Mathematics and Statistics 11(2): 373-378, 2023
DOI: 10.13189/ms.2023.110216 http://www.hrpub.org

# Investigation on Isotropic Bezier Sweeping Surface "IBSS" with Bishop Frame

**W. M. Mahmoud[1,*], M. A. Soliman[2], Esraa. M. Mohamed[1]**

[1]Department of Mathematics, Faculty of Science, Aswan University, Aswan, Egypt
[2]Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt

*Received December 7, 2022; Revised February 13, 2023; Accepted March 12, 2023*

***Cite This Paper in the Following Citation Styles***

***(a):*** *[1] W. M. Mahmoud, M. A. Soliman, Esraa. M. Mohamed , "Investigation on Isotropic Bezier Sweeping Surface "IBSS" with Bishop Frame," Mathematics and Statistics, Vol. 11, No. 2, pp. 373 - 378, 2023. DOI: 10.13189/ms.2023.110216.*

***(b):*** *W. M. Mahmoud, M. A. Soliman, Esraa. M. Mohamed (2023). Investigation on Isotropic Bezier Sweeping Surface "IBSS" with Bishop Frame. Mathematics and Statistics, 11(2), 373 - 378. DOI: 10.13189/ms.2023.110216.*



**Abstract** This research aims to study the Sweeping surface which is generated by the motion of the straight line (the profile curve) while this movement of the plane in the space is in the same direction as the normal to a cubic Bezier curve (spine curve). In geometrical modeling, sweeping is an essential and useful tool and has some applications, especially in geometric design. The idea depends on choosing a geometrical object which is the straight line, that is called the generator, and sweeping it along a cubic Bezier curve (spine curve), which is called trajectory, along the Cubic Bezier curve (spine curve) in an isotropic space has produced an Isotropic Bezier Sweeping Surfaces (IBSS). This study discusses Isotropic Bezier Sweeping Surfaces (IBSS) with the Bishop frame. We studied a special case of a surface sweep, which is the cylindrical surface resulting from a path curve that is a straight line. We have calculated the 1st fundamental and 2nd fundamental forms for this surface. The parametric description of the Weingarten Isotropic Bezier Sweeping Surfaces (IBSS) is also calculated in terms of Gaussian and mean curvatures. Mathematica 3D visualizations were used to create these curvatures. Finally, we characterized new associated surfaces according to the Bishop frame on (IBSS), such as studying minimal and developable isotropic Bezier sweeping surfaces (IBSS).

**Keywords** Bishop Frame, Bezier Curve, Isotropic Space, Sweeping Surface

## 1. Introduction

When the profile curve is traveling in the same direction as the normal plane, a sweeping surface has been created by the motion of a plane curve (also known as a generatrix) through space along a path curve [1],[2]. A cylindrical sweeping surface results from a path curvature that is a straight line. This study focuses on the geometrical properties of Isotropic Bezier Sweeping Surfaces (IBSS) with Bishop frames.

The use of the sweeping surface in numerous practical computer-aided geometric design applications makes it significant. The Frenet frame can be used to parameterize the sweeping surface. However, it has no definitions at inflection points or along straight curve stretches when the curvature disappears [3],[4].

L. Bishop began developing the Bishop frames in 1975 in an effort to develop some new frames with advantages over Serret-Frenet frames. Recently, Euclidean space has been used extensively in published papers dealing with this subject [4],[5].

All differential geometry, kinematics, robotics, and engineering require curve theory. Bezier curves are a popular type of curve. Bezier curves have been employed in numerous domains, including computer-aided geometric design (CAGD) [6],[7],[8]. Control points are distinctly identified for each Bezier curve [10]. The most well-liked research areas have been employed in CAGD to create Bezier curves using shape control parameters. The control points of a Bezier curve and surface could be determined to be linearly related [6],[7],[9],[10]. A Bezier curve was established by the control points, as it is well-known from the pertinent literature. When control points have been specified, we can use the Bernstein polynomial and the De Castaljeu algorithm.

In this study, we are able to derive new linked surfaces by employing the Bishop frame in isotropic space $E_3^1$ , we will study an Isotropic Bezier sweeping surface's characteristics.

## 2. Preliminaries

The fundamentals of differential geometry for space curves and sweeping surfaces in isotropic- space were reviewed in this section. Let $\rho$ be an isotropic regular curve indicated by $\{\mathrm{T}, M, \mathrm{B}\}$ and $\{\mathrm{T}, \mathrm{N}_1, \mathrm{N}_2\}$ the Bishop Frame (or parallel transport frame) and the Frenet frame, respectively, are located on the unit speed curve $\rho$ Then, Bishop Frame and Ferret formula are provided by

$$\begin{bmatrix} \acute{\mathrm{T}} \\ \acute{M} \\ \acute{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathrm{T} \\ M \\ \mathrm{B} \end{bmatrix} \tag{1}$$

And

$$\begin{bmatrix} \acute{\mathrm{T}} \\ \acute{\mathrm{N}}_1 \\ \acute{\mathrm{N}}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{T} \\ \mathrm{N}_1 \\ \mathrm{N}_2 \end{bmatrix} \tag{2}$$

After that, Frenet frame and Bishop frame have a relationship that is

$$\begin{bmatrix} \mathrm{T} \\ M \\ \mathrm{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} \mathrm{T} \\ \mathrm{N}_1 \\ \mathrm{N}_2 \end{bmatrix} \tag{3}$$

Where

$$\vartheta = \tan^{-1}\frac{k_2}{k_1}\ ,\ \tau = \acute{\vartheta}\ \ ,\ \kappa = \sqrt{{k_1}^2 + {k_2}^2} \tag{4}$$

$$\mathrm{N}_2 = T \wedge \mathrm{N}_1$$

Conversely, Bishop Darboux vector and Bishop curvatures are described by

$$\left.\begin{matrix} k_1 = \kappa\cos\vartheta \\ k_2 = \kappa\sin\vartheta \end{matrix}\right] \tag{5}$$

The curve $\rho$ is a slant helix if the unit vector $\mathrm{N}_2$ forms a constant angle with the fixed unit vector. In addition, if and only if $\frac{k_2}{k_1}$ is constant, $\rho$ is a slant helix [11].

Metrics and movements Isotropic geometry are based on the $\boldsymbol{G_6}$ affine transformations $(x,y,z) \to (\acute{x},\acute{y},\acute{z})$ in $R^3$ [11]

$$\left.\begin{matrix} \acute{x} \ = \ p_1 \ + \ x\cos(\psi) - \ y\sin(\psi) \\ \acute{y} \ = \ p_2 \ + \ x\sin(\psi) \ + \ y\cos(\psi) \\ \acute{z} \ = \ p_3 \ + \ p_4\, x \ + \ p_5\, y \ + \ z \end{matrix}\right] \tag{6}$$

Where $p_1, p_2, p_3, p_4, p_5, \psi \in \mathbb{R}$

Affine transformations are called isotropic congruence transformations or isotropic motions, The Euclidean distance between two points $P = (\lambda_1, \mu_1, \chi_1)$ and $Q = (\lambda_2, \mu_2, \chi_2)$ was defined as their isotropic distance.

$$d(P,Q) = \sqrt{(\lambda_1 - \lambda_2)^2 + (\mu_1 - \mu_2)} \tag{7}$$

In $I_3^1$, let points $X = (\lambda_1, \mu_1, \chi_1)$ and $Y = (\lambda_2, \mu_2, \chi_2)$. $X$ and 's isotropic inner product was defined by

$$\langle X, Y\rangle = \begin{cases} \chi_1\,\chi_2 & if\ \lambda_i = \mu_i \\ \lambda_1\lambda_2 + \mu_1\,\mu_2 & if\ otherwise \end{cases} \tag{8}$$

The 1st fundamental form I and the 2nd fundamental form II of the sweeping surfaces we are given by

$$I = g_{11}d\eta^2 + 2g_{12}\,d\eta\,d\xi + g_{22}d\xi^2 \tag{9}$$

$$g_{11} = \langle\psi_\eta, \psi_\eta\rangle\ \ ,\ g_{12} = \langle\psi_\eta, \psi_\xi\rangle\ ,\ g_{22} = \langle\psi_\xi, \psi_\xi\rangle \tag{10}$$

$$II = h_{11}d\eta^2 + 2h_{12}\,d\eta d\xi + h_{22}d\xi^2 \tag{11}$$

$$h_{11} = \langle\psi_{\eta\eta}, n\rangle\ , h_{12} = \langle\psi_{\eta\xi}, n\rangle\ , h_{22} = \langle\psi_{\xi\xi}, n\rangle \tag{12}$$

$n = (0,0,1)$ denotes the isotropic unit normal vector field, the Gaussian Curvature $\mathcal{K}$ and the mean curvature $\mathcal{H}$ of cylindrical surface $\mathcal{M}$ have been given by, respectively,

$$\mathcal{K} = \frac{h_{11}h_{22} - {h_{12}}^2}{g_{11}g_{22} - {g_{12}}^2}\ ,\ \mathcal{H} = \frac{h_{11}g_{22} + g_{11}h_{22} - 2\,g_{12}h_{12}}{2(g_{11}g_{22} - {g_{12}}^2)} \tag{13}$$

Surfaces having no Gaussian curvature could be developable. If and only if the mean curvature vanishes identically, a surface $\mathcal{M} \subset \mathbb{R}^3$ counts as minimal [11],[2],[3].

A Bezier Curve is composed of Bernstein polynomial and control points and is defined as

$$\rho(\eta) = \sum_{i=0}^{n} P_i\, B_{i.n}(\eta)\ \ ,\eta \in [0,1] \tag{14}$$

Where $P_i$ is the i<sup>th</sup> control point, $n$ is the polynomial order of the curve, and $B_{i.n}(\eta)$ is the basis function, called Bernstein polynomial and defined as

$$B_{i.n}(\eta) = \binom{n}{i}\eta^i(1-\eta)^{n-i}\ \ ,\ \eta \in [0,1] \tag{15}$$

$\binom{n}{i}$ is the binomial coefficient defined as

$$\binom{n}{i} = \begin{cases} \frac{n!}{i!(n-i)!} & 0 \le i \le n \\ 0 & else \end{cases} \tag{16}$$

Bernstein polynomial become $B_0^0(\eta) = 1$ and $B_j^n(\eta) = 1$ for $j \notin \{0, \dots, n\}$ in special conditions. For cubic Bezier curve $n = 3$ and the curve defined by four control points. It is derivable from [13]

$$\rho(\eta) = \begin{bmatrix} \eta^3 & \eta^2 & \eta & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{17}$$

Given four control points, $P_0 = [0,0,0], P_1 = [0,-1,1]$, $P_2 = [0,0,1], P_0 = [0,3,0]$, then a cubic Bezier curve on the $\mathrm{yz-plane}$ defined by

$$\rho(\eta) = (0,\ 6\eta^2 - 3\eta, -3\eta^2 + 3\eta) \tag{18}$$

# 3. Bezier Sweeping Surface in Isotropic Space

The cylindrical surface is considered a special state of the sweeping surface. The swept surface formed is a cylindrical surface if the path curve is a straight line, and could be written in the form:

$$\mathcal{M}:(\eta,\xi)=\rho(\eta,\xi)+\xi w \quad (19)$$

$w$ is the straight line's vector, while $\eta$ and $\xi$ are the surface's parametric coordinates. we studied the cylindrical surface generated by the Bezier curve, and the cylindrical surface generated by extruding a cubic bezier curve on the yz − plane along the x-direction for 5 units

$$\rho(\eta)=(0,\ 6\eta^2-3\eta,\ -3\eta^2+3\eta) \quad (20)$$

The extrusion vector $w$ is

$$w=\{5,0,0\} \quad (21)$$

the parametric equation of the Bezier cylindrical surface is

$$\psi(\eta,\xi)=(5\xi\ ,\ 6\eta^2-3\eta,\ -3\eta^2+3\eta) \quad (22)$$

### 3.1. The Properties of IBSS

Within this section, we will get acquainted with the properties of the surface in terms of the minimal surface, the developable surface and some other characteristics.

To compute the 1st fundamental form of $\mathcal{M}$, we have to calculate the following

$$\psi_\eta=\Big(3\eta\big((1-2\eta)k_1(\eta)+(-1+\eta)k_2(\eta)\big),\ -3+12\eta+5\xi k_1(\eta),\ 3-6\eta+5\xi k_2(\eta)\Big)$$

$$\psi_\xi=\{5,0,0\}$$

The coefficients of 1st fundamental form are equal.

$$g_{11}=(-3+12\eta+5\xi k_1(\eta))^2+9\eta^2((-1+2\eta)k_1(\eta)-(-1+\eta)k_2(\eta))^2$$

$$g_{12}=-15\eta((-1+2\eta)k_1(\eta)-(-1+\eta)k_2(\eta))$$

$$g_{22}=25$$

To compute the 2nd fundamental form of $\mathcal{M}$, we have to calculate the following

$$\psi_{\eta\eta}=((6-24\eta)k_1(\eta)-5\xi k_1(\eta)^2+6(-1+2\eta)k_2(\eta)-5\xi k_2(\eta)^2+3\eta\big((1-2\eta)k_1{}'(\eta)+(-1+\eta)k_2{}'(\eta)\big),\ 12+3\eta k_1(\eta)\big((1-2\eta)k_1(\eta)+(-1+\eta)k_2(\eta)\big)+5\xi k_1{}'(\eta),\ -6+3\eta k_2(\eta)((1-2\eta)k_1(\eta)+(-1+\eta)k_2(\eta))+5\xi k_2{}'(\eta))$$

$$\psi_{\eta\xi}=(0,\ 5k_1(\eta),\ 5k_2(\eta))\ \psi_{\xi\xi}=(0,\ 0,\ 0)$$

As a result, the 2nd fundamental form element is as follows:

$$h_{11}=-6+3\eta k_2(\eta)((1-2\eta)k_1(\eta)+(-1+\eta)k_2(\eta))+5\xi k_2{}'(\eta)$$

$$h_{12}=5k_2(\eta)\ ,\ h_{22}=0$$

Then, Figures (1) and (2) illustrate the Gaussian Curvature $\mathcal{K}$ and the mean curvature $\mathcal{H}$ , respectively, as follows

$$\mathcal{K}=-\frac{k_2(\eta)^2}{\left(-3+12\eta+5\xi k_1(\eta)\right)^2} \quad (23)$$

$$\mathcal{H}=\frac{-6-3\eta k_2(\eta)\big((1-2\eta)k_1(\eta)+(-1+\eta)k_2(\eta)\big)}{2\left(-3+12\eta+5\xi k_1(\eta)\right)^2}+\frac{5\xi k_2{}'(\eta)}{2\left(-3+12\eta+5\xi k_1(\eta)\right)^2} \quad (24)$$

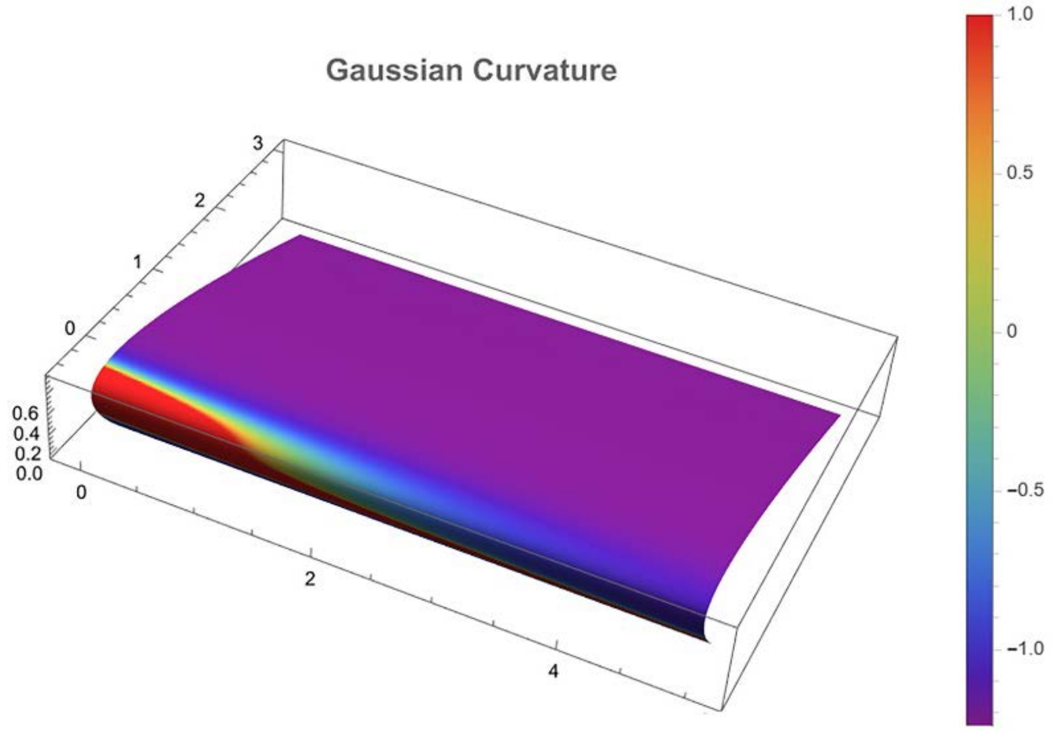


**Figure 1.** Gaussian curvature function graphic above and the variations of Gaussian curvature on isotropic Bezier sweeping surface (23). In the color gradient, the maximum value of gaussian curvature $(\mathcal{K} > 0)$ is in the red color, the minimum value in the violet color $(\mathcal{K} < 0)$, and it in the milky color the gaussian curvature vanishes $(\mathcal{K} = 0)$

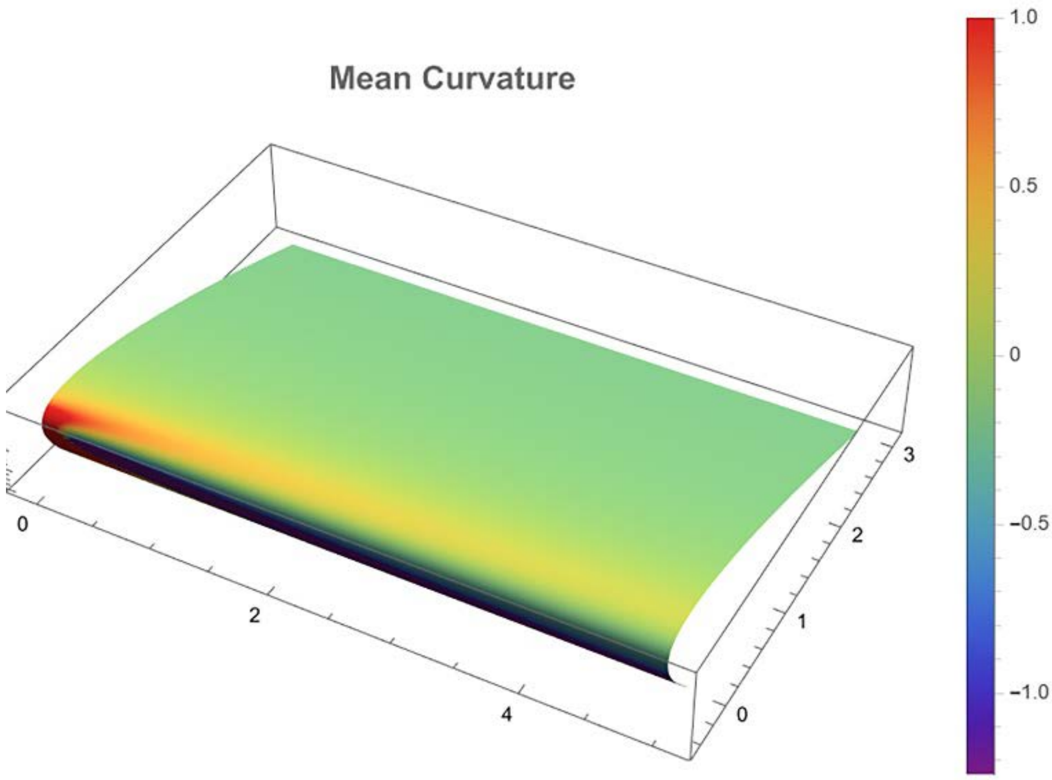


**Figure 2.** Mean curvature function graphic above and the variations of mean curvature on isotropic Bezier sweeping surface (24). In the color gradient, the maximum value of mean curvature $(\mathcal{H} > 0)$ is in the red color, the minimum value in the violet color $(\mathcal{H} < 0)$, and it in the milky color the mean curvature vanishes $(\mathcal{H} = 0)$

**Theorem 3.1.1** Let $\mathcal{M}:\psi(\eta,\xi)$ be an Isotropic Bezier sweeping surface, then it is a developable surface if and only if

$$k_2(\eta)=0$$

Proof: - Isotropic Bezier sweeping surface satisfies the equation $\mathcal{K}(\eta,\xi)=0$, from the Eq. (23) we get

$$0=-\frac{k_2(\eta)^2}{(-3+12\eta+5\xi k_1(\eta))^2}$$

We have

$$k_2(\eta)=0$$

Then The Bezier sweeping surface $\mathcal{M}$ is developable.

**Theorem 3.1.2** The Isotropic Bezier sweeping surface $\mathcal{M}$ is minimal surface under the following condition:

$$k_1(\eta)=\frac{6-3\eta k_2(\eta)^2+3\eta^2k_2(\eta)^2-5\xi k_2{}'(\eta)}{3\eta(-1+2\eta)k_2(\eta)}$$

Proof: - The Isotropic Bezier sweeping surface satisfies $\mathcal{H}(\eta,\xi)=0$, From the Eq. (24) we have

$$0=\frac{-6-3\eta k_2(\eta)((1-2\eta)k_1(\eta)+(-1+\eta)k_2(\eta))+5\xi k_2{}'(\eta)}{2(-3+12\eta+5\xi k_1(\eta))^2}$$

We get

$$0=-6-3\eta k_2(\eta)((1-2\eta)k_1(\eta)+(-1+\eta)k_2(\eta))+5vk_2{}'(\eta)$$

Then

$$k_1(\eta)=\frac{6-3\eta k_2(\eta)^2+3\eta^2k_2(\eta)^2-5vk_2{}'(\eta)}{3\eta(-1+2\eta)k_2(\eta)}$$

The IBSS $\mathcal{M}$ is a minimal surface.

**Theorem 3.1.3** The isotropic Bezier sweeping surface given by (22) is Weingarten surface if

$$k_2(\eta)=0$$

Proof:- Weingarten by differential Eq. (23),(24)

$$\mathcal{H}_\eta=\left(\frac{1}{2(-3+12\eta+5\xi k_1[\eta])^3}\right)\Bigg((-2\Big(12+5\xi k_1{}'(\eta)\Big)\Big(-6-3\eta k_2(\eta)\big((1-2\eta)k_1(\eta)+(-1+\eta)k_2(\eta)\big)+5\xi k_2{}'(\eta)\Big)+\big(-3+12\eta+5\xi k_1(\eta)\big)\Big(-3k_2(\eta)\Big((1-4\eta)k_1(\eta)+(-1+2\eta)\big(k_2(\eta)-\eta k_1{}'(\eta)\big)\Big)\Big)-3\eta((1-2\eta)k_1(\eta)+2(-1+\eta)k_2(\eta))k_2{}'(\eta)+5\xi k_2{}''(\eta))\Bigg)$$

$$\mathcal{H}_\xi=\frac{30k_1(\eta)(2+\eta k_2(\eta)((1-2\eta)k_1(\eta)+(-1+\eta)k_2(\eta)))}{2(-3+12\eta+5\xi k_1(\eta))^3}+\frac{5(-3+12\eta-5\xi k_1(\eta))k_2{}'(\eta)}{2(-3+12\eta+5\xi k_1(\eta))^3}$$

$$\mathcal{K}_\eta=\frac{2k_2(\eta)(k_2(\eta)(12+5\xi k_1{}'(\eta))+(3-12\eta-5\xi k_1(\eta))k_2{}'(\eta))}{(-3+12\eta+5\xi k_1(\eta))^3}$$

$$\mathcal{K}_\xi=\frac{10k_1(\eta)k_2(\eta)^2}{(-3+12\eta+5\xi k_1(\eta))^3}$$

By apply$\mathcal{K}_\eta\mathcal{H}_\xi-\mathcal{K}_\xi\mathcal{H}_\eta=0$, this equation leads to

$$5k_2(\eta)\Bigg(k_2{}'(\eta)\Big(k_2(\eta)\big(12+5\xi k_1{}'(\eta)\big)+3(1-4\eta)k_2{}'(\eta)\Big)+3k_1(\eta)^2k_2(\eta)\Big((1-4\eta)k_2(\eta)+\eta(-1+2\eta)k_2{}'(\eta)\Big)+k_1(\eta)\Bigg(k_2{}'(\eta)\Big(-12+5\xi k_2{}'(\eta)\Big)+k_2(\eta)\Big(3(-1+2\eta)k_2(\eta)\Big(k_2(\eta)-\eta k_1{}'(\eta)\Big)-5\xi k_2{}''(\eta)\Big)\Bigg)\Bigg)=0$$

Then we have

$$k_2(\eta)=0$$

Then the isotropic Bezier sweeping surface is the Weingarten surface.

# 4. Conclusions

This paper studied the Bishop frame that is associated with a Bezier curve on the sweeping surfaces that are generated by this curve. Additionally, the conditions for surfaces to be both minimal and developable are analyzed. Finally, we looked at some of this surface's characteristics.

# Acknowledgments

The author expresses gratitude to the referee for his insightful comments, which helped to improve the paper's final draft.

# REFERENCES

[1] Nazra S. H, Abdel-Baky R. A, "Sweeping Surfaces in the Three-Dimensional Lie Group," Symmetry, vol.14, no.4, pp.698, 29 March 2022. https://doi.org/10.3390/sym14040

698

[2] MOFARREH F., ABDEL-BAKY R., ALLUHAIBI N., "SWEEPING SURFACES WITH DARBOUX FRAME IN EUCLIDEAN 3-SPACE $E^3$, " vol. 18, no. 1, pp. 10, 2021.

[3] Nazra S. H, Abdel-Baky R. A., "Time-Like Sweeping Surfaces with a Bishop Frame in the Minkowski 3-Space," Mathematical Problems in Engineering, vol. 2022, 2022. https://doi.org/10.1155/2022/4189643

[4] Dede M., Ekici C., Tozak H., "Directional tubular surfaces," International Journal of Algebra, vol. 9, no. 12, pp. 527-535, 2015.

[5] Ebrahimi M., Butscher A., Cheong H., "A low order, torsion deformable spatial beam element based on the absolute nodal coordinate formulation and Bishop frame," Multibody System Dynamics, vol. 51, no. 3, pp. 247-278, 2021.

[6] Said Mad Zain S. A. A. A., Misro M. Y., Miura K. T., "Generalized fractional Bézier curve with shape parameters," Mathematics, vol. 9, no. 17, pp. 2141, 2021, https://doi.org/10.3390/math9172141

[7] Baydas S., Karakas B., "Defining a curve as a Bezier curve," Journal of Taibah University for Science, vol. 13, no. 1, pp. 522-528, 2019. https://doi.org/10.1080/16583655.2019.1601913

[8] Vassil K., Bazanovk B., "The effect of plyometric training program on young volleyball players in their usual training period," Journal of Human Sport and Exercise, vol. 7, no. 1, pp. S34-S40, 2012.

[9] Berry, T. G., Patterson R. R., "The Uniqueness Of Bézier Control Points," Computer Aided Geometric Design, vol. 14, no. 9, pp. 877-879, 1997.

[10] Hu G., Wu J., Qin X., "A novel extension of the Bézier model and its applications to surface modeling," Advances in Engineering Software, vol. 125, pp. 27-54, 2018.

[11] Mahmoud W. M., Abd ElHafez S. M., "Weingarten Isotropic Embankment Surfaces According to Adapted Frame," Information Sciences Letters, vol. 11, no. 1, pp. 12, 2022.

[12] Masal M., Azak Masal A. Z., "Ruled surfaces according to Bishop frame in the Euclidean 3-space," Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, vol. 89, no.2, pp. 415-424, 2019.

[13] Chang K. H., "GEOMETRIC MODELING," in e-Design: computer-aided engineering design., 1 st ed, Academic Press, 2016, pp. 43-124.