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Inequalities for Forgotten Index of Duplication and Double Duplication of Graphs

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Abstract Molecular descriptors act as an important part in mathematical chemistry, in investigating quantitative structure-property relationship and quantitative structure-activity relationship. A topological descriptors, also called a molecular descriptor, is a mathematical formula applied to any graph which produces new molecular structure. In medicine mathematical model, the chemical compound is represented as an undirected graph, where each vertex represents an atom and each edge indicates a chemical bond between these atoms. The Wiener index is the first topological index to be used in chemistry introduced by Harold Wiener [1947]. It is used to compare the boiling points of some alkane isomers. There are various topological indices which are applied in chemistry. Among them, our interest is on Forgotten index which is degree based topological index introduced by Furtula and Gutman in 2015[2], defined as $F(G) = \sum_{u \in V(G)} d_u^3$ where d_u is the degree of vertex u in G. The mathematicians and chemists have studied several general properties of Forgotten index which may help the chemical and pharmaceutical industry to achieve the significance details by quantitative methods than by experiments.Vaidya et al (2009) proposed the concept of duplication of a vertex by an edge and duplication of an edge by a vertex of graphs. Shobana et al. proposed the double duplication of graphs (2017) [6]. Only connected, simple, undirected and finite graphs are considered throughout this article. Also, some inequalities are obtained by comparing the duplication and double duplication of graphs using Forgotten index which can also be used by chemists to generate new antidrug in future.

Keywords Forgotten Index, Duplication, Double Duplication, Inequality

1 Introduction

All graphs considered in this article are simple and connected graphs. A topological index is a type of a molecular descriptor. It is determined based on the molecular graph of a chemical compound structure. The First Zagreb index $M_1(G)$ of a graph G introduced by Gutman and Trinajstic in 1972 [5] near 75 years ago is defined as

$$M_1(G) = \sum_{u \in V(G)} d_u^2 \tag{1}$$

The Forgotten index $F(G)$ of a graph G is one of the oldest popular connectivity index and widely used on QSPR/QSAR studies. It was introduced by Furtula and Gutman in 2015 [2] which is defined as

$$F(G) = \sum_{u \in V(G)} d_u^3 \tag{2}$$

Definition 1.1. *[8]* ***Duplication of a node (vertex) by a line (edge)*** *was introduced by Vaidya et al in 2009 which is defined as a duplication of a vertex v_l by a new edge $e = v'v''$ in a graph G resulting in a new graph G' such that $N(v') = \{v_l, v''\}$ and $N(v'') = \{v_l, v'\}$.*

Definition 1.2. *[8]* ***Duplication of a line (edge) by a node (vertex)*** *is defined as a duplication of an edge $e = v_i v_{i+1}$ by a vertex v' in a graph G resulting in a new graph G' such that $N(v') = \{v_i, v_{i+1}\}$.*

In this paper, duplication of all nodes (vertices) by lines (edges) of G and duplication of all edges by vertices of G are notated by $D_V(G)$ and $D_E(G)$ respectively.

Definition 1.3. *[6]* ***The double duplication of a node (vertex) by a line (edge) of a graph*** *was introduced by Roopa and Shobana in 2017 which is defined as a duplication of a vertex x_l by $e = x_l'x_l''$ in G resulting in G', in which $N(x_l') = \{x_l, x_l''\}$ and $N(x_l'') = \{x_l, x_l'\}$. Again duplication of vertices x_l, x_l' and x_l'' by $e' = y_l z_l$, $e'' = y_l' z_l'$ and $e''' = y_l'' z_l''$ respectively in G' resulting in G'' such that, $N(y_l) = \{z_l, x_l\}$, $N(z_l) = \{y_l, x_l\}$, $N(y_l') = \{z_l', x_l'\}$, $N(z_l') = \{y_l', x_l'\}$, $N(y_l'') = \{z_l'', x_l''\}$ and $N(z_l'') = \{y_l'', x_l''\}$. Double duplication of nodes (vertices) by lines (edges) respectively G is denoted by $DD_{VV}(G)$.*

Definition 1.4. *[6]* ***The double duplication of node (vertex) by a line (edge) followed by line (edge) by node (vertex) of a graph*** *is defined as, a dulpication of a vertex v_l by an edge $e = v_l'v_l''$ in a graph G resulting in a graph G' in which $N(v_l') = \{v_l, v_l''\}$ and $N(v_l'') = \{v_l, v_l'\}$. Again duplication of edges $v_lv_l', v_l'v_l''$ and v_lv_l'' by vertices u_l, u_l' and u_l'' respectively in G' resulting in a new graph G'' such that, $N(u_l) = \{v_l, v_l'\}$, $N(u_l') = \{v_l', v_l''\}$ and $N(u_l'') = \{v_l, v_l''\}$. Double duplication: vertices by edges, followed with edges by vertices respectively of a graph G are denoted by $DD_{VE}(G)$.*

Definition 1.5. *[6]* ***The double duplication of a line (edge) by a node (vertex) followed with node (vertex) by a line (edge) of a graph*** *is defined as, a duplication of an edge $e_l = v_lv_{l+1}$ by a vetex v_l' in a graph G resulting in a graph G' in which $N(v_l') = \{v_l, v_{l+1}\}$. Again duplication of v_l, v_{l+1} and v_l' by the edges $e' = u_lw_l$, $e'' = u_{l+1}w_{l+1}$ and $e''' = u_l'w_l'$ in G' resulting in a new graph G'' such that $N(u_l) = \{v_l, w_l\}$, $N(w_l) = \{u_l, v_l\}$, $N(u_{l+1}) = \{v_{l+1}, w_{l+1}\}$, $N(w_{l+1}) = \{u_{l+1}, v_{l+1}\}$, $N(u_l') = \{v_l', w_l'\}$ and $N(w_l') = \{u_l', v_l'\}$. Double duplication: lines (edges) by nodes (vertices), followed with nodes (vertices) by lines (edges) of G are denoted by $DD_{EV}(G)$.*

Definition 1.6. *[6]* ***The double duplication of a line (edge) by node (vertex) followed with line (edge) by node (vertex) of a graph*** *is defined as, a duplication of an edge $e = v_lv_l'$ by a vertex v_l'' in a graph G resulting in a graph G' in which $N(v_l'') = \{v_l, v_l'\}$. Again duplication of the edges v_lv_l', $v_l'v_l''$ and v_lv_l'' by the vertices u_l'', u_l' and u_l respectively in G' resulting in a new graph G'' such that $N(u_l) = \{v_l, v_l''\}$, $N(u_l') = \{v_l', v_l''\}$ and $N(u_l'') = \{v_l, v_l'\}$. Double duplication of lines (edges) by nodes (vertices) of G is denoted by $DD_{EE}(G)$.*

2 Preliminaries

The following results are taken from [7] which are used to prove inequalities in section 3.

- $F(D_V(G)) = F(G) + 6M_1(G) + 24(n + m)$
- $F(D_E(G)) = 8F(G) + 8m$
- $F(DD_{VV}(G)) = F(G) + 240n + 12M_1(G) + 96m$
- $F(DD_{EE}(G)) = 64F(G) + 88m$
- $F(DD_{VE}(G)) = 8(F(G) + 6M_1(G) + 27n + 25m)$
- $F(DD_{EV}(G)) = 8(F(G) + 3M_1(G) + 3n + 16m)$

3 Main results

In this article, some graph inequalities on duplication and double duplication graphs using Forgotten index are verified.

Theorem 3.1. *For any connected graph $G(n, m)$, $F(D_V(G)) < F(DD_{VV}(G)) < F(DD_{VE}(G))$.*

Proof. Let G be a graph with n vertices and m edges. The graphs $D_V(G)$, $DD_{VV}(G)$, $DD_{VE}(G)$ contain $3n$ vertices and $m+3n$ edges, $9n$ vertices and $m+12n$ edges and $6n+m$ vertices and $9n+3m$ edges respectively.

case (i): To get 1^{st} minimum, compare $F(D_V(G))$, $F(DD_{VV}(G))$ and $F(DD_{VE}(G))$.

- $F(D_V(G)) - F(DD_{VV}(G)) = -6(M_1(G) + 36n + 12m) < 0$
  $\implies F(D_V(G)) < F(DD_{VV}(G))$.
- $F(D_V(G)) - F(DD_{VE}(G)) = -(7F(G) + 42M_1(G) + 192n + 176m) < 0$
  $\implies F(D_V(G)) < F(DD_{VE}(G))$.

case (ii): The 2^{nd} minimum is obtained by comparing $F(DD_{VV}(G))$ and $F(DD_{VE}(G))$.

- $F(DD_{VV}(G)) - F(DD_{VE}(G)) = -(7F(G) + 36M_1(G) - 24n + 104m) < 0$.
  Thus, $F(DD_{VV}(G)) < F(DD_{VE}(G))$

From the above cases, it follows that $F(D_V(G)) < F(DD_{VV}(G)) < F(DD_{VE}(G))$.

□

Theorem 3.2. *For any connected graph G with n vertices and m edges, $F(D_E(G)) < F(DD_{EV}(G)) < F(DD_{EE}(G))$.*

Proof. Let G be a graph with n vertices and m edges.

The graphs $D_E(G)$, $DD_{EV}(G)$, $DD_{EE}(G)$ contain $(n + m)$ vertices and $3m$ edges, $(3n + 3m)$ vertices and $(3n + 6m)$ edges and $(n + 4m)$ vertices and $9m$ edges respectively.

case (i): In order to get the 1^{st} minimum, it is necessary to compare $F(D_E(G))$, $F(DD_{EV}(G))$ and $F(DD_{EE}(G))$.

- $F(D_E(G)) - F(DD_{EV}(G)) = -24(M_1(G) + n + 5m) < 0$
  $\implies F(D_E(G)) < F(DD_{EV}(G))$
- $F(D_E(G)) - F(DD_{EE}(G)) = -8(7F(G) + 10m) < 0$
  $\implies F(DD_{EV}(G)) < F(DD_{EE}(G))$

case (ii): For the 2^{nd} minimum, compare $F(DD_{EV}(G))$ and $F(DD_{EE}(G))$.

- $F(DD_{EV}(G)) - F(DD_{EE}(G)) = -8(7F(G) - 3M_1(G) - 3n - 5m) < 0$
  Therefore, $F(DD_{EV}(G) < F(DD_{EE}(G))$

From case (i) and case (ii), it follows that $F(D_E(G)) < F(DD_{EV}(G) < F(DD_{EE}(G)$.

□

Theorem 3.3. *Let $P_n(n \geq 4)$ be a path graph with n vertices, then $F(D_E(P_n)) < F(D_V(P_n)) < F(DD_{EV}(P_n)) < F(DD_{VV}(P_n)) < F(DD_{EE}(P_n)) < F(DD_{VE}(P_n))$*

Proof. A path graph P_n is a graph whose vertices are listged in the order $v_1, v_2, ...v_n$ such that the edges obtained are v_iv_{i+1} where $i = 1, 2, ..., n-1$.
To prove the inequalities, the following cases are to be considered.

case (i): To obtain $F(D_E(P_n))$ as 1^{st} minimum, compare it with $F(D_V(P_n))$, $F(DD_{EV}(P_n))$, $F(DD_{VV}(P_n))$, $F(DD_{EE}(P_n))$ and $F(DD_{VE}(P_n))$.

- $F(D_E(P_n)) - F(D_V(P_n)) = -8n - 58 < 0$ $\Longrightarrow F(D_E(P_n)) < F(D_V(P_n))$
- $F(D_E(P_n)) - F(DD_{EV}(P_n)) = -240n + 264 < 0$ $\Longrightarrow F(D_E(P_n)) < F(DD_{EV}(P_n))$
- $F(D_E(P_n)) - F(DD_{VV}(P_n)) = -320n + 62 < 0$ $\Longrightarrow F(D_E(P_n)) < F(DD_{VV}(P_n))$
- $F(D_E(P_n)) - F(DD_{EE}(P_n)) = -528n + 864 < 0$ $\Longrightarrow F(D_E(P_n)) < F(DD_{EE}(P_n))$
- $F(D_E(P_n)) - F(DD_{VE}(P_n)) = -216n + 480 < 0$ $\Longrightarrow F(D_E(P_n)) < F(DD_{VE}(P_n))$

case (ii): To get $D_V(P_n)$ as the 2^{nd} minimum, compare it with $F(DD_{EV}(P_n))$,$F(DD_{VV}(P_n))$,$F(DD_{EE}(P_n))$ and $F(DD_{VE}(P_n))$

- $F(D_V(P_n)) - F(DD_{EV}(P_n)) = -232n + 310 < 0$ $\Longrightarrow F(D_V(P_n)) < F(DD_{EV}(P_n))$
- $F(D_V(P_n)) - F(DD_{VV}(P_n)) = -312n + 108 < 0$ $\Longrightarrow F(D_V(P_n)) < F(DD_{VV}(P_n))$
- $F(D_V(P_n)) - F(DD_{EE}(P_n)) = -568n + 910 < 0$ $\Longrightarrow F(D_V(P_n)) < F(DD_{EE}(P_n))$
- $F(D_V(P_n)) - F(DD_{VE}(P_n)) = -592n + 526 < 0$ $\Longrightarrow F(D_V(P_n)) - F(DD_{VE}(P_n))$

case (iii): Comparing $F(DD_{VV}(P_n))$,$F(DD_{EE}(P_n))$ and $F(DD_{VE}(P_n))$ to get $DD_{EV}(P_n)$ as the 3^{rd} minimum of the F-index.

- $F(DD_{EV}(P_n)) - F(DD_{VV}(P_n)) = -80n - 202 < 0$ $\Longrightarrow F(DD_{EV}(P_n)) < F(DD_{VV}(P_n))$
- $F(DD_{EV}(P_n)) - F(DD_{EE}(P_n)) = -288n + 600 < 0$ $\Longrightarrow F(D_E(P_n)) < F(D_{EE}(P_n))$
- $F(DD_{EV}(P_n)) - F(DD_{VE}(P_n)) = -360n + 216 < 0$ $\Longrightarrow F(DD_{EV}(P_n)) < F(DD_{VE}(P_n))$.

case (iv): In order to obtain $F(DD_{VV}(P_n)$ as 4^{th} minimum, it is necessary to compare it with $F(DD_{EE}(P_n))$ and $F(DD_{VE}(P_n))$.

- $F(DD_{VV}(P_n)) - F(DD_{EE}(P_n)) = -208n + 802 < 0$ $\Longrightarrow F(DD_{VV}(P_n)) < F(DD_{EE}(P_n))$
- $F(DD_{VV}(P_n)) - F(DD_{VE}(P_n)) = -328n + 418 < 0$ $\Longrightarrow F(DD_{VV}(P_n)) < F(DD_{VE}(P_n))$

case (v): To get 5^{th} minimum namely $F(DD_{EE}(P_n))$, compare it with $F(DD_{VE}(P_n))$.

- $F(DD_{EE}(P_n)) - F(DD_{VE}(P_n)) = -72n - 384 < 0$ $\Longrightarrow F(DD_{EE}(P_n)) < F(DD_{VE}(P_n))$

From the above discussed cases, it follows $F(D_E(P_n)) < F(D_V(P_n)) < F(DD_{EV}(P_n)) < F(DD_{VV}(P_n)) < F(DD_{EE}(P_n)) < F(DD_{VE}(P_n))$. □

Observation 3.3.1. The theorem 3.3 holds good for $n \geq 4$ and the following inequalities are discussed for $n \leq 3$.
(i) For $n = 2$; $F(D_E(P_n)) < F(D_V(P_n)) < F(DD_{EE}(P_n)) < F(DD_{EV}(P_n)) < F(DD_{VV}(P_n)) < F(DD_{VE}(P_n))$.
(ii) For $n = 3$; $F(D_E(P_n)) < F(D_V(P_n)) < F(DD_{EV}(P_n)) < F(DD_{EE}(P_n)) < F(DD_{VV}(P_n)) < F(DD_{VE}(P_n))$.

Theorem 3.4. *Let $C_n(n \geq 3)$ be a cycle graph with n vertices, then $F(D_E(C_n)) < F(D_V(C_n)) < F(DD_{EV}(C_n)) < F(DD_{VV}(C_n)) < F(DD_{EE}(C_n)) < F(DD_{VE}(C_n))$*

Proof. A simple graph of n vertices ($n \geq 3$) and n edges forming a cycle of length n is called as a cycle graph denoted by C_n. In a cycle graph, all the vertices are of degree 2.
To prove the inequalities,the following cases are to be taken into account.

case (i): To get $F(D_E(C_n))$ as 1^{st} minimum, compare it with $F(D_V(C_n)$, $F(DD_{EV}(C_n))$, $F(DD_{VV}(C_n))$, $F(DD_{EE}(C_n))$ and $F(DD_{VE}(C_n))$.

- $F(D_E(C_n)) - F(D_V(C_n)) = -8n < 0$ $\Longrightarrow F(D_E(C_n)) < F(D_V(C_n))$
- $F(D_E(C_n)) - F(DD_{EV}(C_n)) = -240n < 0$ $\Longrightarrow F(D_E(C_n)) < F(DD_{EV}(C_n))$
- $F(D_E(C_n)) - F(DD_{VV}(C_n)) = -320n < 0$ $\Longrightarrow F(D_E(C_n)) < F(DD_{VV}(C_n))$
- $F(D_E(C_n)) - F(DD_{EE}(C_n)) = -528n < 0$ $\Longrightarrow F(D_E(C_n)) < F(DD_{EE}(C_n))$
- $F(D_E(C_n)) - F(DD_{VE}(C_n)) = -600n < 0$ $\Longrightarrow F(D_E(C_n)) < F(DD_{VE}(C_n))$

case (ii): To obtain $D_V(C_n)$ as 2^{nd} minimum of F-index, compare it with $F(DD_{EV}(C_n))$, $F(DD_{VV}(C_n))$, $F(DD_{EE}(C_n))$ and $F(DD_{VE}(C_n))$.

- $F(D_V(C_n)) - F(DD_{EV}(C_n)) = -232n < 0$ $\Longrightarrow F(D_V(C_n)) < F(DD_{EV}(C_n))$
- $F(D_V(C_n)) - F(DD_{VV}(C_n)) = -312n < 0$ $\Longrightarrow F(D_V(C_n)) < F(DD_{VV}(C_n))$
- $F(D_V(C_n)) - F(DD_{EE}(C_n)) = -520n < 0$ $\Longrightarrow F(D_V(C_n)) < F(DD_{EE}(C_n))$

- $F(D_V(C_n)) - F(DD_{VE}(C_n)) = -592n < 0$
$\implies F(D_V(C_n)) - F(DD_{VE}(C_n))$

case (iii): To get $DD_{EV}(C_n)$ as 3^{rd} minimum of F-index, compare it with $F(DD_{VV}(C_n))$, $F(DD_{EE}(C_n))$ and $F(DD_{VE}(C_n))$.

- $F(DD_{EV}(C_n)) - F(DD_{VV}(C_n)) = -80n < 0$
$\implies F(DD_{EV}(C_n)) < F(DD_{VV}(C_n))$
- $F(DD_{EV}(C_n)) - F(DD_{EE}(C_n)) = -288n < 0$
$\implies F(D_E(C_n)) < F(D_{EE}(C_n))$
- $F(DD_{EV}(C_n)) - F(DD_{VE}(C_n)) = -360n < 0$
$\implies F(DD_{EV}(C_n)) < F(DD_{VE}(C_n))$.

case (iv): $DD_{VV}(C_n)$ is obtained as the 4^{th} minimum from the following comparison.
Let $DD_{VV}(C_n)$ be the forth minimum of F-index.

- $F(DD_{VV}(C_n)) - F(DD_{EE}(C_n)) = -208n < 0$
$\implies F(DD_{VV}(C_n)) < F(DD_{EE}(C_n))$
- $F(DD_{VV}(C_n)) - F(DD_{VE}(C_n)) = -328n < 0$
$\implies F(DD_{VV}(C_n)) < F(DD_{VE}(C_n))$

case (v): To get 5^{th} minimum as $F(DD_{EE}(C_n))$, compare it with $F(DD_{VE}(C_n))$.

- $F(DD_{EE}(C_n)) - F(DD_{VE}(C_n)) = -72n < 0$
$\implies F(DD_{EE}(C_n)) < F(DD_{VE}(C_n))$

From the above discussed cases, it follows $F(D_E(C_n)) < F(D_V(C_n)) < F(DD_{EV}(C_n)) < F(DD_{VV}(C_n)) < F(DD_{EE}(C_n)) < F(DD_{VE}(C_n))$. ☐

Theorem 3.5. *Let $S_n(n \geq 10)$ be a star graph with n vertices, then $F(D_V(S_n)) < F(DD_{VV}(S_n)) < F(D_E(S_n)) < F(DD_{EV}(S_n)) < F(DD_{VE}(S_n)) < F(DD_{EE}(S_n))$.*

Proof. A star graph S_n is a tree with one vertex of degree $(n-1)$ and all the remaining vertices with degree 1. To prove the inequalities, the following cases are to be verified.

case (i): To get the 1^{st} minimum as $F(D_V(S_n))$, compare it with $F(DD_{VV}(S_n))$, $F(D_E(S_n))$, $F(DD_{EV}(S_n))$, $F(DD_{VE}(S_n))$ and $F(DD_{EE}(S_n))$.

- $F(D_V(S_n)) - F(DD_{VV}(S_n)) = -6n^2 - 282n - 72 < 0$
$\implies F(D_V(S_n)) < F(DD_{VV}(S_n))$
- $F(D_V(S_n)) - F(D_E(S_n)) = -7n^3 + 27n^2 + 6n - 2 < 0$
$\implies F(D_V(S_n)) < F(D_E(S_n))$
- $F(D_V(S_n)) - F(DD_{EV}(S_n)) = -7n^3 + 3n^2 - 114n + 118 < 0$
$\implies F(D_V(S_n)) < F(D_{EV}(S_n))$
- $F(D_V(S_n)) - F(DD_{VE}(S_n)) = -7n^3 - 21n^2 - 354n + 190 < 0$
$\implies F(D_V(S_n)) < F(DD_{VE}(S_n))$
- $F(D_V(S_n)) - F(DD_{EE}(S_n)) = -63n^3 + 195n^2 - 298n + 62 < 0$
$\implies F(D_V(S_n)) < F(DD_{EE}(S_n))$

case (ii): To get $F(DD_{VV}(S_n))$ as the 2^{nd} minimum of the F-index, compare $F(DD_{VV}(S_n))$, $F(D_E(S_n))$,$F(DD_{EV}(S_n))$, $F(DD_{VE}(S_n))$ and $F(DD_{EE}(S_n))$.

- $F(DD_{VV}(S_n)) - F(D_E(S_n)) = -7n^3 + 33n^2 + 288n - 74 < 0$
$\implies F(DD_{VV}(S_n)) < F(D_E(S_n))$
- $F(D_{VV}(S_n)) - F(DD_{EV}(S_n)) = -7n^3 + 9n^2 + 168n + 46 < 0$
$\implies F(D_{VV}(S_n)) < F(DD_{EV}(S_n))$
- $F(D_{VV}(S_n)) - F(DD_{VE}(S_n)) = -7n^3 - 15n^2 - 72n + 118 < 0$
$\implies F(D_{VV}(S_n)) < F(DD_{VE}(S_n))$
- $F(D_{VV}(S_n)) - F(DD_{EE}(S_n)) = -63n^3 + 201n^2 - 16n + 118 < 0$
$\implies F(D_{VV}(S_n)) - F(DD_{EE}(S_n))$

case (iii): To obtain $D_E(S_n)$ as the 3^{rd} minimum of the F-index, compare $F(DD_{EV}(S_n))$,$F(DD_{VE}(S_n))$ and $F(DD_{EE}(S_n))$

- $F(D_E(S_n)) - F(DD_{EV}(S_n)) = -24n^2 - 120n + 120 < 0$
$\implies F(D_E(S_n)) < F(DD_{EV}(S_n))$
- $F(D_E(S_n)) - F(DD_{VE}(S_n)) = -48n^2 - 360n + 192 < 0$
$\implies F(D_E(S_n)) < F(D_{VE}(S_n))$
- $F(D_E(S_n)) - F(DD_{EE}(S_n)) = -56n^3 + 168n^2 - 304n + 192 < 0$
$\implies F(D_E(S_n)) < F(DD_{EE}(S_n))$.

case (iv): In order to get the 4^{th} minimum, it is necessary to compare $D_{EV}(S_n)$ with $F(DD_{VE}(S_n))$ and $F(DD_{EE}(S_n))$

- $F(D_{EV}(S_n)) - F(DD_{VE}(S_n)) = -24n^2 - 240n + 72 < 0$
$\implies F(D_{EV}(S_n)) < F(DD_{VE}(S_n))$
- $F(D_{EV}(S_n)) - F(DD_{EE}(S_n)) = -56n^3 + 192n^2 - 184n + 72 < 0$
$\implies F(D_{EV}(S_n)) < F(DD_{EE}(S_n))$

case (v): To obtain 5^{th} minimum, compare $F(DD_{VE}(S_n))$ with $F(DD_{EE}(S_n))$ and $F(DD_{VE}(S_n))$.

- $F(DD_{VE}(S_n)) - F(DD_{EE}(S_n)) = -56n^3 + 216n^2 + 56n < 0$
$\implies F(DD_{VE}(S_n)) < F(DD_{EE}(S_n))$

From the above discussed cases, it follows $F(D_V(S_n)) < F(DD_{VV}(S_n)) < F(D_E(S_n)) < F(DD_{EV}(S_n)) < F(DD_{VE}(S_n)) < F(DD_{EE}(S_n))$. ☐

Observation 3.5.1. The theorem 3.5 holds good for $n \geq 10$ and the following are discussed for $n \leq 9$.
(i) For $n = 2$; $F(D_E(S_n)) < F(D_V(S_n)) <$

$F(DD_{EE}(S_n)) < F(DD_{EV}(S_n)) < F(DD_{VV}(S_n)) < F(DD_{VE}(S_n))$.
(ii) For $n = 3$; $F(D_E(S_n)) < F(D_V(S_n)) < F(DD_{EV}(S_n)) < F(DD_{EE}(S_n)) < F(DD_{VV}(S_n)) < F(DD_{VE}(S_n))$.
(iii) For $n = 4$; $F(D_E(S_n)) < F(D_V(S_n)) < F(DD_{EV}(S_n)) < F(DD_{VV}(S_n)) < F(DD_{EE}(S_n)) < F(DD_{VE}(S_n))$.
(iv) For $n = 5$; $F(D_V(S_n)) < F(D_E(S_n)) < F(DD_{EV}(S_n)) < F(DD_{VV}(S_n)) < F(DD_{VE}(S_n)) < F(DD_{EE}(S_n))$.
(iii) For $n = 6, 7, 8$ and 9; $F(D_V(S_n)) < F(D_E(S_n)) < F(DD_{VV}(S_n)) < F(DD_{EV}(S_n)) < F(DD_{VE}(S_n)) < F(DD_{EE}(S_n))$.

4 Conclusion

In this article, some inequalities on duplication and double duplication graphs of Forgotten index of general graphs are proved. To study and analyze more inequalities using forgotten index is our future work.

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