$$\left(x_{k,i}\right) \to y_i \text{ as } k \to \infty.$$

If $y = (y_1, y_2, y_3)$, then

$$d(x_k, y)^2 = \sum_{i=1}^{n} \left(x_{k,i} - y_i\right)^2 \to 0 \text{ as } k \to \infty.$$

For each coordinate position i, we have

$$\left(x_{k,i} - y_i\right)^2 \le d(x_k, y)^2 \to 0.$$

Thus, we have every Cauchy sequence (x_n) in $\mathbb{R}^3$ is converges to y element of $\mathbb{R}^3$. In other words, $\mathbb{R}^3$ is complete and $\mathbb{R}^3$ is a Hilbert space.

Then we have unitary element in $\mathbb{R}^3$,

$$\mathbf{1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which is multiplied by itself will produce 1. Thus, $\mathbb{R}^3$ is Hilbert space with unitary element.

Based on Example 1, we can find some possible unitary element in $\mathbb{R}^3$,

$$\mathbf{1}^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{1}^{**} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{1}^{***} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

which is multiplied by itself will produce 1. In other words, the unitary element in Hilbert space is not unique.

One important thing about the development of measure theory in probability that it relies on the concept of ordering. In measurement theory, a random variable is a function of the sampling space. A random variable is said to be larger than another random variable if it is larger than the set of points in the sample space with probability 1. Thus, an ordered relationship is needed between elements in Hilbert space so that the following concept of partial ordered set is introduced.

Definition 2. [14] Given $\mathcal{H}$ be a Hilbert space. Given $\succcurlyeq$ be a relation defined on $\mathcal{H}$. Furthermore, relation $\succcurlyeq$ is said to be a partial ordering of $\mathcal{H}$ if satisfies the following properties:

i. Antisymmetry
   For all $\boldsymbol{a}$ and $\boldsymbol{b}$ in H, $\boldsymbol{a} \succcurlyeq \boldsymbol{b}$ and $\boldsymbol{b} \succcurlyeq \boldsymbol{a}$ if and only if $\boldsymbol{a} = \boldsymbol{b}$.
ii. Transitivity
   For all $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ in H, if $\boldsymbol{a} \succcurlyeq \boldsymbol{b}$ and $\boldsymbol{b} \succcurlyeq \boldsymbol{c}$ then $\boldsymbol{a} \succcurlyeq \boldsymbol{c}$.

The following is a simple example of partial ordering relation in Hilbert space $\mathcal{H}$.

Example 2. Let $\mathcal{H} = \mathbb{R}$ be Hilbert space. We define an inner product as

$$\langle x, y \rangle = xy$$

and the metric is

$$d(x, y) = x - y,$$

We define " $<$ " as a relation in $\mathcal{H}$. Relation " $<$ " is antisymmetric because it holds $\forall x, y \in \mathcal{H}, x < y$ and $y < x$ if and only if $x = y$. Relation " $<$ " holds transitivity properties because $\forall x, y, z \in \mathcal{H}$ it applies $x < y$ and $y < z$ then $x < z$. Thus, " $<$ " is partial ordering relation in $\mathcal{H}$.

The following introduces the concept of upper and lower bounds of a set with a partial ordering.

Definition 3. [14] Let $\mathcal{H}$ be a Hilbert space with a partial ordering relation $\succcurlyeq$. An element $\boldsymbol{\alpha} \vee \boldsymbol{\beta}$ is said to be the smallest upper bound of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ if:

i. $\boldsymbol{\alpha} \vee \boldsymbol{\beta} \succcurlyeq \boldsymbol{\alpha}$ and $\boldsymbol{\alpha} \vee \boldsymbol{\beta} \succcurlyeq \boldsymbol{\beta}$.
ii. If $\boldsymbol{\omega} \succcurlyeq \boldsymbol{\alpha}$ and $\boldsymbol{\omega} \succcurlyeq \boldsymbol{\beta}$ then $\boldsymbol{\omega} \succcurlyeq \boldsymbol{\alpha} \vee \boldsymbol{\beta}$, in other words ω is the smallest element.

Otherwise, an element $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ is said to be the greatest lower bound of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ if:

i. $\boldsymbol{\alpha} \succcurlyeq \boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ *and* $\boldsymbol{\beta} \succcurlyeq \boldsymbol{\alpha} \wedge \boldsymbol{\beta}$.
ii. If $\boldsymbol{\alpha} \succcurlyeq \boldsymbol{\omega}$ *and* $\boldsymbol{\beta} \succcurlyeq \boldsymbol{\omega}$ *then* $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \succcurlyeq \boldsymbol{\omega}$, *in other words,* $\boldsymbol{\omega}$ *is the greatest element.*

The antisymmetric property of partially ordered relation means that the greatest lower bound and the smallest upper bound are unique.

The following is an example of the smallest upper bound and greatest lower bound in Hilbert space $\mathcal{H}$.

Example 3. Let $\mathcal{H} = [0,1]$ be Hilbert space. Note that an inner product is defined by

$$\langle x, y \rangle = xy$$

and the metric is

$$d(x, y) = x - y$$

then " $<$ " is a partial ordering relation in $\mathcal{H}$. Thus, $\boldsymbol{\alpha} \vee \boldsymbol{\beta}$ is 1 as the smallest upper bound and $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ is 0 as the greatest lower bound.

If the random variable is in the usual definition, i.e., defined as a function of the sample space to real numbers then we can find the smallest upper bound and the largest lower bound pointwise easily. However, it turns out that we cannot guarantee that a partially ordered set has the smallest upper bound or the largest lower bound. So that we will further introduce the concept of lattice.

Definition 4. (Small and McLeish, 1994) Given $\mathcal{H}$ be a Hilbert space with partial ordering relation $\succcurlyeq$. Lattice is a Hilbert space with partial ordering relation that every pair of its element has smallest upper bound and greatest lower bound. The following properties indicate that a lattice is distributive:

i. $\boldsymbol{a} \wedge (\boldsymbol{b} \vee \boldsymbol{c}) = (\boldsymbol{a} \wedge \boldsymbol{b}) \vee (\boldsymbol{a} \wedge \boldsymbol{c})$
ii. $\boldsymbol{a} \vee (\boldsymbol{b} \wedge \boldsymbol{c}) = (\boldsymbol{a} \vee \boldsymbol{b}) \wedge (\boldsymbol{a} \vee \boldsymbol{c})$
   for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ element of $\mathcal{H}$.

Example 4. Since $\mathcal{H} = [0,1]$ in Example 3 that every pair of its element has smallest upper bound and greatest lower bound then $\mathcal{H}$ is lattice.

From Definition 4, we have the following Corrolary.

Corrolary 1. A lattice of Hilbert space $\mathcal{H}$ is said to be associative for all $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ element of $\mathcal{H}$ if:

i. $\boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) = (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \wedge \boldsymbol{\gamma}$
ii. $\boldsymbol{\alpha} \vee (\boldsymbol{\beta} \vee \boldsymbol{\gamma}) = (\boldsymbol{a} \vee \boldsymbol{b}) \vee \boldsymbol{c}$

Proof:

i. Let $\boldsymbol{a} = (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \wedge \boldsymbol{\gamma}$ and $\boldsymbol{b} = \boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \wedge \boldsymbol{\gamma})$. From definition, $\boldsymbol{a} = (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \wedge \boldsymbol{\gamma}$ then $\boldsymbol{\alpha} \succcurlyeq \boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ and $\boldsymbol{a} \succcurlyeq \boldsymbol{\gamma}$. So, we have $\boldsymbol{a} \succcurlyeq \boldsymbol{\alpha}$ and $\boldsymbol{a} \succcurlyeq \boldsymbol{\gamma}$. Similarly, $\boldsymbol{b} = \boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \wedge \boldsymbol{\gamma})$ then $\boldsymbol{b} \succcurlyeq \boldsymbol{\alpha}$ and $\boldsymbol{b} \succcurlyeq \boldsymbol{\beta} \wedge \boldsymbol{\gamma}$. Thus $\boldsymbol{b} \succcurlyeq \boldsymbol{\alpha}, \boldsymbol{b} \succcurlyeq \boldsymbol{\beta}, \boldsymbol{b} \succcurlyeq \boldsymbol{\gamma}$. Based on the relationship and definition of $\wedge$, we have $\boldsymbol{a} \succcurlyeq \boldsymbol{b}$ and $\boldsymbol{b} \succcurlyeq \boldsymbol{a}$. Thus $\boldsymbol{a} = \boldsymbol{b}$.

ii. In a similar way, let $\boldsymbol{a} = (\boldsymbol{\alpha} \vee \boldsymbol{\beta}) \vee \boldsymbol{\gamma}$ and $\boldsymbol{b} = \boldsymbol{\alpha} \vee (\boldsymbol{\beta} \vee \boldsymbol{\gamma})$. From definition, $\boldsymbol{a} = (\boldsymbol{\alpha} \vee \boldsymbol{\beta}) \vee \boldsymbol{\gamma}$ then $\boldsymbol{a} \succcurlyeq \boldsymbol{\alpha} \vee \boldsymbol{\beta}$ and $\boldsymbol{a} \succcurlyeq \boldsymbol{\gamma}$. So, we have $\boldsymbol{a} \succcurlyeq \boldsymbol{\alpha}$ and $\boldsymbol{a} \succcurlyeq \boldsymbol{\gamma}$. Similarly, $\boldsymbol{b} = \boldsymbol{\alpha} \vee (\boldsymbol{\beta} \vee \boldsymbol{\gamma})$ then $\boldsymbol{b} \succcurlyeq \boldsymbol{\alpha}$ and $\boldsymbol{b} \succcurlyeq \boldsymbol{\beta} \wedge \boldsymbol{\gamma}$. Thus $\boldsymbol{b} \succcurlyeq \boldsymbol{\alpha}, \boldsymbol{b} \succcurlyeq \boldsymbol{\beta}, \boldsymbol{b} \succcurlyeq \boldsymbol{\gamma}$. Based on the relationship and definition of $\vee$, we have $\boldsymbol{a} \succcurlyeq \boldsymbol{b}$ and $\boldsymbol{b} \succcurlyeq \boldsymbol{a}$. Thus $\boldsymbol{a} = \boldsymbol{b}$.

Through mathematical induction, it can be proven that every element of a finite set has the smallest upper bound and the largest lower bound. Noted that this property only applies to finite sets of elements.

Definition 5. [14] Given H be a Hilbert space with partial ordering relation $\succcurlyeq$. The completeness of a lattice if any set of elements has a least upper bound and greatest lower bound.

Example 5. From Example 3, $\mathcal{H} = [0,1]$ be Hilbert space is complete lattice.

The purpose of this paper is to show that a set of fundamental conditions on the Hilbert space ensures that the same algebraic and topological properties apply as the random variable Hilbert space determined on the previously mentioned probability space. Thus, the following is defined probability Hilbert space from the point of view of partially ordering set.

Definition 6. [14] Given a Hilbert space $\mathcal{H}$ with unitary element. Given partial ordered relation $\succcurlyeq$. A triplet $(\mathcal{H}, \mathbf{1}, \succcurlyeq)$ is said probability Hilbert space if:

i. $(\mathcal{H}, \mathbf{1})$ is a Hilbert space with unitary element.
ii. $(\mathcal{H}, \succcurlyeq)$ is a lattice.
iii. $\boldsymbol{\alpha} \succcurlyeq \mathbf{0}$ and $\boldsymbol{\beta} \succcurlyeq \mathbf{0}$ then $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \geq 0$ with equality if and only if $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \mathbf{0}$.
iv. For any positive scalar ε and any elements $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ the following identities hold:

$$\varepsilon(\boldsymbol{\alpha} \vee \boldsymbol{\beta}) = (\varepsilon\boldsymbol{\alpha}) \vee (\varepsilon\boldsymbol{\beta})$$

and

$$(\boldsymbol{\alpha} \vee \boldsymbol{\beta}) + \boldsymbol{\gamma} = (\boldsymbol{\alpha} + \boldsymbol{\gamma}) \vee (\boldsymbol{\beta} + \boldsymbol{\gamma})$$

Example 6. Let $\xi = [0,1]$ be unit interval in real space $\mathbb{R}$. Let $\mathcal{H}$ be Hilbert space with an inner product is defined by:

$$\langle t, u \rangle = \int_0^1 t(x) \cdot u(x) dx$$

Let an unitary element be constant function,

$$\mathbf{1}(x) = c, \forall x \in [0,1]$$

If we defined partial ordered relation " $\geq$ " then $t \geq u$ if and only if there are t and u such that $t(x) \geq u(x), \forall x \in [0,1]$.

The four properties in Definition 6 distinguish ordinary random variables that can be integrated into the probability space. In fact, henceforth we will work on elements of non-negative random variables, so we need a definition of the absolute value in the Hilbert probability space.

Definition 7. [14] The absolute value in probability Hilbert space is defined as $|\boldsymbol{a}| = \boldsymbol{a}^+ + \boldsymbol{a}^-$ where $\boldsymbol{a}^+ = \boldsymbol{a} \vee \mathbf{0}$ and $\boldsymbol{a}^- = (-\boldsymbol{a})^+$.

The following is a proposition about absolute values that will be useful in σ-algebra.

Proposition 1. *Given a Hilbert space $\mathcal{H}$. For all $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{H}$:*

i. $|\boldsymbol{a}| = \boldsymbol{a} \vee (-\boldsymbol{a})$
ii. $\boldsymbol{a} = \boldsymbol{a}^+ - \boldsymbol{a}^-$
iii. $\boldsymbol{a} + \boldsymbol{b} = (\boldsymbol{a} \vee \boldsymbol{b}) + (\boldsymbol{a} \wedge \boldsymbol{b})$
iv. $\||\boldsymbol{a}|\| = \|\boldsymbol{a}\|$

Proof:

i.
$$\begin{aligned} |\boldsymbol{a}| &= \boldsymbol{a}^+ - \boldsymbol{a}^- \\ &= (\boldsymbol{a} \vee \mathbf{0}) + (-\boldsymbol{a})^+ \\ &= (\boldsymbol{a} \vee \mathbf{0}) + \big((-\boldsymbol{a}) \vee \mathbf{0}\big) \\ &= (\boldsymbol{a} + \mathbf{0}) \vee (-\boldsymbol{a} + \mathbf{0}) \\ &= \big(\boldsymbol{a} \vee (-\boldsymbol{a})\big) + \mathbf{0} \\ &= \boldsymbol{a} \vee (-a) \end{aligned}$$

ii.
$$\begin{aligned} \boldsymbol{a}^+ - \boldsymbol{a}^- &= \boldsymbol{a}^+ - (-\boldsymbol{a})^+ \\ &= (\boldsymbol{a} \vee \mathbf{0}) - \big((-\boldsymbol{a}) \vee \mathbf{0}\big) \\ &= (\boldsymbol{a} \vee \mathbf{0}) + (-(-\boldsymbol{a}) \vee \mathbf{0}) \\ &= (\boldsymbol{a} + \mathbf{0}) \vee (-(-\boldsymbol{a}) + \mathbf{0}) \\ &= \boldsymbol{a} \vee \big(-(-\boldsymbol{a})\big) \\ &= \boldsymbol{a} \vee \boldsymbol{a} \\ &= \boldsymbol{a} \end{aligned}$$

iii.
$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= (\boldsymbol{a} + \boldsymbol{b}) \vee (\boldsymbol{b} + \boldsymbol{a}) \\ &= \big(\boldsymbol{a} \vee (\boldsymbol{b} + \boldsymbol{a})\big) + \big(\boldsymbol{b} \vee (\boldsymbol{b} + \boldsymbol{a})\big) \\ &= (\boldsymbol{a} \vee \boldsymbol{b}) + (\boldsymbol{a} \vee \boldsymbol{a}) + (\boldsymbol{b} \vee \boldsymbol{b}) + (\boldsymbol{b} \vee \boldsymbol{a}) \\ &= \big((\boldsymbol{a} \vee \boldsymbol{b}) + \boldsymbol{a}\big) + \big(\boldsymbol{b} + (\boldsymbol{b} \wedge \boldsymbol{a})\big) \\ &= (\boldsymbol{a} \vee \boldsymbol{b}) + (\boldsymbol{a} \wedge \boldsymbol{b}) \end{aligned}$$

iv.
$$\begin{aligned} \||\boldsymbol{a}|\| &= \|\boldsymbol{a}^+ + \boldsymbol{a}^-\| \\ &= \|(\boldsymbol{a} \vee \mathbf{0}) + (-\boldsymbol{a}^+ \vee \mathbf{0})\| \\ &= \|\big(\boldsymbol{a} \vee (-\boldsymbol{a})\big) + (\boldsymbol{a} - \boldsymbol{a})\| \\ &= \|\boldsymbol{a}\| \end{aligned}$$

The following propositions are obtained regarding the convergence of lattice sequences.

Proposition 2. Let a probability Hilbert space $(H, \mathbf{1}, \succcurlyeq)$. Let $\boldsymbol{x}_n$ and $\boldsymbol{y}_n$ be convergence random variable sequences to the $\boldsymbol{x}$ and $\boldsymbol{y}$. Thus $\boldsymbol{x}_n \vee \boldsymbol{y}_n$ convergence to $(\boldsymbol{x} \vee \boldsymbol{y})$ and $\boldsymbol{x}_n \wedge \boldsymbol{y}_n$ convergence to $(\boldsymbol{x} \wedge \boldsymbol{y})$.

Proof:
Let $(\boldsymbol{x}_n) \to \boldsymbol{x}$ and $(\boldsymbol{y}_n) \to \boldsymbol{y}$. Now, we will prove that $(\boldsymbol{x}_n \vee \boldsymbol{y}_n) \to (\boldsymbol{x} \vee \boldsymbol{y})$.
Let $k \in [0, \infty]$ such that

$$|(\boldsymbol{x}_n \vee \boldsymbol{y}_n) - (\boldsymbol{x} \vee \boldsymbol{y})| \leq |\boldsymbol{x}_n - \boldsymbol{x}| + |\boldsymbol{y}_n - \boldsymbol{y}| \leq k\|\boldsymbol{x}_n - \boldsymbol{x}\| + k\|\boldsymbol{y}_n - \boldsymbol{y}\| \to 0,$$

at $n \to \infty$.

We now give definitions of indicators that are useful in constructing σ -algebra.

Definition 8. [14] Suppose there is a random variable $\boldsymbol{y}$ on Hilbert space $\mathcal{H}$. Thus, $\boldsymbol{y}$ is said to be an indicator if $\mathbf{1} \succcurlyeq \boldsymbol{y} \succcurlyeq \mathbf{0}$, and if $\boldsymbol{y} \vee (\mathbf{1} - \boldsymbol{y})$ and $\boldsymbol{y} \wedge (\mathbf{1} - \boldsymbol{y})$ equal $\mathbf{1}$ and $\mathbf{0}$, respectively.

Example 7. Let Ω for the indicator $\mathbf{1}$ and $\emptyset$ for the indicator $\mathbf{0}$.

The following defines the relationship between Boolean algebra and σ-algebra.

Definition 9. [14] *Suppose a lattice has the largest element $\boldsymbol{m}$ and the smallest element $\boldsymbol{n}$. Two elements are said to be complementary if for every $\boldsymbol{x}$ there exists an element $\boldsymbol{y}$ such that $\boldsymbol{x} \vee \boldsymbol{y} = \boldsymbol{m}$ and $\boldsymbol{x} \wedge \boldsymbol{y} = \boldsymbol{n}$. Furthermore, a distributive complement lattice is said to be a Boolean algebra. If a Boolean algebra is countable, it is called σ -algebra.*

In the following, we generate a proposition about the σ-algebra formed by a set of indicators in the Hilbert probability space.

Proposition 3. *If there is a set of indicators in the Hilbert probability space, then σ-algebra can be formed.*

Proof:
The step that needs to be done is to show that the set of indicators forms a lattice first. Suppose there are any two events $\mathcal{P}$ and $\mathcal{Q}$. Let $\boldsymbol{a} = \mathbf{1}_\mathcal{P} \vee \mathbf{1}_\mathcal{Q}$ where $\mathbf{1}_\mathcal{P}$ and $\mathbf{1}_\mathcal{Q}$ are indicators. Thus applies $\mathbf{1} \succcurlyeq \mathbf{1}_\mathcal{P} \succcurlyeq \mathbf{0}$ dan $\mathbf{1} \succcurlyeq \mathbf{1}_\mathcal{Q} \succcurlyeq \mathbf{0}$. In other words, $\mathbf{1}_\mathcal{P} \vee (\mathbf{1} - \mathbf{1}_\mathcal{P}) = \mathbf{1}$ and $\mathbf{1}_\mathcal{P} \wedge (\mathbf{1} - \mathbf{1}_\mathcal{P}) = \mathbf{0}$. Similarly, $\mathbf{1}_\mathcal{Q} \vee \left(\mathbf{1} - \mathbf{1}_\mathcal{Q}\right) = \mathbf{1}$ and $\mathbf{1}_\mathcal{Q} \wedge \left(\mathbf{1} - \mathbf{1}_\mathcal{Q}\right) = \mathbf{0}$. We will show that $\boldsymbol{a}$ is indicator. In a similar way, $\boldsymbol{b} = \mathbf{1}_\mathcal{P} \wedge \mathbf{1}_\mathcal{Q}$ is an indicator too. Since $\boldsymbol{a} \vee \boldsymbol{b} \succcurlyeq \mathbf{0}$ dan $\mathbf{1} \succcurlyeq \boldsymbol{a} \vee \boldsymbol{b}$, the class containing the indicator forms a lattice. Lattice has minimum and maximum elements are $\mathbf{0}$ and $\mathbf{1}$, fulfills $\boldsymbol{a} \vee \boldsymbol{b} \succcurlyeq \mathbf{0}$ and $\boldsymbol{a} \wedge \boldsymbol{b} \succcurlyeq \mathbf{1}$ where $\boldsymbol{a} = \mathbf{1}_\mathcal{P} \vee \mathbf{1}_\mathcal{Q}$.

In an indicator class, let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, the left distributive property

$$\boldsymbol{a} \wedge (\boldsymbol{b} \vee \boldsymbol{c}) = \boldsymbol{a} \wedge \mathbf{0} = \mathbf{1}$$

equivalent with

$$(\boldsymbol{a} \wedge \boldsymbol{b}) \vee (\boldsymbol{a} \wedge \boldsymbol{c}) = \mathbf{1} \vee \mathbf{1} = \mathbf{1}$$

and the right distribution

$$\boldsymbol{a} \vee (\boldsymbol{b} \wedge \boldsymbol{c}) = \boldsymbol{a} \vee \mathbf{1} = \mathbf{0}$$

equivalent with

$$(\boldsymbol{a} \vee \boldsymbol{b}) \wedge (\boldsymbol{a} \vee \boldsymbol{b}) = \mathbf{0} \wedge \mathbf{0} = \mathbf{0}$$

are satisfied.

Thus, the indicator class is a Bollean algebra.

Based on Definition 9, we need to show that the class of indicators is countably complete. Let $\boldsymbol{a}_k, k = 1,2,3, \cdots$ are indicator sequences. It will be shown that this sequence has the least upper bound and greatest lower bound. First, we show that indicator sequences have the least upper bound. Let Cauchy sequence $\boldsymbol{b}_n = \bigvee_{k=1}^{n} \boldsymbol{a}_k$. For all $m > n$ where $m, n \in \mathbb{N}$,

$$\langle \boldsymbol{b}_m - \boldsymbol{b}_n, \boldsymbol{b}_m - \boldsymbol{b}_n \rangle \leq \cdots \leq \langle \boldsymbol{b}_m, 1 \rangle - \langle \boldsymbol{b}_n, 1 \rangle$$

Note that,
$\langle \boldsymbol{b}_m, 1 \rangle - \langle \boldsymbol{b}_n, 1 \rangle$ is increasing sequence and bounded by $\mathbf{1}$ then $\lim_{m,n \to \infty} \langle \boldsymbol{b}_m, 1 \rangle - \langle \boldsymbol{b}_n, 1 \rangle = 0$.

Thus, $\boldsymbol{b}_n$ is Cauchy sequence where $\boldsymbol{b} \in \mathcal{H}$ is limit of $\boldsymbol{b}_n$.

Lastly, we will show that $\boldsymbol{b}$ is the smallest upper bound and also an indicator. Based on Proposition 2, the set of all random variables related to $\succcurlyeq \boldsymbol{b}_n$ is closed set for all n. Thus, $\boldsymbol{b} \succcurlyeq \boldsymbol{b}_n$ for all n. Let $\boldsymbol{c} \succcurlyeq \boldsymbol{b}_n$ for all n. Since the set of random variables is less than equal to $\boldsymbol{c}$ is a closed set then $\boldsymbol{c} \succcurlyeq \boldsymbol{b}$. Thus, $\boldsymbol{b}$ is the least upper bound. In the same way, $\boldsymbol{b}$ is an indicator.

3. Dataset Application

In this section, partial ordering and probability distribution are explained in the shares data set of 42 energy companies in Indonesia that belong to the development board category from www.idx.co.id. The development board is intended for companies that have not been able to meet the main board listing requirements and have not recorded a net profit.

Figure 1 shows the ordering relations of the 42 companies. It appears that the highest share value is at the fourth point, namely Astrindo Nusantara Infrastrukt Tbk. company. Next, the probability distribution will be determined according to the dataset. We tried continuous probability distributions including the Normal probability distribution, the Log Normal probability distribution, and the Cauchy distribution.

Table 1. Shares of Development Board Energy Companies

Code	Company	Shares
AIMS	Akbar Indo Makmur Stimec Tbk.	220,000,000
ARII	Atlas Resources Tbk.	3,431,000,000
ARTI	Ratu Prabu Energi Tbk.	7,840,000,000
BIPI	Astrindo Nusantara Infrastrukt Tbk.	57,918,360,917
BSSR	Baramulti Suksessarana Tbk.	2,616,500,000
BULL	Buana Lintas Lautan Tbk.	14,117,801,449
CANI	Capitol Nusantara Indonesia Tbk.	833,440,000
CNKO	Exploitasi Energi Indonesia Tbk.	8,956,361,206
ETWA	Eterindo Wahanatama Tbk.	4,668,671,400
GTBO	Garda Tujuh Buana Tbk.	2,500,000,000
IATA	MNC Energy Investments Tbk.	25,238,221,508
ITMA	Sumber Energi Andalan Tbk.	999,053,167
KOPI	Mitra Energi Persada Tbk.	697,266,668
MBAP	Mitrabara Adiperdana Tbk.	1,227,271,952
MITI	Mitra Investindo Tbk.	3,540,735,503
MTFN	Capitalinc Investment Tbk.	31,842,082,852
PKPK	Perdana Karya Perkasa Tbk.	600,000,000
RIGS	Rig Tenders Indonesia Tbk.	609,130,000
RUIS	Radiant Utama Interinsco Tbk.	770,000,000
SMMT	Golden Eagle Energy Tbk.	3,150,000,000
SMRU	SMR Utama Tbk.	12,499,385,782
SUGI	Sugih Energy Tbk.	24,811,541,414
TPMA	Trans Power Marine Tbk.	2,633,300,000
TRAM	Trada Alam Minera Tbk.	49,643,627,934
TAMU	Pelayaran Tamarin Samudra Tbk.	37,500,000,000
FIRE	Alfa Energi Investama Tbk.	1,475,363,179
DWGL	Dwi Guna Laksana Tbk.	9,252,820,991
BOSS	Borneo Olah Sarana Sukses Tbk.	1,400,000,000
JSKY	Sky Energy Indonesia Tbk.	2,032,540,000
INPS	Indah Prakasa Sentosa Tbk.	650,000,000
TCPI	Transcoal Pacific Tbk.	5,000,000,000
SURE	Super Energy Tbk.	1,497,576,771
WOWS	Ginting Jaya Energi Tbk.	2,475,720,000
BESS	Batulicin Nusantara Maritim Tbk.	3,435,183,958
SGER	Sumber Global Energy Tbk.	3,727,301,785
BSML	Bintang Samudera Mandiri Lines Tbk.	1,850,225,000
ADMR	Adaro Minerals Indonesia Tbk.	40,882,331,500
SEMA	Semacom Integrated Tbk.	1,347,258,842
SICO	Sigma Energy Compressindo Tbk.	910,038,262
COAL	Black Diamond Resources Tbk.	6,250,000,000
CBRE	Cakra Buana Resources Energi Tbk.	4,538,000,000

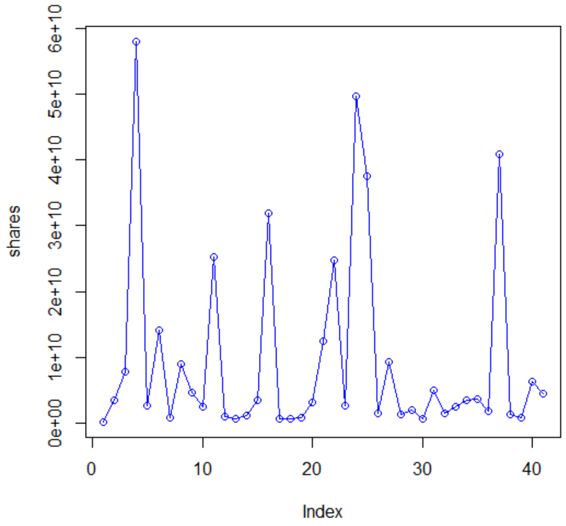

Figure 1. Ordered relation of 42 energy companies

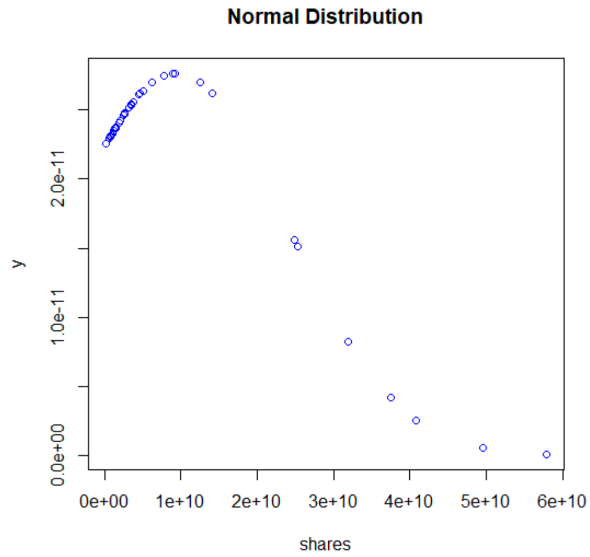

Figure 2. Normal distribution plot

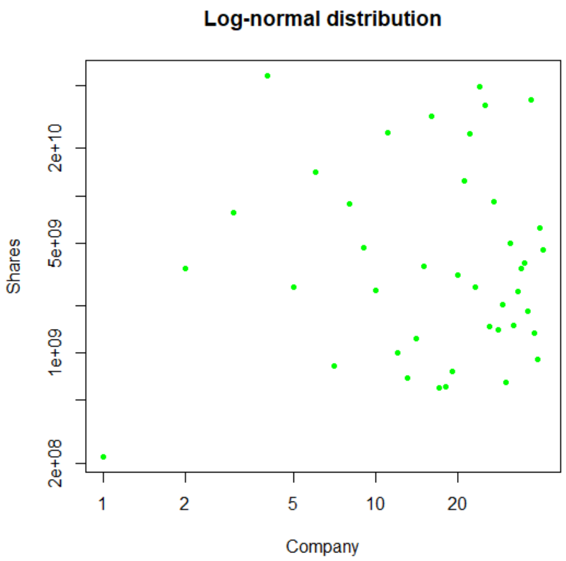

Figure 3. Log-Normal distribution plot

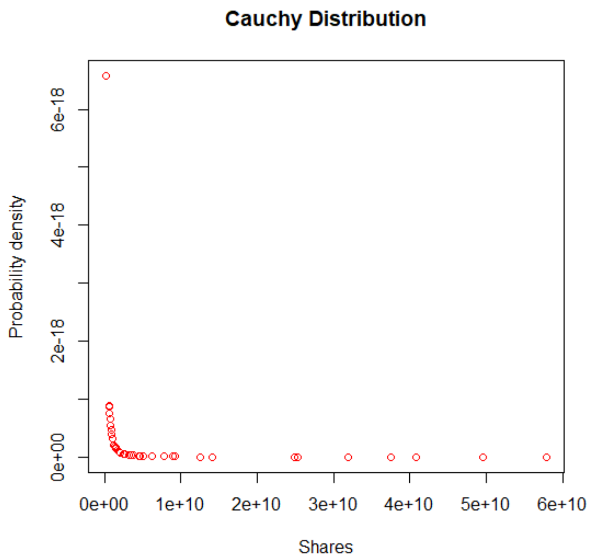

Figure 4. Cauchy distribution plot

Based on Figure 2, it can be seen that the distribution of the Normal distribution plots. It can be said that the data is symmetrical even though the symmetry on the left side is truncated. This means that the data can be further analyzed parametrically under the assumption of normality or not. Next, we try to perform a logarithmic transformation to obtain a Log-Normal distribution plot. It can be seen in Figure 3 that the data does not form the proper distribution pattern. Thus, further analysis with the concept of Log-Normal distribution is not recommended. From the sequence, we can state that the dataset is a Cauchy sequence that converges to 0 as shown in Figure 4. This means that the data can be analyzed using the concept of probability Hilbert space which will be discussed in further research.

4. Conclusion

From a basic set of conditions on the Hilbert space ensures that the same algebraic and topological properties apply as the random variable Hilbert space determined on the previously mentioned probability space. We can define probability Hilbert space from the point of view of partially ordering set. The application to energy company data shares shows that the data is a convergent Cauchy sequence and can be formed as a random variable in the probability Hilbert space.

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