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Numerical Solution of Linear and Nonlinear Second Order Initial Value Problems Using Three-Step Generalized Off-Step Hybrid Block Method

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Abstract The numerical of second order initial value problems (IVPs) has garnered a lot of attention in literature, with recent studies ensuring to develop new methods with better accuracy than previously existing approaches. This led to the introduction of hybrid block methods which is a class of block methods capable of directly solving second order IVPs without reduction to a system of first order IVPs. Its hybrid characteristic features the addition of off-step points in the derivation of this block method, which has shown remarkable improvement in the accuracy of the block method. This article proposes a new three-step hybrid block method with three generalized off-step points to find the direct solution of second order IVPs. To derive the method, a power series is adopted as an approximate solution and is interpolated at the initial point and one off-step point while its second derivative is collocated at all points in the interval to obtain the main continuous scheme. The analysis of the method shows that the developed method is of order 7, zero-stable, consistent, and hence convergent. The numerical results affirm that the new method performs better than the existing methods it is compared with, in terms of error accuracy when solving the same IVPs of second order ordinary differential equations.

Keywords Linear, Nonlinear, Second Order, Initial Value Problems, Three-Step, Generalized Off-Step, Hybrid Block Method

1. Introduction

Consider the following second order initial value problem (IVP) of ordinary differential equations (ODEs)

$$y'' = f(x, y, y'), y(a) = \eta_0, y'(a) = \eta_1 \quad (1)$$

with $x \in [a, b]$. Equation (1) could either be linear or nonlinear depending on the properties of the dependent variable, and various numerical methods have been developed to solve either linear, nonlinear, or both. Some of these numerical methods are seen in studies such as [1] where second order Bratu-type IVPs were solved using a sixth order Runge-Kutta method while [2] adopted predictor-corrector method of the same type of IVPs. [3] developed a new class of variable coefficients and two-step semi-hybrid methods to solve problems in the form of Equation (1), [4] enhanced the conventional Numerov method to solve second order IVPs with better accuracy, and [5] solved Equation (1) problems using a novel Lie group based neural network method. In [6], a review of some numerical methods for solving second order IVPs was conducted and the methods under consideration were the third order convergence numerical method, successive approximation method and Adomian decomposition method. Other studies that have considered the numerical solution of linear, nonlinear, or both, for second order IVPs include [7-11]. In their works, they adopted numerical approaches such as the use of Bernoulli polynomials, hybrid linear multistep methods, predictor-corrector methods, and block methods. However, there is still room for improvement in terms of accuracy. Hence, the aim of this article is to develop a new three-step hybrid block method with generalized off-step points for solving equations in the form of (1).

2. Development of the Method

Assume that the approximate solution of Equation (1) is the power series polynomial

$$y(x) = \sum_{j=0}^{i+c-1} a_j \left(\frac{x-x_n}{h}\right)^j \tag{2}$$

where $x \in (x_n, x_{n+3}]$ for $n = 0,3,6,\ldots,N-3, h = x_{\delta+1} - x_\delta, \delta = 0,1,2,\ldots,N$ in the interval $[a,b]$, with i and c denoting the number of interpolation and collocation points, respectively.

The next step in deriving the proposed method is finding the second derivative of Equation (2) which is given by

$$y''(x) = f(x,y,y') = \sum_{j=2}^{i+c-1} a_j \frac{j(j-1)}{h^2}\left(\frac{x-x_n}{h}\right)^{j-2}. \tag{3}$$

Combining both interpolating Equation (2) at x_n, x_{n+p} and collocating Equation (3) at all points in the interval produces nine equations which can be represented in the following matrix form

$$AX = B \tag{4}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & p & p^2 & p^3 & p^4 & p^5 & p^6 & p^7 & p^8 \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{6p}{h^2} & \frac{12p^2}{h^2} & \frac{20p^3}{h^2} & \frac{30p^4}{h^2} & \frac{42p^5}{h^2} & \frac{56p^6}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6q}{h^2} & \frac{12q^2}{h^2} & \frac{20q^3}{h^2} & \frac{30q^4}{h^2} & \frac{42q^5}{h^2} & \frac{56q^6}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6r}{h^2} & \frac{12r^2}{h^2} & \frac{20r^3}{h^2} & \frac{30r^4}{h^2} & \frac{42r^5}{h^2} & \frac{56r^6}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{12}{h^2} & \frac{48}{h^2} & \frac{160}{h^2} & \frac{480}{h^2} & \frac{1344}{h^2} & \frac{3584}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{18}{h^2} & \frac{108}{h^2} & \frac{540}{h^2} & \frac{2430}{h^2} & \frac{10206}{h^2} & \frac{40824}{h^2} \end{pmatrix}$$

$$X = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix}, \quad and \ B = \begin{pmatrix} y_n \\ y_{n+p} \\ f_n \\ f_{n+p} \\ f_{n+q} \\ f_{n+r} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix}.$$

To obtain the values of $a_j, (j = 0,1,\ldots,8)$, Equation (4) is solved using Gaussian elimination and then substituted back into Equation (2) to get the approximate solution. The derivation of the new method can be described by the following procedures.

Procedure 1: Evaluating approximate solution equation at the non-interpolating point, i.e., at $x_{n+q}, x_{n+r}, x_{n+1}, x_{n+2}$ and x_{n+3}.

Procedure 2: Evaluating the first derivative of approximate solution equation at all points, i.e., at $x_n, x_{n+p}, x_{n+q}, x_{n+r}, x_{n+1}, x_{n+2}$, and x_{n+3}.

Procedure 3: Combining equations from Procedure 1 and Procedure 2 to give a block of the form as follows.

$$\begin{aligned} A^{3[3]_2} Y^{3[3]_2} = B_1^{3[3]_2} R_1^{3[3]_2} + B_2^{3[3]_2} R_2^{3[3]_2} \\ + h^2 \Big[D^{3[3]_2} R_3^{3[3]_2} \\ + E^{3[3]_2} R_4^{3[3]_2} \Big] \end{aligned} \tag{5}$$

where

$$Y^{3[3]_2} = \begin{pmatrix} y_{n+p} \\ y_{n+q} \\ y_{n+r} \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix}, R_1^{3[3]_2} = \begin{pmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}, R_2^{3[3]_2} = \begin{pmatrix} y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix},$$

$$R_3^{3[3]_2} = \begin{pmatrix} f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}, R_4^{3[3]_2} = \begin{pmatrix} f_{n+p} \\ f_{n+q} \\ f_{n+r} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix},$$

$$A^{3[3]_2} = \frac{1}{hp}\begin{pmatrix} -hq & 1 & 0 & 0 & 0 & 0 \\ -hr & 0 & 1 & 0 & 0 & 0 \\ -h & 0 & 0 & 1 & 0 & 0 \\ -2h & 0 & 0 & 0 & 1 & 0 \\ -3h & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_1^{3[3]_2} = \frac{1}{hp}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & h(p-q) \\ 0 & 0 & 0 & 0 & 0 & h(p-r) \\ 0 & 0 & 0 & 0 & 0 & h(p-1) \\ 0 & 0 & 0 & 0 & 0 & h(p-2) \\ 0 & 0 & 0 & 0 & 0 & h(p-3) \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$B_2^{3[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$D^{3[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & D_{16} \\ 0 & 0 & 0 & 0 & 0 & D_{26} \\ 0 & 0 & 0 & 0 & 0 & D_{36} \\ 0 & 0 & 0 & 0 & 0 & D_{46} \\ 0 & 0 & 0 & 0 & 0 & D_{56} \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{pmatrix},$$

$$E^{3[3]_2} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} \\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46} \\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} \\ E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66} \end{pmatrix}.$$

The elements of matrix $D^{3[3]_2}$ and matrix $E^{3[3]_2}$ are given in Appendix A.

Now, multiplying (5) by the inverse of $A^{3[3]_2}$ gives

$$I^{3[3]_2}Y^{3[3]_2} = \bar{B}_1^{3[3]_2}R_1^{3[3]_2} + \bar{B}_2^{3[3]_2}R_2^{3[3]_2} + h^2\left[\bar{D}^{3[3]_2}R_3^{3[3]_2} + \bar{E}^{3[3]_2}R_4^{3[3]_2}\right] \quad (6)$$

where

$$I^{3[3]_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\bar{B}_1^{3[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\bar{B}_2^{3[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & hp \\ 0 & 0 & 0 & 0 & 0 & hq \\ 0 & 0 & 0 & 0 & 0 & hr \\ 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 2h \\ 0 & 0 & 0 & 0 & 0 & 3h \end{pmatrix},$$

$$\bar{D}^{3[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \bar{D}_{16} \\ 0 & 0 & 0 & 0 & 0 & \bar{D}_{26} \\ 0 & 0 & 0 & 0 & 0 & \bar{D}_{36} \\ 0 & 0 & 0 & 0 & 0 & \bar{D}_{46} \\ 0 & 0 & 0 & 0 & 0 & \bar{D}_{56} \\ 0 & 0 & 0 & 0 & 0 & \bar{D}_{66} \end{pmatrix},$$

$$\bar{E}^{3[3]_2} = \begin{pmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} & \bar{E}_{14} & \bar{E}_{15} & \bar{E}_{16} \\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} & \bar{E}_{24} & \bar{E}_{25} & \bar{E}_{26} \\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} & \bar{E}_{34} & \bar{E}_{35} & \bar{E}_{36} \\ \bar{E}_{41} & \bar{E}_{42} & \bar{E}_{43} & \bar{E}_{44} & \bar{E}_{45} & \bar{E}_{46} \\ \bar{E}_{51} & \bar{E}_{52} & \bar{E}_{53} & \bar{E}_{54} & \bar{E}_{55} & \bar{E}_{56} \\ \bar{E}_{61} & \bar{E}_{62} & \bar{E}_{63} & \bar{E}_{64} & \bar{E}_{65} & \bar{E}_{66} \end{pmatrix},$$

which leads to the following equations and the elements of $\bar{D}^{3[3]_2}$ and $\bar{E}^{3[3]_2}$ are shown in Appendix B.

$$\begin{aligned} y_{n+p} = y_n + hpy'_n + h^2\Big[&\frac{p^2}{5040qr}(168p^2 - 154p^3 + 48p^4 - 5p^5) - 420pq - 84p^3q + 308p^2q + 8p^4q \\ &- 420pr + 308p^2r - 84p^3r + 8p^4r + 1680qr - 770pqr + 168p^2qr \\ &- 14p^3qr)f_n + \frac{p^4}{1680(-1+p)(-1+q)(-1+r)}(84p^2 - 40p^3 + 5p^4 - 168pq \\ &+ 70p^2q - 8p^3q - 168pr + 70p^2r - 8p^3r + 420qr - 140pqr + 14p^2qr)f_{n+1} \\ &- \frac{p^4}{1680(-2+p)(q-2)(r-2)}(42p^2 - 32p^3 + 5p^4 - 84pq + 56p^2q - 8p^3q - 84pr \\ &+ 56p^2r - 8p^3r + 210qr - 112pqr + 14p^2qr)f_{2+n} + \frac{p^4}{5040(p-3)(q-3)(r-3)}(28p^2 \\ &- 24p^3 + 5p^4 - 56pq + 42p^2q - 8p^3q - 56pr + 42p^2r - 8p^3r + 140qr - 84pqr \\ &+ 14p^2qr)f_{n+3} + \frac{p^2}{840(11p - 6 - 6p^2 + p^3)(p-q)(p-r)}(-252p^2 + 308p^3 - 120p^4 \\ &+ 15p^5 + 420pq - 462p^2q + 168p^3q - 20p^4q + 420pr - 462p^2r + 168p^3r \\ &- 20p^4r - 840qr + 770pqr - 252p^2qr + 28p^3qr)f_{n+p} \\ &- \frac{p^4}{840(-1+q)q(-p+q)(6-5q+q^2)(q-r)}(-168p + 154p^2 - 48p^3 + 5p^4 + 420r \\ &- 308pr + 84p^2r - 8p^3r)f_{n+q} + \frac{p^4}{840(q-r)(-1+r)r(-p+r)(6-5r+r^2)}(154p^2 \\ &- 168p - 48p^3 + 5p^4 + 420q - 308pq + 84p^2q - 8p^3q)f_{n+r}\Big] \end{aligned} \quad (7)$$

$$y_{n+q} = y_n + hqy'_n + h^2\Bigg[\frac{q^2}{5040pr}(-420pq + 168q^2 + 308pq^2 - 154q^3 - 84pq^3 + 48q^4 + 8pq^4 - 5q^5 + 1680pr - 420qr - 770pqr + 308q^2r + 168pq^2r - 84q^3r - 14pq^3r + 8q^4r)f_n - \frac{q^4}{1680(-1+p)(-1+q)(r-1)}(168pq - 84q^2 - 70pq^2 + 40q^3 + 8pq^3 - 5q^4 - 420pr + 168qr + 140pqr - 70q^2r - 14pq^2r + 8q^3r)f_{n+1} + \frac{q^4}{1680(p-2)(q-2)(r-2)}(84pq - 42q^2 - 56pq^2 + 32q^3 + 8pq^3 - 5q^4 - 210pr + 84qr + 112pqr - 56q^2r - 14pq^2r + 8q^3r)f_{2+n} - \frac{q^4}{5040(-3+p)(-3+q)(-3+r)}(56pq - 28q^2 - 42pq^2 + 24q^3 + 8pq^3 - 5q^4 - 140pr + 56qr + 84pqr - 42q^2r - 14pq^2r + 8q^3r)f_{3+n} - \frac{q^4}{840p(-6+11p-6p^2+p^3)(p-q)(p-r)}(154q^2 - 168q - 48q^3 + 5q^4 + 420r - 308qr + 84q^2r - 8q^3r)f_{n+p} - \frac{q^2}{840(-1+q)(-p+q)(6-5q+q^2)(q-r)}(252q^2 - 420pq + 462pq^2 - 308q^3 - 168pq^3 + 120q^4 + 20pq^4 - 15q^5 + 840pr - 420qr - 770pqr + 462q^2r + 252pq^2r168q^3r - 28pq^3r + 20q^4r)f_{n+q}\frac{q^4(6-5r+r^2)^{-1}}{840(q-r)(r-1)r(r-p)}(420p - 168q - 308pq + 154q^2 + 84pq^2 - 48q^3 - 8pq^3 + 5q^4)f_{n+r} \tag{8}$$

$$y_{n+r} = y_n + hry'_n + h^2\Bigg[\frac{r^2}{5040pq}(1680pq - 70(6q + p(6+11q))r + 28(6 + 11q + p(11 + 6q))r^2 14(11 + 6q + p(6+q))r^3 + 8(6+p+q)r^4 - 5r^5)f_n + \frac{r^2}{5040}\Bigg(\frac{3r^2}{(p-1)(q-1)(r-1)}(420pq\ 28(6q + p(6+5q))r + 14(6 + 5q + p(5+q))r^2 - 8(5+pq)r^3 + 5r^4)f_{n+1} - \frac{3r^2}{(p-2)(q-2)(r-2)}(210pq - 28(3q + p(3+4q))r + 14(3 + 4q + p(4+q))r^2 - 8(4+p+q)r^3 + 5r^4)f_{n+2} + \frac{r^2}{(p-3)(q-3)(r-3)}(140pq - 28(2q + p(2+3q))r + 14(2 + 3q + p(3+q))r^2 8(3+p+q)r^3 + 5r^4)f_{n+3} + \frac{6r^2}{(p-3)(p-2)(p-1)p(p-q)(p-r)}(-(r-4)r(42 + r(5r-28)) + 4q(r(77 + r(-21+2r)) - 105))f_{n+p} + \frac{6r^2}{(p-q)(-3+q)(-2+q)(-1+q)q(q-r)}((r-4)r(42 + r(5r-28)) + p(420 - 4r(77 + r(2r-21))))f_{n+q} + \frac{6}{(q-r)(r-3)(r-2)(r-1)(r-p)}(-70(6q + p(6+11q))r + 840pq + 42(6 + 11q + p(11 + 6q))r^2 - 28(11 + 6q + p(6+q))r^3 + 20(6+p+q)r^4 - 15r^5)f_{n+r})\Bigg] \tag{9}$$

$$y_{n+1} = y_n + hy'_n + h^2\Big[\frac{1}{5040pqr}(-49 + q\,(106 - 294r) + 106r + 2p\,(53 - 147r + 7q(97r - 21)))\,f_n + \frac{1}{5040}\Big(\frac{3}{(p-1)\,(q-1)\,(r-1)}(q\,(132 - 83 - 238r) + 132r + 2p\,(66 - 119r + 7q\,(-17 + 38r)))f_{n+1} - \frac{3}{(-2+p)\,(-2+q)\,(-2+r)}\,3\,(-19 + q\,(34 - 70r) + 34r + 2p(17 - 35r + 7q(-5 + 13r)))f_{2+n} + \frac{(-11 + q(20 - 42r) + 20r + 2p(10 - 21r + 7q(-3 + 8r)))}{(-3+p)(-3+q)(-3+r)}f_{n+3} + \frac{6(106q - 49 + 106r - 294qr)}{p\,(-6 + 11p - 6p^2 + p^3)\,(p-q)\,(p-r)}f_{n+p} + \frac{6}{(p-q)\,q\,(-6 + 11q - 6q^2 + q^3)(q-r)}(49 - 106p - 106r + 294pr)f_{n+q} + \frac{6}{r(r-p)(r-q)(11r - 6 - 6r^2 + r^3)}(106p - 49 + 106q - 294pq)f_{n+r}\Big)\Big] \tag{10}$$

$$y_{n+2} = y_n + 2hy'_n + h^2\Big[\frac{2}{315pqr}(4 + 4q + 4r - 21qr + 4p - 21pr + 7pq\,(14r - 3))f_n + \frac{2}{315}\Big(\frac{3}{(-1+p)(-1+q)(-1+r)}(-56 + q\,(52 - 56r) + 52r + p\,(52 - 56r + 7q(-8 + 11r)))f_{n+1} - \frac{3}{(-2+p)(-2+q)(-2+r)}(32 - 18r + q(-18 + 7r) + p(-18 + 7r + 7q(1 + r)))f_{n+2} + \frac{(p(-4 + 7qr) - 4(-2 + q + r))}{(p-3)(q-3)(r-3)}f_{n+3} + \frac{6}{p\,(-6 + 11p - 6p^2 + p^3)\,(p-q)(p-r)}(4 + 4q + 4r - 21qr)f_{n+p} + \frac{6(21pr - 4 - 4p - 4r)}{(p-q)q(11q - 6 - 6q^2 + q^3)(q-r)}f_{n+q} + \frac{6(4 + 4p + 4q - 21pq)}{(p-r)(q-r)r(11r - 6 - 6r^2 + r^3)}f_{n+r}\Big)\Big] \tag{11}$$

$$y_{n+3} = y_n + 3hy'_n + h^2\Big[\frac{3}{560pqr}(3(-9 + q(6 - 14r) + 6r) + 2p(9 - 21r + 7q(-3 + 13r)))f_n + \frac{3}{560}\Big(\frac{9}{(p-1)(q-1)(r-1)}(6q(6 - 7r) + 9(-3 + 4r) + p\,(36 - 42r + 14q(-3 + 4r)))f_{n+1} + \frac{9}{(p-2)(q-2)(r-2)}(-189 + q\,(90 - 42r) + 2p\,(45 + 7q\,(-3 + r) - 21r) + 90r)f_{n+2} + \frac{1}{(p-3)(q-3)(r-3)}(-42qr + 27\,(-11 + 4q + 4r) + 2p\,(54 - 21r + 14qr - 21q))f_{n+3} - \frac{18(9 - 6q - 6r + 14qr)}{p(-6 + 11p - 6p^2 + p^3)(p-q)(p-r)}f_{n+p} + \frac{18}{(p-q)q(-6 + 11q - 6q^2 + q^3)(q-r)}(9 - 6p - 6r + 14pr)f_{n+q} + \frac{18}{(p-r)r(r-q)(11r - 6 - 6r^2 + r^3)}(9 - 6p - 6q + 14pq)f_{n+r}\Big)\Big] \tag{12}$$

Replacing Equation (7) into the first derivatives of the discrete schemes yields the following first derivatives of the block

$$\acute{Y}^{3[3]_2} = \acute{B}_2^{3[3]_2} R_2^{3[3]_2} + h\left[\acute{D}^{3[3]_2} R_3^{3[3]_2} + \acute{E}^{3[3]_2} R_4^{3[3]_2}\right] \quad (13)$$

where

$$\acute{Y}^{3[3]_2} = \begin{pmatrix} y'_{n+p} \\ y'_{n+q} \\ y'_{n+r} \\ y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \end{pmatrix}, \acute{B}_2^{3[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\acute{D}^{3[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \acute{D}_{16} \\ 0 & 0 & 0 & 0 & 0 & \acute{D}_{26} \\ 0 & 0 & 0 & 0 & 0 & \acute{D}_{36} \\ 0 & 0 & 0 & 0 & 0 & \acute{D}_{46} \\ 0 & 0 & 0 & 0 & 0 & \acute{D}_{56} \\ 0 & 0 & 0 & 0 & 0 & \acute{D}_{66} \end{pmatrix},$$

$$\acute{E}^{3[3]_2} = \begin{pmatrix} \acute{E}_{11} & \acute{E}_{12} & \acute{E}_{13} & \acute{E}_{14} & \acute{E}_{15} & \acute{E}_{16} \\ \acute{E}_{21} & \acute{E}_{22} & \acute{E}_{23} & \acute{E}_{24} & \acute{E}_{25} & \acute{E}_{26} \\ \acute{E}_{31} & \acute{E}_{32} & \acute{E}_{33} & \acute{E}_{34} & \acute{E}_{35} & \acute{E}_{36} \\ \acute{E}_{41} & \acute{E}_{42} & \acute{E}_{43} & \acute{E}_{44} & \acute{E}_{45} & \acute{E}_{46} \\ \acute{E}_{51} & \acute{E}_{52} & \acute{E}_{53} & \acute{E}_{54} & \acute{E}_{55} & \acute{E}_{56} \\ \acute{E}_{61} & \acute{E}_{62} & \acute{E}_{63} & \acute{E}_{64} & \acute{E}_{65} & \acute{E}_{66} \end{pmatrix}.$$

The elements of $\acute{D}^{3[3]_2}$ and $\acute{E}^{3[3]_2}$ are shown in Appendix C.

3. Properties of the Developed Method

3.1. Order of the Method

The linear difference operator L associated with Equation (6) is defined as

$$\begin{aligned} L[y(x);h] = I^{3[3]_2} Y^{3[3]_2} - \bar{B}_1^{3[3]_2} R_1^{3[3]_2} \\ - \bar{B}_2^{3[3]_2} R_2^{3[3]_2} \\ - h^2 \Big[\bar{D}^{3[3]_2} R_3^{3[3]_2} \\ + \bar{E}^{3[3]_2} R_4^{3[3]_2}\Big] \end{aligned} \quad (14)$$

where $y(x)$ is an arbitrary test function continuously differentiable on $[a,b]$. Meanwhile $Y^{3[3]_2}$ and $R_4^{3[3]_2}$ components are expanded in Taylors series respectively and its terms are collected in powers of h to give

$$\begin{aligned} L[y(x);h] = \bar{C}_0^{3[3]_2} y(x) + \bar{C}_1^{3[3]_2} h y'(x) \\ + \bar{C}_2^{3[3]_2} h^2 y''(x) + \cdots \end{aligned} \quad (15)$$

Definition 3.1 Hybrid block method (6) and associated linear operator in (14) are said to be of order d if $\bar{C}_0^{3[3]_2} = \bar{C}_1^{3[3]_2} = \bar{C}_2^{3[3]_2} = \cdots = \bar{C}_{d+1}^{3[3]_2} = 0$ and $\bar{C}_{d+2}^{3[3]_2} \neq 0$ with error vector constants $\bar{C}_{d+2}^{3[3]_2}$, (Refer [12]).

By using Taylor series expansion about x_n for (14), it is found that the order of method is $[7,7,7,7,7,7]^T$.

The new three-step hybrid block method (6) is said to be consistent if its order is greater than or equal to one (1). Therefore, this new method is consistent since its order is greater than 1.

3.2. Zero Stability

The new three-step hybrid block method in (6) is said to be *zero-stable* if no root of the first characteristic polynomial $\pi(\omega) = \left|\omega I_{6\times6} - \bar{B}_1^{3[3]_2}\right|$ is having a modulus greater than one and every root of modulus one is simple, where $I_{6\times6}$ is identity matrix and $\bar{B}_1^{3[3]_2}$ $\bar{B}_1^{[1]_2}$ is the coefficients matrix of y_n function. By setting determinant $\pi(\omega) = 0$, then

$$\pi(\omega) = \left| \omega \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = \mathbf{0}$$

$$= \omega^5(\omega - 1) = 0$$

which implies $\omega = 0,0,0,0,0,1$. Hence, the newly developed method is zero-stable.

3.3. Consistency and Convergence

Theorem 3.1 Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent, (see [12]).

Based on the above theorem, the method in (6) is consistent and zero-stable and thus is convergent.

4. Tested Problems

To determine the performance of the newly developed method in comparison with the existing methods, the following second order ODE problems considered in previous studies are tested.

For the following problems, the errors obtained by the new method and existing methods are tabulated in Tables 1 – 3 while Figures 1 – 3 show another presentation of the error comparisons by plotting the efficiency curves of

$$e(x) = \frac{1}{|\log error|}$$

against the x values.

Problem 1:

$$y'' - y = 0, \qquad y(0) = 1, y'(0) = 1, 0 < x < 1$$
$$\text{with } h = 1/10.$$

Exact solution: $y(x) = e^x$. **Source:** [13] and [14]

Table 1. Comparison of errors obtained by the new method, $p = 1/16. q = 1/3, r = 1/2$, with the previous studies for solving Problem 1

x	New method	AbdelRahim [13]	Sagir [14]
0.1	6.661338e-16	1.345370e^{-9}	-
0.2	5.417888e-14	1.961963e-08e^{-8}	-e^{-7}
0.3	4.334311e-13	6.068528e^{-7}	5.7600e^{-6}10
0.4	3.828271e-12	e^{-7}	1.6413e^{-6}
0.5	8.202772e-12	2.186774e^{-7}	1.7001e^{-6}
0.6	1.192690e-11	3.424175e^{-7}	2.3905e^{-6}
0.7	2.218714e-11	5.014861e^{-6}	3.4705e^{-6}
0.8	3.276979e-11	7.008304e^{-6}	4.4925e^{-6}
0.9	4.269163e-11	9.461535e^{-6}	4.1569e^{-6}
1.0	6.170309e-11	1.244008e^{-6}6	4.4590e^{-6}

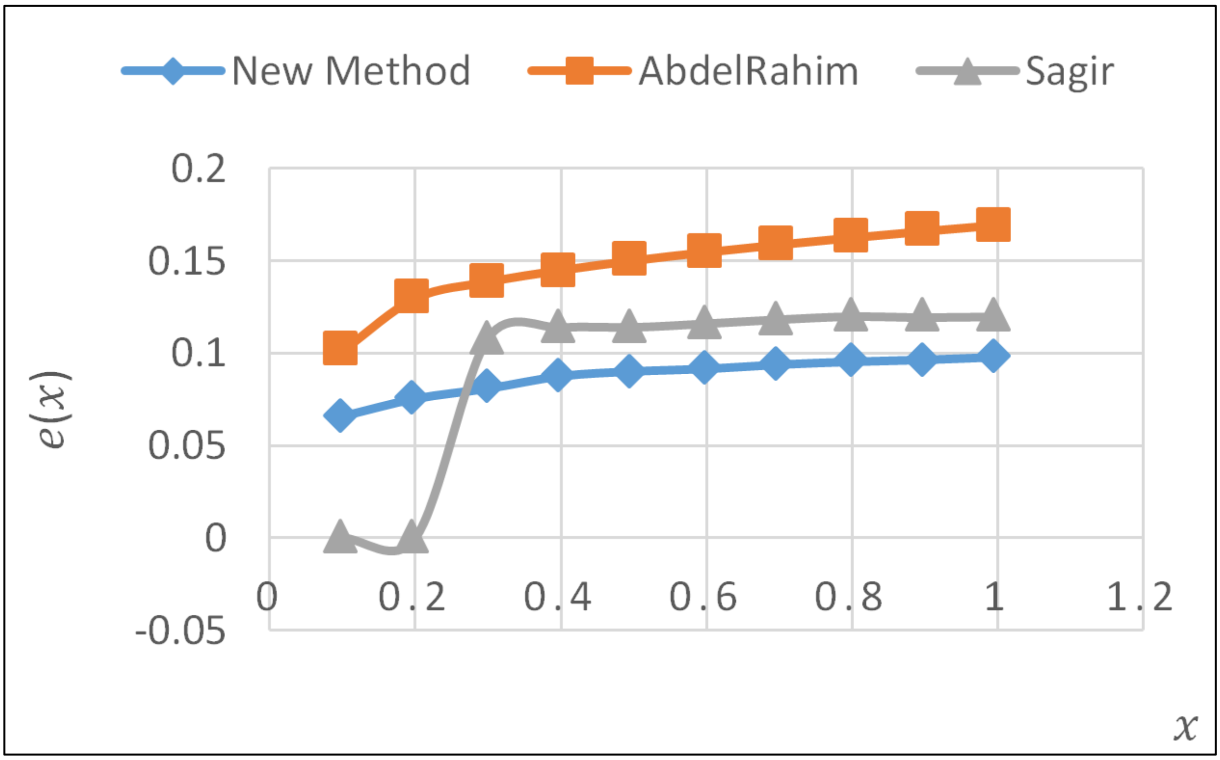

Figure 1. The accuracy curves of the methods and their comparisons for Problem 1

Problem 2:

$$y'' - y' = 0, \qquad y(0) = 1, y'(0) = -1, 0 < x < 1$$

with $h = 1/10$.

Exact solution: $y(x) = 1 - e^x$. **Source:** [15] and [16]

Table 2. Comparison of errors obtained by the new method, $p = 1/16. q = 1/3, r = 1/2$, with the previous studies for solving Problem 2

x	New method	Kuboye [15]	Mohammed [16]
0.1	4.996004e-16	2.508826e-13	2.198000000e-05
0.2	7.729928e-14	6.493175e-11	6.070400000e-06
0.3	1.288192e-12	1.683146e-09	1.005100000e-05
0.4	1.783057e-10	1.700635e-08	1.402530000e-05
0.5	3.740463e-10	1.025454e-07	1.799340000e-05
0.6	5.918914e-10	2.558711e-06	2.161620000e-05
0.7	1.069789e-09	5.273300e-06	2.799300000e-05
0.8	1.598091e-09	8.275935e-06	3.456100000e-05
0.9	2.184004e-09	1.161667e-16	4.111400000e-05
1.0	3.151647e-09	1.542187e-05	4.765600000e-05

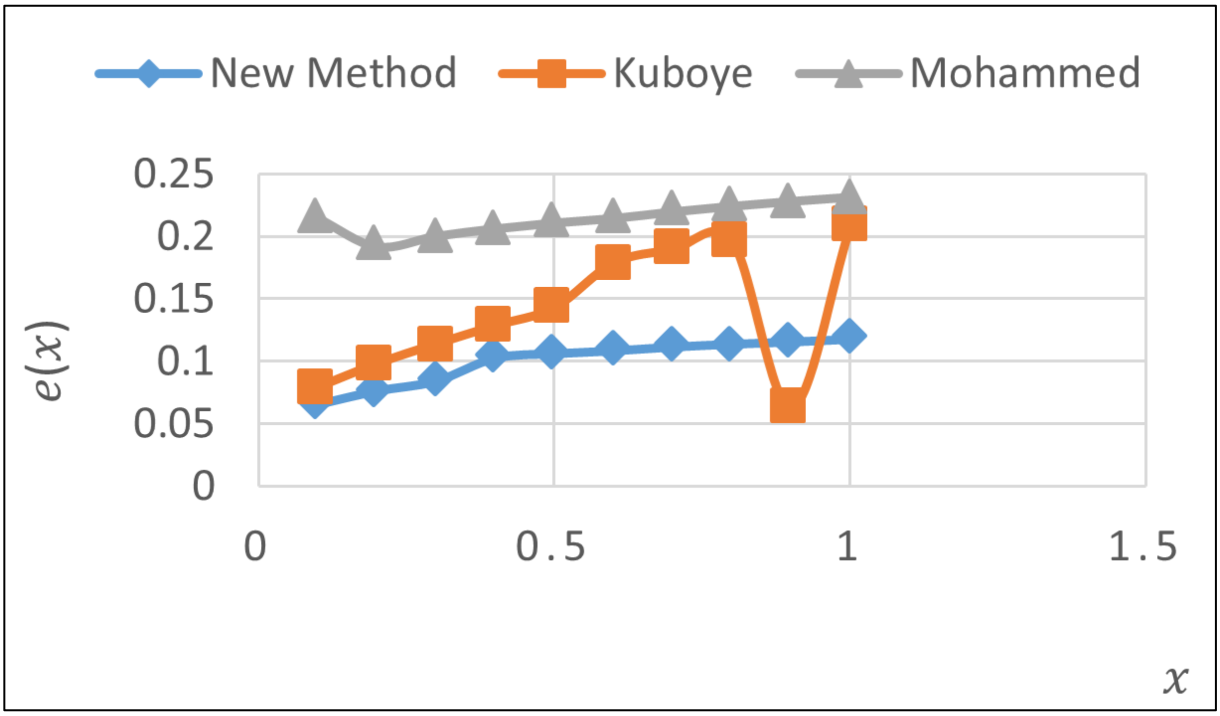

Figure 2. The accuracy curves of the methods and their comparisons for problem 2

Problem 3:

$$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = 1/2, 0 < x < 1$$

with $h = 1/10$.

Exact solution: $y(x) = 1 + (1/2)ln|(2 + x)/(2 - x)|$.

Source: [17]

Table 3. Comparison of errors obtained by the new method, $p = 1/16. q = 1/3, r = 1/2$, with the previous studies for solving Problem 3

x	New method	Kuboye [15]	Adeniyi & Alabi [17]
0.1	5.329071e-14	9.577668e-10	0.1329867326e-09
0.2	4.394263e-12	2.368709e-09	0.5872691257e-08
0.3	3.592149e-11	3.732243e-09	0.1327845616e-07
0.4	3.289591e-11	5.475119e-09	0.2317829012e-07
0.5	1.380163e-11	1.142189e-08	0.3218793564e-07
0.6	1.368163e-10	4.567944e-08	0.6871246012e-07
0.7	3.666144e-09	2.055838e-06	0.1012728156e-06
0.8	7.890375e-09	4.248299e-06	0.1231093271e-06
0.9	1.180054e-08	6.660458e-06	0.2019286712e-06
1.0	5.347980e-08	9.445166e-06	0.2990871645e-06

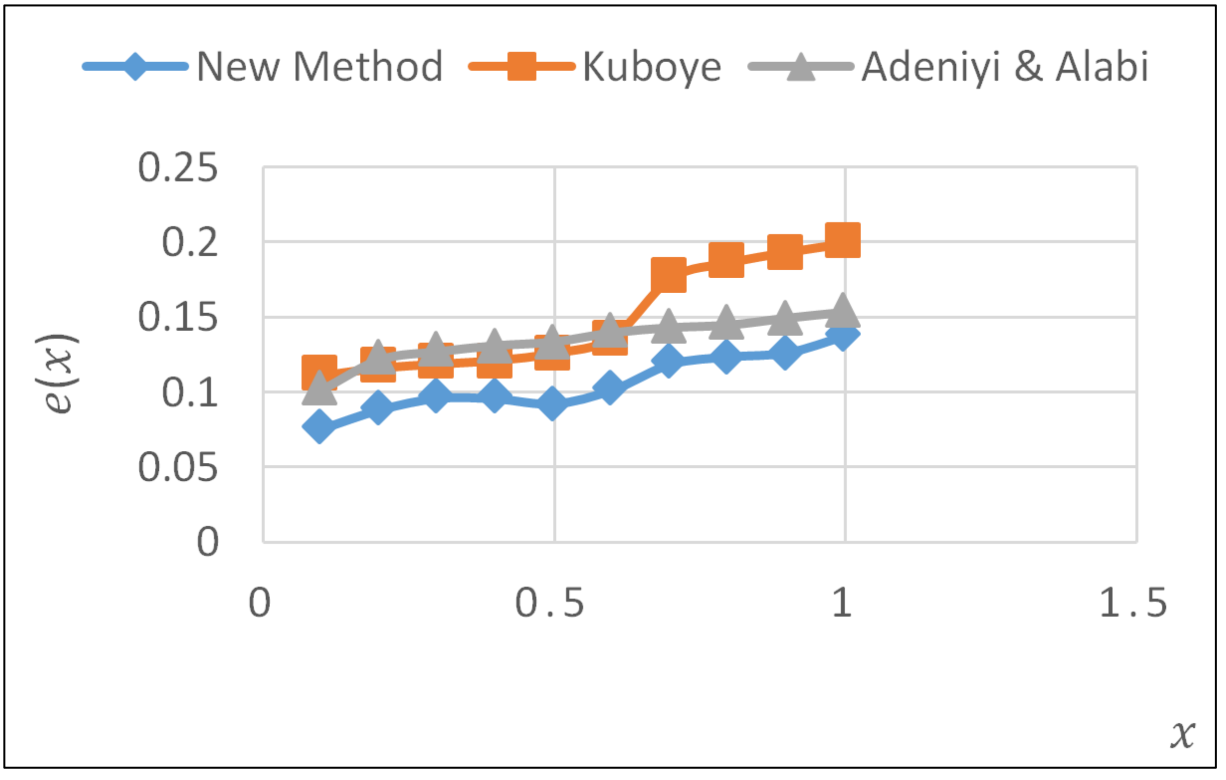

Figure 3. The accuracy curves of the methods and their comparisons for problem 3

5. Conclusions

A new generalized three-step hybrid block method with generalized three off-step points in first step for directly solving second order ordinary differential equations using interpolation and collocation approach has been successfully derived. Numerical results indicate that the new method outperforms the existing methods in terms of errors. These numerical results are seen in Problem 1 where a linear IVP was considered. Problem 1 was solved by [13] and [14] where both studies developed hybrid block methods to solve this problem. Table 1 shows the newly developed method in this article having better accuracy, in terms of absolute error, over the results obtained by [13] and [14]. This improved accuracy is also seen in Figure 1 where the accuracy line representing the new method is closer to the value 0 than the other methods. The dip at the beginning of the curve for Sagir [14] is because of the unavailable values at 0.1 and 0.2. This implies that the new method had better accuracy at all points. For Problem 2, the solution of another linear IVP was compared to [15] and [16] where multistep block methods were developed by authors. Table 2 and Figure 2 also show the newly developed method performing better in the comparison of absolute error over [15] and [16]. Specifically, the method in [15] displayed inconsistency as seen in the graph in Figure 2. A nonlinear problem was considered in Problem 3 and the results from the newly developed method were compared with the block method by [15] and the collocation method by [17]. Although the methods by [15] and [17] competed closely to the new method in this article, there is still a clear improvement in accuracy by the new method in this article.

Therefore, the developed method is not only a new approach with better accuracy for solving second order IVPs, but it is also capable of approximating the numerical solutions at three points simultaneously. The method is also flexible in the sense that all possible values in first step are considered.

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Appendix

Appendix A

$$D_{16}=\frac{1}{5040pr}\,(5p^7-1680p^2qr-8p^6\,(6+q+r)\;+\;14p^5\,(11\;+6r\;+q\,(6\;+r))-28p^4\,(6+11r+q\,(11+6r))+2pq^2\,(4q^4+840r-7q^3(6+r)+14q^2(11+6r)-35q(6+11r))+70p^3\,(6r+q(6+11r))+q^3(8q^3(6+r)-5q^4-420r-154q^2-84q^2r+28q(6+11r))),$$

$$D_{26}=\frac{1}{5040pq}(5p^7-1680p^2qr-8p^6\,(6\;+q\;+r)\;+\;14p^5\,(11\;+6r\;+q\,(6+r))-\;28p^4\,(6\;+11r+q\,(11+6r))+70p^3\,(6r\;+q\,(6+11r))-2pr^2\,(2r\,(105\;-77r+21r^2-2r^3)+7q\,(-120+55r-12r^2+r^3))+r^3\,(r\,(168-154r+48r^2-5r^3)+4q(-105+77r-21r^2+2r^3))),$$

$$D_{36}=\frac{(-1+p)}{5040pqr}(49+5p^6-106r-p^5(43+8q+8r)+2q(-53+147r)+p^4(111+76r+2q(38+7r))\;-p^3(57+232r+2q(116+77r))\;+p^2(188r-57+4q(47+154r))+p(-57+188r-4q(-47+266r))),$$

$$D_{46}=\frac{(-2+p)}{2520pqr}(5p^6-2p^5(19+4q+4r)+2p^4(39+34r+q(34+7r))+8(-4(1+r)+q(21r-4))-4p^3(3+43r+q(43+35r))-4p(12-38r+q(175r-38))+p^2(-24+76r+q(76+490r))),$$

$$D_{56}=\frac{(-3+p)}{1680pqr}(5p^6-p^5\,(33+8q+8r)+9(9-6r+2q(7r-3))\;+p^4(55+60r+2q(30+7r))-\;9p\,(3\;-12r+4q\,(-3+14r))\;-\;p^3\,(3+128r+2q(64+63r))+p^2(-9+36r+4q(9+98r)))$$

$$D_{66}=\frac{p}{5040hqr}(5p^5-1680qr-,48p^4-\;8p^4q-8p^4r+14p^3(11+6r+q(6+r))-168p^2+308p^2r+28p^2q(11+6r)+70p(6r+q(6+11r))),$$

$$E_{11}=\frac{q}{840p(-6+11p-6p^2+p^3)(p-r)}(-15p^6+5p^5(q+4(6+r))+p^4(5q^2-8q(6+r)-\;28\,(11+6r))+p^3\,(5q^3-8q^2(6+r)+14q\,(11+6r)+42\,(6+11r))+p^2\,(5q^4-420r-8q^3\,(6+r)+14q^2\,(11+6r)-28q\,(6+11r))+pq(5q^4+420r-8q^3(6+r)+14q^2\,(11+6r)-28q\,(6+11r))+q^2\,(5q^4+420r-8q^3\,(6+r)+14q^2(11+6r)-28q(6+11r))),$$

$$E_{12}=\frac{1}{840(-6+11q-6q^2+q^3)(q-r)}\;(p^5(-5q+8(6+r))-5p^6+p^4\,(-5q^2+8q(6+r)-14\,(11+6r))+p^3\,(-5q^3+8q^2\,(6+r)-14q\,(11+6r)+28\,(6+11r))+15q^6+420q^2r\;-20q^5\,(6+r)\;+\;28q^4\,(11+6r)\;-42q^3\,(6+11r)\;+p^2\,(-5q^4-420r+8q^3(6+r)-14q^2\,(11+6r)+28q\,(6+11r))\;+pq\,(-5q^4\;-420r+8q^3(6+r)-14q^2(11+6r)+28q(6+11r))),$$

$$E_{13}=\frac{q}{840r(-p+r)(r-q)(11r-6-6r^2+r^3)}(5p^7+420p^3q-8p^6(6+q)+14p^5\,(11+6q)-28p^4(6+11q)+q^4(168-154q+48q^2-5q^3)+4pq^3(77q-105-21q^2+2q^3))$$

$$E_{14} = \frac{q}{1680(-1+p)(q-1)(r-1)}(8p^6(5+q+r) - 5p^7 - 420p^3qr - 14p^5(6+5r+q(5+r)) + 28p^4(6r+q(6+5r)) + q^4(5q^3 - 168r - 8q^2(5+r) + 14q(6+5r)) - 2pq^3(4q^3 - 210r - 7q^2(5+r) + 14q(6+5r))),$$

$$E_{15} = \frac{q}{1680(-2+p)(q-2)(r-2)}(5p^7 + 210p^3qr - 8p^6(4+q+r) + 14p^5(3+4r+q(4+r)) + q^4(-5q^3 + 84r + 8q^2(4+r) - 14q(3+4r)) - 28p^4(3r+q(3+4r)) + 2pq^3(4q^3 - 105r - 7q^2(4+r) + 14q(3+4r))),$$

$$E_{16} = \frac{q}{5040(-3+p)(-3+q)(-3+r)}(-5p^7 - 140p^3qr + 8p^6(3+q+r) - 14p^5(2+3r+q(3+r)) + 28p^4(2r+q(2+3r)) + q^4(5q^3 - 56r - 8q^2(3+r) + 14q(2+3r)) - 2pq^3(4q^3 - 70r - 7q^2(3+r) + 14q(2+3r))),$$

$$E_{21} = \frac{r}{840p(-6+11p-6p^2+p^3)(p-q)}(-15p^6 + 5p^5(24+4q+r) - p^4(308+48r-5r^2+8q(21+r)) + p^3(252+154r-48r^2+5r^3+q(462+84r-8r^2)) - p^2(r(168-154r+48r^2-5r^3) + 4q(105+77r-21r^2+2r^3)) + pr(q(420-308r+84r^2-8r^3) + r(-168+154r-48r^2+5r^3)) + r^2(q(420-308r+84r^2-8r^3) + r(-168+154r-48r^2+5r^3))),$$

$$E_{22} = \frac{r}{840(p-q)q(-6+11q-6q^2+q^3)(q-r)}(8p^6(6+r) - 5p^7 - 420p^3r - 14p^5(11+6r) + 28p^4(6+11r) + 4pr^3(105-77r+21r^2-2r^3) + r^4(154r-168-48r^2+5r^3)),$$

$$E_{23} = \frac{1}{840(q-r)(-6+11r-6r^2+r^3)}(5p^6 + p^5(-48-8q+5r) + p^4(154+q(84-8r) - 48r+5r^2) + p^3(-168+154r-48r^2+5r^3-4q(77-21r+2r^2)) + p^2(q(420-308r+84r^2-8r^3) + r(154r-168-48r^2+5r^3)) + pr(q(420-308r+84r^2-8r^3) + r(-168+154r-48r^2+5r^3)) + r^2(r(252-308r+120r^2-15r^3) + 2q(-210+231r-84r^2+10r^3))),$$

$$E_{24} = \frac{r}{1680(p-1)(q-1)(-1+r)}(8p^6(5+q+r) - 5p^7 - 420p^3qr - 14p^5(6+5r+q(5+r)) + 28p^4(6r+q(6+5r)) + 2pr^3(r(-84+35r-4r^2) + 7q(30-10r+r^2)) + r^4(-2q(84-35r+4r^2) + r(84-40r+5r^2))),$$

$$E_{25} = \frac{r}{1680(-2+p)(q-2)(r-2)}(5p^7 + 210p^3qr - 8p^6(4+q+r) + 14p^5(3+4r+q(4+r)) - 28p^4(3r+q(3+4r)) - 2pr^3(7q(15-8r+r^2) - 2r(21-14r+2r^2)) + r^4(r(-42+32r-5r^2) + q(84-56r+8r^2))),$$

$$E_{26} = \frac{r}{5040(-3+p)(q-3)(r-3)}(8p^6(3+q+r) - 5p^7 - 140p^3qr - 14p^5(2+3r+q(3+r)) + 28p^4(2r+q(2+3r)) + 2pr^3(r(-28+21r-4r^2) + 7q(10-6r+r^2)) + r^4(q(-56+42r-8r^2) + r(28-24r+5r^2))),$$

$$E_{31} = -\frac{1}{840p(6-5p+p^2)(p-q)(p-r)}(15p^6 - 49 + q(106-294r) + 106r - 5p^5(21+4q+4r) + p(-49+q(106-294r)+106r) + p^4(203+148r+4q(37+7r)) - p^3(49+314r+q(314+224r)) + p^2(-49+106r+2q(53+273r))),$$

$$E_{32} = \frac{1}{840\,(p-q)q\,(-6+11q-6q^2+q^3)\,(q-r)}(49 - 5p^7 - 106r - 420p^3r + 8p^6(6+r) - 14p^5(11+6r) + 28p^4(6+11r) + 2p(-53+147r)),$$

$$E_{33} = \frac{1}{840r(r-p)(r-q)(11r-6-6r^2+r^3)}(5p^7 - 49 + p\,(106-294q) + 106q + 420p^3q - 8p^6(6+q) + 14p^5(11+6q) - 28p^4(6+11q)),$$

$$E_{34} = \frac{1}{1680\,(-1+q)\,(-1+r)}(83 - 5p^6 - 132r + p^5(35+8q+8r) + p\,(-49+q(106-294r)+106r) + p^2(-49+q(106-294r)+106r) + 2q\,(-66+119r) - p^4(49+62r+2q(31+7r)) + p^3(-49+106r+2q(53+63r))),$$

$$E_{35} = \frac{1}{1680\,(-2+p)\,(-2+q)\,(-2+r)}(19 + 5p^7 - 34r + 210\,p^3qr - 8p^6(4+q+r) + q(70r-34) + p\,(q(70-182r) - 34 + 70r) + 14p^5(3+4r+q(4+r)) - 84p^4r + 28p^4q(3+4r)),$$

$$E_{36} = \frac{1}{5040(-3+p)(-3+q)(-3+r)}(q(20-42r) - 11 - 5p^7 + 20r - 140p^3qr + 8p^6(3+q+r) - 14p^5(2+3r+q(3+r)) + 28p^4(2r+q(2+3r)) + 2p\,(10-21r+7q(8r-3))),$$

$$E_{41} = -\frac{1}{420p(3-4p+p^2)(p-q)(p-r)}(15p^6 - 10p^5(9+2q+2r) + 8(4+4q+4r-21qr) + 4p\,(4+4q+4r-21qr) + p^2(8+8q+8r+378qr) + 4p^4(32(1+r)+q(32+7r)) - 2p^3(-2+103r+q(103+98r))),$$

$$E_{42} = -\frac{1}{420(p-q)q(-6+11q-6q^2+q^3)(q-r)}(5p^7 + p\,(64-336r) + 420p^3r + 64(1+r) - 8p^6(6+r) + 14p^5(11+6r) - 28p^4(6+11r)),$$

$$E_{43} = \frac{1}{420r(-p+r)(-q+r)(-6+11r-6r^2+r^3)}(5p^7 + p\,(64-336q) + 420p^3q + 64(1+q) - 8p^6(6+q) + 14p^5(11+6q) - 28p^4(6+11q)),$$

$$E_{44} = \frac{1}{840(-1+p)(-1+q)(-1+r)}(8p^6(5+q+r) - 5p^7 - 420p^3qr + 64(-14+q(13-14r)+13r) - 14p^5(6+5r+q(5+r)) + 28p^4(6r+q(6+5r)) + 16p(52-56r+7q(-8+11r))),$$

$$E_{45} = \frac{1}{840(-2+q)(-2+r)}(5p^6 - 2p^5(11+4q+4r) - 4p^3(1+q+r+21qr) + 8(32-18r+q\,(7r-18)) + 2p^4(20r-1+q(20+7r)) + 4p\,(-4(1+r)+q(21r-4)) + p^2(-8(1+r)+q(-8+42r))),$$

$$E_{46} = \frac{1}{2520\,(-3+p)\,(-3+q)\,(-3+r)}8p^6(3+q+r) - 5p^7 - 140p^3qr - 64\,(-2+q+r) + 16p(-4+7qr) - 14p^5(2+3r+q(3+r)) + 28p^4(2r+q(2+3r)),$$

$$E_{51} = \frac{1}{280p(2-3p+p^2)(p-q)(p-r)}(5p^5(15+4q+4r) - 15p^6 + 9(9-6r+2q(7r-3)) + 3p(9-6r+2q(-3+7r)) - p^4(83+108r+4q\,(27+7r)) + 3p^3(1+46r+q(46+56r)) - p^2(-9+6r+q(6+266r))),$$

$$E_{52} = \frac{1}{280\,(p-q)q\,(-6+11q-6q^2+q^3)\,(q-r)}(-5p^7 + 81\,(3-2r) - 420p^3r + 8p^6(6+r) - 14p^5(11+6r) + 54p(-3+7r) + 28p^4(6+11r)),$$

$$E_{53}=\frac{1}{280r(-p+r)(-q+r)(-6+11r-6r^2+r^3)}(5p^7+420p^3q-8p^6(6+q)+81(-3+2q)+14p^5(11+6q)-54p(-3+7q)-28p^4(6+11q)),$$

$$E_{54}=\frac{1}{560(-1+p)(-1+q)(-1+r)}(-5p^7-420p^3qr+8p^6(5+q+r)-14p^5(6+5r+q(5+r))+54p(18-21r+7q(-3+4r))+28p^4(6r+q(6+5r))-81(9-12r+2q(-6+7r))),$$

$$E_{55}=\frac{1}{560(-2+p)(-2+q)(r-2)}(5p^7+54p(45+7q(r-3)-21r)+210p^3qr-8p^6(4+q+r)+14p^5(3+4r+q(4+r))-28p^4(3r+q(3+4r))-81(63-30r+2q(-15+7r))),$$

$$E_{56}=-\frac{1}{1680(-3+q)(-3+r)}(5p^6-p^5(9+8q+8r)-9(99-36r+2q(-18+7r))+3p(9-6r+2q(-3+7r))+p^2(9-6r+2q(-3+7r))+p^4(1+18r+2q(9+7r))-p^3(-3+2r+q(2+42r))),$$

$$E_{61}=-\frac{p}{840h(-6+11p-6p^2+p^3)(p-q)(p-r)}(15p^5-840qr-20p^4(6+q+r)+28p^3(11+6r+q(6+r))-42p^2(6+11r+q(11+6r))+70p(6r+q(6+11r)))$$

$$E_{62}=-\frac{p^3}{840h(p-q)q(-6+11q-6q^2+q^3)(q-r)}(5p^4+420r-8p^3(6+r)+14p^2(11+6r)-28p(6+11r)),$$

$$E_{63}=\frac{p^3}{840hr(-p+r)(-q+r)(11r-6-6r^2+r^3)}(5p^4+420q-8p^3(6+q)+14p^2(11+6q)-28p(6+11q)),$$

$$E_{64}=-\frac{p^3}{1680h(p-1)(q-1)(r-1)}(5p^4+420qr-8p^3(5+q+r)+14p^2(6+5r+q(5+r))-28p(6r+q(6+5r))),$$

$$E_{65}=\frac{p^3}{1680h(-2+p)(-2+q)(r-2)}(5p^4+210qr-8p^3(4+q+r)+14p^2(3+4r+q(4+r))-28p(3r+q(3+4r))),$$

$$E_{66}=-\frac{p^3}{5040h(p-3)(q-3)(r-3)}(5p^4+140qr-8p^3(3+q+r)+14p^2(2+3r+q(3+r))-28p(2r+q(2+3r))).$$

Appendix B

$$\overline{D}_{16}=-\frac{p^2}{5040qr}(5p^5-1680qr-8p^4(6+q+r)+14p^3(11+6r+q(6+r))-28p^2(6+11r+q(11+6r))+70p(6r+q(6+11r))),$$

$$\overline{D}_{26}=\frac{q^2}{5040pr}(2p(4q^4+840r-7q^3(6+r)+14q^2(11+6r)-35q(6+11r))+q(-5q^4-420r+8q^3(6+r)-14q^2(11+6r)+28q(6+11r))),$$

$$\overline{D}_{36}=\frac{r^2}{5040pq}(-2p(2r(105-77r+21r^2-2r^3)+7q(-120+55r-12r^2+r^3))+r(r(168-154r+48r^2-5r^3)+4q(-105+77r-21r^2+2r^3))),$$

$$\overline{D}_{46}=\frac{1}{5040pqr}(-49+q(106-294r)+106r+2p(53-147r+7q(-21+97r))),$$

$$\bar{D}_{56}=\frac{2}{315pqr}(4+4q+4r-21qr+p(4-21r+7q(-3+14r))),$$

$$\bar{D}_{66}=\frac{3}{560pqr}(3(-9+q(6-14r)+6r)+2p(9-21r+7q(-3+13r))),$$

$$\bar{E}_{11}=\frac{p^2}{840(-6+11p-6p^2+p^3)(p-q)(p-r)}(15p^5-840qr-20p^4(6+q+r)+28p^3(11+6r+q(6+r))-42p^2(6+11r+q(11+6r))+70p(6r+q(6+11r))),$$

$$\bar{E}_{12}=\frac{p^4}{840\,(p-q)\,q\,(-6+11q-6q^2+q^3)\,(q-r)}(5p^4+420r-8p^3\,(6+r)+14p^2\,(11+6r)-28p(6+11r)),$$

$$\bar{E}_{13}=-\frac{p^4}{840r(-p+r)(-q+r)(11r-6-6r^2+r^3)}(5p^4+420q-8p^3(6+q)+14p^2(11+6q)-28p(6+11q)),$$

$$\bar{E}_{14}=\frac{p^4}{1680(-1+p)(-1+q)(-1+r)}(5p^4+420qr-8p^3(5+q+r)+14p^2\,(6+5r+q(5+r))-28p(6r+q(6+5r))),$$

$$\bar{E}_{15}=-\frac{p^4}{1680(-2+p)(q-2)(r-2)}(5p^4+210qr-8p^3(4+q+r)+14p^2\,(3+4r+q\,(4+r))-28p(3r+q(3+4r))),$$

$$\bar{E}_{16}=\frac{p^4}{5040(-3+p)(-3+q)(-3+r)}(5p^4+140qr-8p^3\,(3+q+r)+14p^2(2+3r+q(3+r))-28p(2r+q(2+3r))),$$

$$\bar{E}_{21}=\frac{q^4}{840p(-6+11p-6p^2+p^3)(p-q)(p-r)}(-5q^4-420r+8q^3\,(6+r)-14q^2\,(11+6r)+28q(6+11r)),$$

$$\bar{E}_{22}=\frac{q^2}{840(p-q)\,(-6+11q-6q^2+q^3)\,(q-r)}\,(2p\,(10q^4+420r-14q^3\,(6+r)+21q^2\,(11+6r)-35q\,(6+11r))+q\,(-15q^4-420r+20q^3\,(6+r)-28q^2\,(11+6r)+42q(6+11r))),$$

$$\bar{E}_{23}=\frac{q^4}{840r\,(-p+r)\,(-q+r)\,(-6+11r-6r^2+r^3)}(q\,(168-154q+48q^2-5q^3)+4p\,(-105+77q-21q^2+2q^3)),$$

$$\bar{E}_{24}=\frac{q^4}{1680(-1+p)(-1+q)(-1+r)}(q\,(5q^3-168r-8q^2(5+r)+14q(6+5r))-(8pq^3-420pr-14pq^2(5+r)+14q(6+5r))),$$

$$\bar{E}_{25}=\frac{q^4}{1680(-2+p)(-2+q)(-2+r)}(q(84r-5q^3+8q^2(4+r)-14q(3+4r))+2p(4q^3-105r-7q^2(4+r)+14q(3+4r))),$$

$$\bar{E}_{26}=\frac{q^4}{5040(-3+p)(-3+q)(-3+r)}(q(5q^3-56r-8q^2\,(3+r)+14q(2+3r))-2p(4q^3-70r-7q^2(3+r)+14q(2+3r))),$$

$$\bar{E}_{31}=\frac{r^4}{840p\,(-6+11p-6p^2+p^3)\,(p-q)\,(p-r)}(r\,(168-154r+48r^2-5r^3)+4q\,(-105+77r-21r^2+2r^3)),$$

$$\bar{E}_{32}=\frac{r^4}{840(p-q)\,q\,(-6+11q-6q^2+q^3)(q-r)}(p\,(420\;-\;308r\;+\;84r^2\;-\;8r^3)\;+\;r\,(-168\;+\;154r-48r^2+5r^3)),$$

$$\bar{E}_{33}=\frac{r^2}{840(p-r)(q-r)(-6+11r-6r^2+r^3)}(2p\,(r\,(210-231r+84r^2\;-10r^3)\;+7q\,(-60+55r\;-18r^2+\;2r^3)\,)\;+\;r\,(\,q\,(420\;-462r\;+168r^2\;-20r^3)\;+\;r\,(-252+308r-120r^2+15r^3))),$$

$$\bar{E}_{34}=\frac{r^4}{1680(p-1)(q-1)(r-1)}(2p\,(r(-84+35r-4r^2)+7q(30-10r+r^2))+r(r(84-40r+5r^2)-2q(84-35r+4r^2))),$$

$$\bar{E}_{35}=\frac{r^4}{1680(p-2)(q-2)(r-2)}(-2p(7q(15-8r+r^2)-2r(21-14r+2r^2))+r^2(84q-56qr+8qr^2-42+32r-5r^2)),$$

$$\bar{E}_{36}=\frac{r^4}{5040(-3+p)(q-3)(r-3)}(2p(r(-28+21r-4r^2)+7q(10-6r+r^2))+qr(42r-56-8r^2)+r(28-24r+5r^2)),$$

$$\bar{E}_{41}=\frac{1}{840p(-6+11p-6p^2+p^3)(p-q)(p-r)}(-49+q(106-294r)+106r),$$

$$\bar{E}_{42}=\frac{1}{840(p-q)q(-6+11q-6q^2+q^3)(q-r)}(49-106r+2p(-53+147r))$$

$$\bar{E}_{43}=\frac{1}{840r(-p+r)(-q+r)(-6+11r-6r^2+r^3)}(-49+p(106-294q)+106q)$$

$$\bar{E}_{44}=\frac{1}{1680(p-1)(q-1)(r-1)}(q\,(132-238r)-83+132r+2p(66-119r+266qr-119q))$$

$$\bar{E}_{45}=\frac{1}{1680(-2+p)(q-2)(r-2)}(19-34r+q(70r-34)+p(q(70-182r)-34+70r))$$

$$\bar{E}_{46}=\frac{1}{5040(-3+p)(q-3)(r-3)}(q(20-42r)-11+20r+2p(10-21r+7q(8r-3)))$$

$$\bar{E}_{51}=\frac{4}{105p(-6+11p-6p^2+p^3)(p-q)(p-r)}(4+4q+4r-21qr)$$

$$\bar{E}_{52}=\frac{4}{105(p-q)q(-6+11q-6q^2+q^3)(q-r)}(p(-4+21r)-4(1+r))$$

$$\bar{E}_{53}=\frac{4}{105r(-p+r)(-q+r)(-6+11r-6r^2+r^3)}(4+4p+4q-21pq)$$

$$\bar{E}_{54}=\frac{2}{105(-1+p)(-1+q)(-1+r)}(q(52-56r)-56+52r+p(52-56r+7q(11r-8)))$$

$$\bar{E}_{55}=-\frac{2}{105(-2+p)(-2+q)(-2+r)}(32-18r+q(-18+7r)+p(-18+7r+7q(1+r)))$$

$$\bar{E}_{56}=\frac{2}{315(-3+p)(-3+q)(-3+r)}(-4(-2+q+r)+p(-4+7qr))$$

$$\bar{E}_{61}=-\frac{27}{280p(-6+11p-6p^2+p^3)(p-q)(p-r)}\;(9-6r+2q(-3+7r))$$

$$\bar{E}_{62}=\frac{27}{280(p-q)q(-6+11q-6q^2+q^3)(q-r)}(9-6r+2p(-3+7r))$$

$$\bar{E}_{63}=-\frac{27}{280r(r-p)(r-q)(-6+11r-6r^2+r^3)}(9-6q+2p(-3+7q))$$

$$\bar{E}_{64} = \frac{27}{560(-1+p)(-1+q)(r-1)}(36q - 27 + 36r - 42qr + p(36 - 42r + 14q(-3+4r)))$$

$$\bar{E}_{65} = \frac{27}{560(-2+p)(-2+q)(-2+r)}(q(90-42r) - 189 + 2p(45 + 7q(r-3) - 21r) + 90r)$$

$$\bar{E}_{66} = \frac{3}{560(p-3)(q-3)(r-3)}(3(q(36-14r) - 99 + 36r) + 2p(54 - 21r + 7q(2r-3)))$$

Appendix C

$$\acute{D}_{16} = -\frac{p}{2520qr}(-210p^2 + 231p^3 - 84p^4 + 10p^5 + 420pq - 385p^2q + 126p^3q - 14p^4q + 420pr - 385p^2r + 126p^3r - 14p^4r - 1260qr + 770pqr - 210p^2qr + 21p^3qr),$$

$$\acute{D}_{26} = \frac{q}{2520pr}(-420pq + 210q^2 + 385pq^2 - 231q^3 - 126pq^3 + 84q^4 + 14pq^4 - 10q^5 + 1260pr - 420qr - 770pqr + 385q^2r + 210pq^2r - 126q^3r - 21pq^3r + 14q^4r),$$

$$\acute{D}_{36} = \frac{1}{2520pq}r\,(1260pq - 70\,(6q + p(6+11q))r + 35\,(6 + 11q + p(11+6q))r^2 - 21(11 + 6q + p(6+q))r^3 + 14(6+p+q)r^4 - 10r^5),$$

$$\acute{D}_{46} = \frac{1}{2520pqr}(-66 + 7q(17-38r) + 119r + 7p(17 - 38r + q(-38+135r))),$$

$$\acute{D}_{56} = \frac{1}{315pqr}(-2(-18 + 7r + 7q(1+r)) + 7p(-2(1+r) + q(-2+15r))),$$

$$\acute{D}_{66} = \frac{3}{280pqr}(-54 + 7q(3-2r) + 21r + 7p(3 - 2r + q(-2+5r))),$$

$$\acute{E}_{11} = \frac{p}{420\,(-6 + 11p - 6p^2 + p^3)\,(p-q)\,(p-r)}\,(-630p^2 + 924p^3 - 420p^4 + 60p^5 + 840pq - 1155p^2q + 504p^3q - 70p^4q + 840pr - 1155p^2r + 504p^3r - 70p^4r - 1260qr + 1540pqr - 630p^2qr + 84p^3qr),$$

$$\acute{E}_{12} = \frac{p^3}{420(p-q)q(11q - 6 - 6q^2 + q^3)(q-r)}(231p^2 - 210p - 84p^3 + 10p^4 + 420r - 385pr + 126p^2r - 14p^3r),$$

$$\acute{E}_{13} = \frac{p^3}{420(p-r)(r-1)r(r-q)(6 - 5r + r^2)}\,(231p^2 - 210p - 84p^3 + 10p^4 + 420q - 385pq + 126p^2q - 14p^3q),$$

$$\acute{E}_{14} = \frac{p^3}{840\,(-1+p)\,(-1+q)\,(-1+r)}(126p^2 - 70p^3 + 10p^4 - 210pq + 105p^2q - 14p^3q - 210pr + 105p^2r - 14p^3r + 420qr - 175pqr + 21p^2qr),$$

$$\acute{E}_{15} = -\frac{p^3}{840\,(-2+p)\,(-2+q)\,(-2+r)}(63p^2 - 56p^3 + 10p^4 - 105pq + 84p^2q - 14p^3q - 105pr + 84p^2r - 14p^3r + 210qr - 140pqr + 21p^2qr),$$

$$\acute{E}_{16} = \frac{p^3}{2520\,(p-3)(q-3)(r-3)}(42p^2 - 42p^3 + 10p^4 - 70pq + 63p^2q - 14p^3q - 70pr + 63p^2r - 14p^3r + 140qr - 105pqr + 21p^2qr),$$

$$\acute{E}_{21} = \frac{q^3}{420p(11p - 6 - 6p^2 + p^3)(p-q)\,(p-r)}(210q - 231q^2 + 84q^3 - 10q^4 - 420r + 385qr - 126q^2r + 14q^3r),$$

$$\acute{E}_{22} = \frac{q}{420(q-1)(-p+q)(6-5q+q^2)(q-r)}(840pq - 630q^2 - 1155pq^2 + 924q^3 + 504pq^3 - 420q^4 - 70pq^4 + 60q^5 - 1260pr + 840qr + 1540pqr + 1155q^2r + 630pq^2r - 504q^3r - 84pq^3r + 70q^4r),$$

$$\acute{E}_{23} = \frac{q^3}{420\,(p-r)\,(r-1)\,r\,(r-q)\,(6-5r+r^2)}(420p - 210q - 385pq + 231q^2 + 126pq^2 - 84q^3 - 14pq^3 + 10q^4),$$

$$\acute{E}_{24} = -\frac{q^3}{840(-1+p)(-1+q)(-1+r)}(210pq - 126q^2 - 105pq^2 + 70q^3 + 14pq^3 - 10q^4 - 420pr + 210qr + 175pqr - 105q^2r - 21pq^2r + 14q^3r),$$

$$\acute{E}_{25} = \frac{q^3}{840\,(-2+p)\,(-2+q)\,(-2+r)}(105pq - 63q^2 - 84pq^2 + 56q^3 + 14pq^3 - 10q^4 - 210pr + 105qr + 140pqr - 84q^2r - 21pq^2r + 14q^3r),$$

$$\acute{E}_{26} = -\frac{q^3}{2520(-3+p)(-3+q)\,(-3+r)}(70pq - 42q^2 - 63pq^2 + 42q^3 + 14pq^3 - 10q^4 - 140pr + 70qr + 105pqr - 63q^2r - 21pq^2r + 14q^3r),$$

$$\acute{E}_{31} = \frac{r^3}{420(-3+p)\,(-2+p)\,(-1+p)\,p\,(p-q)\,(p-r)}(7q\,(-4+r)\,(15+2\,(-5+r)\,r) + r\,(210 + r(-231+2(42-5r)r))),$$

$$\acute{E}_{32} = \frac{r^3}{420\,(-3+q)\,(-2+q)\,(-1+q)\,q\,(-p+q)\,(q-r)}(7p\,(-4+r)\,(15+2(-5+r)r) + r(210 + r(-231+2(42-5r)r))),$$

$$\acute{E}_{33} = \frac{r}{420(p-r)(q-r)(-3+r)(-2+r)(-1+r)}(-1260pq + 140(6q+p(6+11q))r - 105(6 + 11q + p(11+6q))r^2 + 84(11+6q+p(6+q))r^3 - 70(6+p+q)r^4 + 60r^5),$$

$$\acute{E}_{34} = \frac{r^3}{840\,(-1+p)\,(-1+q)\,(-1+r)}(420pq - 35\,(6q + p\,(6+5q))\,r + 21(6+5q+p(5 + q))r^2 - 14(5+p+q)r^3 + 10r^4),$$

$$\acute{E}_{35} = \frac{r^3}{840\,(-2+p)\,(-2+q)\,(-2+r)}(-210pq + 35(3q+p\,(3+4q))\,r - 21(3+4q+p(4 + q))r^2 + 14(4+p+q)r^3 - 10r^4),$$

$$\acute{E}_{36} = \frac{r^3}{2520\,(-3+p)\,(-3+q)\,(-3+r)}(140pq - 35(2q + p(2+3q))r + 21(2+3q+p(3 + q))r^2 - 14(3+p+q)r^3 + 10r^4)$$

$$\acute{E}_{41} = \frac{1}{420p(-6+11p-6p^2+p^3)(p-q)(p-r)}(-66+119r-7q(-17+38r)),$$

$$\acute{E}_{42} = \frac{1}{420(-1+q)q(-p+q)(6-5q+q^2)(q-r)}(-66+119r-7p(-17+38r)),$$

$$\acute{E}_{43} = \frac{1}{420(p-r)(-1+r)r(-q+r)(6-5r+r^2)}(66-119q+7p(-17+38q)),$$

$$\acute{E}_{44} = \frac{1}{840(p-1)(q-1)(r-1)}7q(40-57r) - 214 + 280r + 7p(40-57r+19q(5r-3)),$$

$$\acute{E}_{45} = \frac{1}{840(p-2)(q-2)(r-2)}(32-49r+7q(-7+12r)+7p(-7+q(12-25r)+12r)),$$

$$\acute{E}_{46} = \frac{1}{2520(-3+p)(-3+q)(-3+r)}(-18+7q(4-7r)+28r+7p(4-7r+q(-7+15r))),$$

$$\acute{E}_{51} = -\frac{4}{105p(-6+11p-6p^2+p^3)(p-q)(p-r)}(-18+7r+7q(1+r)),$$

$$\acute{E}_{52} = -\frac{4}{105(-1+q)q(-p+q)(6-5q+q^2)(q-r)}(-18+7r+7p(1+r)),$$

$$\acute{E}_{53} = \frac{4}{105(p-r)(-1+r)r(-q+r)(6-5r+r^2)}(-18+7q+7p(1+q)),$$

$$\acute{E}_{54} = \frac{2}{105(-1+p)(-1+q)(-1+r)}(q(70-63r)-88+70r+7p(10-9r+q(-9+10r))),$$

$$\acute{E}_{55} = \frac{1}{105(p-2)(q-2)(r-2)}(q(154-84r)-272+154r+7p(22-12r+q(5r-12))),$$

$$\acute{E}_{56} = \frac{2}{315(-3+p)(-3+q)(-3+r)}(24+7q(-2+r)-14r+7p(-2+q+r)),$$

$$\acute{E}_{61} = -\frac{9}{140p(-6+11p-6p^2+p^3)(p-q)(p-r)}(54-21r+7q(-3+2r)),$$

$$\acute{E}_{62} = -\frac{9}{140(-1+q)q(-p+q)(6-5q+q^2)(q-r)}(54-21r+7p(-3+2r)),$$

$$\acute{E}_{63} = \frac{9}{140(p-r)(-1+r)r(-q+r)(6-5r+r^2)}(54-21q+7p(-3+2q)),$$

$$\acute{E}_{64} = \frac{9}{280(-1+p)(-1+q)(-1+r)}(54-21qr+7p(-3r+q(-3+5r))),$$

$$\acute{E}_{65} = \frac{9}{280(p-2)(q-2)(r-2)}(21q(9-4r)+27(7r-16)+7p(27-12r+q(5r-12))),$$

$$\acute{E}_{66} = \frac{3}{280(p-3)(q-3)(r-3)}(q(252-91r)-702+252r+7p(36-13r+q(5r-13))).$$

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