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Killing Vector Fields and Conserved Currents on Rotationally Symmetric Space-time

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Abstract In this paper, we have sketched how Einstein's theory of gravity formulated on $R \times S^3$ topology, i.e, space and time of rotations can be applied to tachyon dynamics and modified gravity. The initiative of formulating physical theories on $R \times S^3$ topology was taken by many physicists in early 1980s among which a first successful attempt was taken by M. Carmeli and S. Malin followed by G. Zet, C. Pasnicu and M. Agop. The main idea of formulating gravity on such topology is due to the fact that the surface of sphere has more symmetries than distance in Minkowskian space-time. Thus, we are making the quantities dependent on angles instead of invariant lengths. Since we have changed the topology on which the theory is formulated, the definition of derivative operators and other differential operators changes. There are two kinds of geometries of $R \times S^3$ topology, the first given by M. Carmeli and S. Malin is of commutative type where the derivatives commute and the other given subsequently by G. Zet, C. Pasnicu and M. Agop is of non-commutative type where the derivatives do not commute and result in an additional term in the equations. Although the Einstein's field equation on $R \times S^3$ topology was already derived [6][8], what we have tried in this paper is to construct Killing vector fields and conserved currents on $R \times S^3$ topology.

Keywords Killing Vector Fields, Conserved Currents, Rotationally Symmetric Space-time

1 Introduction

The idea to formulate theories on $R \times S^3$ topology originates due to the problem of formulating physics where the particle is solely angular dependent rather than distance as in Minkowskian space-time. A very first attempt towards this was taken by M. Carmeli and A. Malka in 1983 a series of seven papers [1]-[7] of which [6] contains theory of gravity on $R \times S^3$ topology. Further attempt to describe gravitation on $R \times S^3$ topology was take by G. Zet, C. Pasnicu and M. Agop [8]. Similarly, Diptiman Sen had formulated supersymmetry on $R \times S^3$ space-time in [9]. To simplify what we meant by formulating theories on $R \times S^3$ topology means, we assume that the tensor quantities that were dependent on the length are now solely angle dependent (such as Euler's angles).

2 Commutative and non-commutative topology

We are going to formulate our theory on commutative type of $R \times S^3$ topology established in [6] because it was the first proposed theory of this type. Non commutative type of theory will be briefly formulated in the end of this paper and will be developed elsewhere.

The main idea of formulating gravity on such topology is due to the fact that the surface of sphere has more symmetries than distance in Minkowskian space-time. Thus, we are making the quantities dependent on angles instead of invariant lengths. Since we have changed the topology on which the theory is formulated, the definition of derivative operators and other differential operators changes. There are two kinds of geometries of R × S3 topology, the first, given by M. Carmeli and S. Malin is of commutative type where the derivatives commute and the other given subsequently by G. Zet, C. Pasnicu and M. Agop is of non-commutative type where the derivatives do not commute and result in an additional term in the equations.

3 Underlying Topology

In the first half of this section, we will provide a brief overview of commutative type $R \times S^3$ topology from [6]. The other half of the section is devoted in providing a brief overview of non commutative $R \times S^3$ topology from [8].

3.1 Commutative topology

To make a transition from flat Minkowskian space-time to S^3 topology, the transnational groups of ordinary flat space-time are replaced with $SU(2)$ groups. Thus, the linear momentum operator $p = -i\hbar\nabla$ is replaced with angular momentum operator $J = -i\hbar\mathbf{L}$ where $\boldsymbol{L} = (L_1, L_2, L_3) = (L_x, L_y, L_z)$ is the corresponding differential operator given by

$$L_1 = \frac{-\sin\phi}{\sin\theta}\frac{\partial}{\partial\psi} - \cos\phi\frac{\partial}{\partial\theta} + \cot\phi\sin\phi\frac{\partial}{\partial\phi}, \tag{3.1}$$

$$L_2 = \frac{-\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi} - \sin\phi\frac{\partial}{\partial\theta} + \cot\phi\cos\phi\frac{\partial}{\partial\phi}, \tag{3.2}$$

$$L_3 = -\frac{\partial}{\partial\phi}. \tag{3.3}$$

The operator $\boldsymbol{L}^2 = L_1^2 + L_2^2 + L_3^2 = L_x^2 + L_y^2 + L_z^2$ is then give by

$$\boldsymbol{L}^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\psi^2} - 2\cos\theta\frac{\partial^2}{\partial\psi\partial\phi} + \frac{\partial^2}{\partial\phi 2}\right). \tag{3.4}$$

We know that angular momentum operator satisfies the following commutation relation

$$[J_x, J_y] = iJ_z. \tag{3.5}$$

Thus, using the definition of $\mathbf{L}$, we get the following commutation relation for angular gradient operator

$$[\mathbf{L}_x, \mathbf{L}_y] = -\varepsilon_{ijk}\mathbf{L}_z. \tag{3.6}$$

In four dimensions, angular gradient operator has the following form:

$$\frac{L}{L\Theta^\mu} = L_\mu = (\partial/\partial t, \mathbf{L}) \tag{3.7}$$

Consider the line element in commutative $R \times S^3$ space-time without gravitation:

$$ds^2 = dt^2 - \left[\gamma_1^{-2}\left(d\Theta^1\right)^2 + \gamma_2^{-2}\left(d\Theta^2\right)^2 + \gamma_3^{-2}\left(d\Theta^3\right)^2\right] \tag{3.8}$$

where

$$d\Theta^1 = \sin\theta\sin\psi d\phi + \cos\psi d\theta, \tag{3.9}$$

$$d\Theta^2 = \sin\theta\sin\psi d\phi - \sin\psi d\theta, \tag{3.10}$$

$$d\Theta^3 = \cos\theta d\phi + d\psi. \tag{3.11}$$

Since ϕ, θ and ψ are Euler's angles, they take the values $0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi$ and $0 \leq \psi \leq 4\pi$. The imperfect differentials appearing in the line element $d\Theta^k$, $k = 1, 2, 3$ are dx^k analogous to the Cartesian differential distances in Euclidean geometry. Thus, we can generalize our line element on commutative $R \times S^3$ topology as

$$ds^2 = g_{\mu\nu}d\Theta^\mu d\Theta^\nu \tag{3.12}$$

where $g_{\mu\nu}$ are functions three Euler angles.

Now, let's take a look at Riemannian geometry formulated on commutative $R \times S^3$ topology. Since we know that the partial derivatives from Cartesian coordinates will be replaced with angular gradient derivatives in 4 dimensions, we can thus write Christoffel symbols as

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(L_\beta g_{\lambda\alpha} + L_\alpha g_{\lambda\beta} - L_\lambda g_{\alpha\beta}\right). \tag{3.13}$$

Similarly, the standard covariant derivative in this case for any contravariant vector V^α will be given by

$$D_\mu V^\alpha = L_\mu V^\alpha + \Gamma^\alpha_{\mu\lambda}V^\lambda. \tag{3.14}$$

where $\Gamma^\alpha_{\mu\lambda}$ are the Christoffel symbols given by (3.13). The outer structure of Einstein's field equation will be same but underlying mathematical form will be different because we will be using angular gradient operators instead of Cartesian partial derivatives. Thus, we have

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{3.15}$$

where $R_{\mu\nu} = R^\sigma_{\alpha\sigma\gamma}$ is the Ricci tensor, R is Ricci scalar, $T_{\mu\nu}$ is the energy-momentum tensor, κ is Einstein's gravitational constant and $R^\rho_{\alpha\beta\gamma}$ is the Riemann Christoffel curvature tensor explicitly given by

$$R^\rho_{\alpha\beta\gamma} = L_\beta\Gamma^\rho_{\alpha\gamma} - L_\gamma\Gamma^\rho_{\alpha\beta} + \Gamma^\delta_{\alpha\gamma}\Gamma^\rho_{\beta\delta} - \Gamma^\delta_{\alpha\beta}\Gamma^\rho_{\gamma\delta} \tag{3.16}$$

where the Christoffel symbols are defined by (3.13). Energy-momentum tensor will have specific forms according to the physics it describes. For the case of perfect fluid, the energy-momentum is formulated in [6].

3.2 Non commutative topology

Let us now recall some already known results on non commutative type of S^3 manifold [8] that we would be referring simultaneously throughout the paper. The sphere S^3 can be parametrized either in terms of Cartesian coordinates x^μ $(\mu = 1, 2, 3, 4)$, complex variables (u, v) or angles (θ, α, β) whose relation with each other is given by

$$u = x^1 + ix^2 = \cos\theta e^{i\alpha} \tag{3.17}$$

$$v = x^3 + ix^4 = \sin\theta e^{i\beta} \tag{3.18}$$

with the volume element

$$d\Omega = \frac{1}{2\pi^2}\sin\theta\cos\theta d\theta d\alpha d\beta. \tag{3.19}$$

S^3 manifold of group $SU(2)$ has $O(4)$ symmetry with $SU(2) \times SU(2)$ algebra. The two subgroups correspond to the left and right multiplication acting on the manifold and thus, as a consequence, we have two independent frames: $e_i^{(1)}$ and $e_i^{(2)}$, $i = 1, 2, 3$. We are going to choose the frame $e_i^{(1)}$ and the corresponding derivatives will be denoted by $\partial_i, i = 1, 2, 3$:

$$\begin{aligned}\partial_1 &= i\left(u\frac{\partial}{\partial v^*} - v\frac{\partial}{\partial u^*} + v^*\frac{\partial}{\partial u} - u^*\frac{\partial}{\partial v}\right)\\ &= -\sin(\alpha+\beta)\frac{\partial}{\partial\theta} + \cos(\alpha+\beta)\left[\tan\theta\frac{\partial}{\partial\alpha} - \cot\theta\frac{\partial}{\partial\beta}\right]\end{aligned} \tag{3.20}$$

$$\begin{aligned}\partial_2 &= u\frac{\partial}{\partial v^*} - v\frac{\partial}{\partial u^*} - v^*\frac{\partial}{\partial u} + u^*\frac{\partial}{\partial v}\\ &= \cos(\alpha+\beta)\frac{\partial}{\partial\theta} + \sin(\alpha+\beta)\left[\tan\theta\frac{\partial}{\partial\alpha} - \cot\theta\frac{\partial}{\partial\beta}\right]\end{aligned} \tag{3.21}$$

$$\partial_3 = i\left(u^*\frac{\partial}{\partial u^*} - u\frac{\partial}{\partial u} + v^*\frac{\partial}{\partial v^*} - v\frac{\partial}{\partial v}\right) = -\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}. \tag{3.22}$$

One can see from the definition of above differential derivatives that they do not commute. Thus

$$[\partial_i, \partial_j] = 2\varepsilon_{ijk}\partial_k = C^k_{ij}\partial_k \tag{3.23}$$

where ε_{ijk} is antisymmetric tensor of rank 3 and $\varepsilon_{123} = 1$. If $\partial_\mu = \partial/\partial\theta^\mu = (\partial_0 = \partial/\partial t, \partial_i)$, then

$$[\partial_\mu, \partial_\nu] = 2\varepsilon_{0\mu\nu\gamma}\partial_\gamma = C^\gamma_{\mu\nu}\partial_\gamma \tag{3.24}$$

where $\varepsilon_{0\mu\nu\gamma}$ is now a antisymmetric tensor of rank 4 with $\varepsilon_{0123} = 1$.

On $R \times S^3$ manifold $\mathcal{M}_{R\times S^3}$, let $\mathcal{T}_{R\times S^3}$ be a tangent bundle. We introduce the metric form $g(X, Y)$ on $\mathcal{T}_{R\times S^3}$ which is symmetric, non degenerate and positive where X and Y are vector fields on $\mathcal{M}_{R\times S^3}$ such that $X = X^\mu\partial_\mu$ and $Y = Y^\mu\partial_\mu$. Thus, the metric g now becomes

$$g(X, Y) = g(X^\mu\partial_\mu, Y^\nu\partial_\nu) = X^\mu Y^\nu g(\partial_\mu, \partial_\nu) = g_{\mu\nu}X^\mu Y^\nu. \tag{3.25}$$

As we have now constructed the definition of metric g on $\mathcal{M}_{R\times S^3}$ using proper Riemannian structure, we can similarly construct the definition of other tensorial quantities like Christoffel symbols, Riemann Christoffel curvature tesor, Ricci tensor and so on. We now readily introduce those quantities below from [8]. We define the connection coefficients on $\mathcal{M}_{R\times S^3}$ as

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} =& \frac{1}{2}g^{\gamma\lambda}\left(\partial_\mu g_{v\gamma} + \partial_v g_{\gamma\mu} - \partial_\gamma g_{\mu\nu}\right)\\ &+ \frac{1}{2}g^{\gamma\lambda}\left(C^\rho_{\mu\nu}g_{\rho\gamma} + C^\rho_{\gamma\mu}g_{\rho\nu} - C^\rho_{\nu\gamma}g_{\rho\mu}\right).\end{aligned} \tag{3.26}$$

From the above definition of Christoffel symbols, one can easily verify that

$$\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} - C^\lambda_{\mu\nu} = 0. \tag{3.27}$$

Covariant derivative in this case for any contravariant vector V^α will be given by

$$D_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\lambda}V^\lambda \tag{3.28}$$

where $\Gamma^\alpha_{\mu\lambda}$ is now defined by (3.26). Riemann Christoffel curvature tensor can be defined as follows:

$$R^\sigma_{\gamma\mu\nu} = \partial_\mu\Gamma^\sigma_{\nu\gamma} - \partial_\nu\Gamma^\sigma_{\mu\gamma} + \Gamma^\rho_{\nu\gamma}\Gamma^\sigma_{\mu\rho} - \Gamma^\rho_{\mu\gamma}\Gamma^\sigma_{\nu\rho} - C^\rho_{\mu\nu}\Gamma^\sigma_{\rho\gamma}. \tag{3.29}$$

From this, the definition of Ricci tensor and scalar curvature is self explanatory. On non commutative $R \times S^3$ topology, one has the same form of Einstein's field equation as Eqn. (3.15) but with mathematical differences which are self explanatory.

3.3 Line element on non commutative topology

Parametrization (3.17), (3.18) of the S^3 sphere leads to the following metric [12]:

$$d\sigma^2 = \cos^2\theta(d\alpha)^2 + \sin^2\theta(d\beta)^2 + (d\theta)^2 \tag{3.30}$$

where $0 \le \theta \le 2\pi$, $0 \le \alpha, \beta \le 2\pi$. The Lorentzian form of the above metric is

$$ds^2 = d\sigma^2 - dt^2. \tag{3.31}$$

or more generally $ds^2 = g_{\mu\nu}d\theta^\mu d\theta^\nu$. Furthermore, one can introduce pseudo-orthonormal tetradic frames as follows [12][13]:

$$e_1 = \frac{1}{a}\left[\cos(\alpha+\beta)\left[\tan\theta\frac{\partial}{\partial\alpha} - \cot\theta\frac{\partial}{\partial\beta}\right] - \sin(\alpha+\beta)\frac{\partial}{\partial\theta}+\right] \tag{3.32}$$

$$e_2 = \frac{1}{a}\left[\sin(\alpha+\beta)\left[\tan\theta\frac{\partial}{\partial\alpha} - \cot\theta\frac{\partial}{\partial\beta}\right] + \cos(\alpha+\beta)\frac{\partial}{\partial\theta}+\right] \tag{3.33}$$

$$e_3 = -\frac{1}{a}\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right) \tag{3.34}$$

$$e_4 = \frac{\partial}{\partial t}. \tag{3.35}$$

The commutation relation for the above pseudo-orthonormal tetradic frames is

$$[e_1, e_2] = D^c_{ab}e_c = \frac{2}{a}\varepsilon^c_{4ab}e_c \tag{3.36}$$

with $\varepsilon_{1234} = -1$. In this case, one can generally write the line element as $ds^2 = g_{\mu\nu}\omega^\mu\omega^\nu = \eta_{\mu\nu}\omega^\mu\omega^\nu$ where ω^μ $\eta_{\mu\nu} = diag(1, 1, 1, -1)$ is the dual orthonormal base corresponding to pseudo-orthonormal tetradic frames.

The flat space-time line element (3.31) will be crucial in solving the field equations for gravitation and also will be referenced numerous times through the paper where we will need some metric for providing applications and examples of our derived equations.

4 Tensor Calculus

4.1 Tensors and covariant derivatives

Using the line element (3.12), we can define a covariant tensor transformation of rank two on commutative $R \times S^3$ topology as follows:

$$T_{\mu\nu} = \frac{L\tilde{\Theta}^\alpha}{L\Theta^\mu}\frac{L\tilde{\Theta}^\beta}{L\Theta^\nu}\tilde{T}_{\alpha\beta}. \quad (4.1)$$

Similarly, we can define contravariant and mixed tensors of rank two as

$$T^{\mu\nu} = \frac{L\Theta^\mu}{L\tilde{\Theta}^\alpha}\frac{L\Theta^\nu}{L\tilde{\Theta}^\beta}\tilde{T}^{\alpha\beta} \quad (4.2)$$

$$T^\mu{}_\nu = \frac{L\Theta^\mu}{L\tilde{\Theta}^\alpha}\frac{L\tilde{\Theta}^\beta}{L\Theta^\nu}\tilde{T}^\alpha_\beta \quad (4.3)$$

respectively. Furthermore, covariant derivatives of the above two rank tensors can be defined as

$$\nabla_\rho T_{\mu\nu} = L_\rho T_{\mu\nu} - \Gamma^\sigma_{\mu\rho}T_{\sigma\nu} - \Gamma^\sigma_{\rho v}T_{\mu\sigma} \quad (4.4)$$

$$\nabla_\rho T^{\mu\nu} = L_\rho T^{\mu\nu} + \Gamma^\mu_{\rho\sigma}T^{\sigma\nu} + \Gamma^\nu_{\sigma\rho}T^{\mu\sigma} \quad (4.5)$$

and

$$\nabla_\rho T^\mu{}_\nu = L_\rho T^\mu{}_\nu + \Gamma^\mu_{\rho\sigma}T^\sigma{}_\nu - \Gamma^\sigma_{v\rho}T^\mu{}_\sigma \quad (4.6)$$

respectively. Note that the Christoffel symbols in this case are defined by (3.13).

4.2 Christoffel symbols and their properties

Using definition (3.13), we can define the Christoffel symbols of first kind using the relation $\Gamma^\lambda_{\alpha\beta} = g^{\lambda\mu}\Gamma_{\mu\alpha\beta}$ as follows:

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(L_\beta g_{\mu\alpha} + L_\alpha g_{\mu\beta} - L_\mu g_{\alpha\beta}\right). \quad (4.7)$$

The Christoffel symbols $\Gamma_{\mu\alpha\beta}$ and $\Gamma^\lambda_{\alpha\beta}$ are symmetric with respect to indices α and β. Furthermore, Christoffel symbols satisfy following properties: *i)*

$$\Gamma_{\mu\alpha\beta} = g_{\lambda\mu}\Gamma^\lambda_{\alpha\beta}, \quad (4.8)$$

ii)

$$\Gamma_{\alpha\mu\beta} + \Gamma_{\beta\mu\alpha} = L_\mu g_{\alpha\beta} \quad (4.9)$$

and *iii)*

$$L_\mu g_{\alpha\beta} = -g^{\beta\rho}\Gamma^\alpha_{\rho\mu} - g^{\alpha\sigma}\Gamma^\beta_{\sigma\mu}. \quad (4.10)$$

$i)$ and $ii)$ are trivial and based on routine calculations. For $iii)$, consider $g^{\alpha\beta}g_{\rho\beta} = \delta^\alpha_\rho$. Differentiate with respect to Θ^μ to get

$$g^{\alpha\beta}L_\mu g_{\rho\beta} + g_{\rho\beta}L_\mu g^{\alpha\beta} = 0. \quad (4.11)$$

Multiplying by $g^{\rho\sigma}$ and rearranging the terms, we get

$$g^{\rho\sigma}g_{\rho\beta}L_\mu g^{\alpha\beta} = -g^{\rho\sigma}g^{\alpha\beta}L_\mu g_{\rho\beta}. \quad (4.12)$$

Now, substitute $ii)$ in right hand side of the equation and contract terms to get the desire result.

Let $\Gamma_{\mu\alpha\beta}$ be a function of coordinate Θ^α and $\tilde{\Gamma}_{\mu\alpha\beta}$ of $\tilde{\Theta}^\alpha$. Then, we can define the transformation law for first kind Christoffel symbols as follows:

$$\tilde{\Gamma}_{\mu\alpha\beta} = \frac{L^2\Theta^\gamma}{L\tilde{\Theta}^\alpha L\tilde{\Theta}^\beta}\frac{L\Theta^\delta}{L\tilde{\Theta}^\mu}g_{\gamma\delta} + \frac{L\Theta^\gamma}{L\tilde{\Theta}^\alpha}\frac{L\Theta^\delta}{L\tilde{\Theta}^\beta}\frac{L\Theta^\rho}{L\tilde{\Theta}^\mu}\Gamma_{\rho\gamma\delta}. \quad (4.13)$$

Similarly, for Christoffel symbols of second kind we have the following transformation:

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \frac{L\tilde{\Theta}^\mu}{L\Theta^\rho}\frac{L^2\Theta^\rho}{L\tilde{\Theta}^\alpha L\tilde{\Theta}^\beta} + \frac{L\tilde{\Theta}^\mu}{L\Theta^\rho}\frac{L\Theta^\gamma}{L\tilde{\Theta}^\alpha}\frac{L\Theta^\delta}{L\tilde{\Theta}^\beta}\Gamma^\rho_{\gamma\delta}. \quad (4.14)$$

4.3 Riemann Christoffel curvature tensor

The Riemann Christoffel curvature tensor or curvature tensor is a 4th order mixed rank tensor defined by (3.16). The curvature tensor is anti symmetric with respect to the indices β and γ. Thus

$$R^\rho_{\alpha\beta\gamma} = -R^\rho_{\alpha\gamma\beta}. \quad (4.15)$$

The curvature tensor also satisfy the following cyclic property:

$$R^\rho_{\alpha\beta\gamma} + R^\rho_{\beta\gamma\alpha} + R^\rho_{\gamma\alpha\beta} = 0. \quad (4.16)$$

Contracting the indices of curvature tensor, we get Ricci tensor as follows:

$$R^\rho_{\alpha\beta\rho} = R_{\alpha\beta} = \frac{L^2\log\sqrt{g}}{L\Theta^\beta L\Theta^\alpha} - \frac{L\Gamma^\rho_{\alpha\beta}}{L\Theta^\rho} + \Gamma^\rho_{\sigma\beta}\Gamma^\sigma_{\alpha\rho} - \Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{\alpha\beta}. \quad (4.17)$$

The curvature tensor can also be contracted in another way as $R^\rho_{\alpha\rho\beta}$ which is related to the above contraction of the curvature tensor and Ricci tensor as follows:

$$R^\rho_{\alpha\rho\beta} = -R^\rho_{\alpha\beta\rho} = -R_{\alpha\beta}. \quad (4.18)$$

The Ricci tensor is symmetric with respect to its two indices. Thus,

$$R_{\alpha\beta} = R_{\beta\alpha}. \quad (4.19)$$

Furthermore, contracting $R_{\alpha\beta}$ gives $R = g^{\alpha\beta}R_{\alpha\beta}$ which is also known as Ricci scalar our scalar curvature.

5 covariant Riemann Christoffel tensor

The covariant Riemann Christoffel tensor or the covariant curvature tensor is defined as

$$R_{\alpha\beta\gamma\rho} = g_{\alpha\delta}R^\delta_{\beta\gamma\rho} = \frac{1}{2}\left(\frac{L^2 g_{\alpha\rho}}{L\Theta^\beta L\Theta^\gamma} + \frac{L^2 g_{\beta\gamma}}{L\Theta^\alpha L\Theta^\rho} - \frac{L^2 g_{\alpha\gamma}}{L\Theta^\beta L\Theta^\rho} - \frac{L^2 g_{\beta\rho}}{L\Theta^\alpha L\Theta^\gamma}\right) + g^{\mu\nu}\Gamma^\mu_{\beta\gamma}\Gamma^\nu_{\alpha\rho} - g^{\mu\nu}\Gamma^\mu_{\beta\rho}\Gamma^\nu_{\alpha\gamma} \quad (5.1)$$

and satisfy following properties:
i)

$$R_{\beta\alpha\gamma\rho} = -R_{\alpha\beta\gamma\rho} \quad (5.2)$$

ii)

$$R_{\alpha\beta\rho\gamma} = -R_{\alpha\beta\gamma\rho} \quad (5.3)$$

iii)

$$R_{\gamma\rho\alpha\beta} = R_{\alpha\beta\gamma\rho} \quad (5.4)$$

iv)

$$R_{\alpha\beta\gamma\rho} + R_{\alpha\gamma\rho\beta} + R_{\alpha\rho\beta\gamma} = 0. \quad (5.5)$$

Both curvature tensor and covariant curvature tensor satisfy the following Bianchi identities:

i)

$$\nabla_\sigma R^\alpha_{\beta\gamma\delta} + \nabla_\gamma R^\alpha_{\beta\delta\sigma} + \nabla_\delta R^\alpha_{\beta\sigma\gamma} = 0 \quad (5.6)$$

ii)

$$\nabla_\sigma R_{\alpha\beta\gamma\rho} + \nabla_\gamma R_{\alpha\beta\rho\sigma} + \nabla_\rho R_{\alpha\beta\sigma\gamma} = 0. \quad (5.7)$$

6 Killing Vectors

6.1 Killing equation

We all know it from the work of Emmy Noether that symmetries lead to conservation laws. So, one might be interested to know how we find symmetries in general relativity because it is so geometric in nature. Mathematically speaking, symmetries in General Relativity occur when metric is same from point to point.

A systematic approach to tackle this symmetries is to introduce the notion of Killing vectors. A Killing vector X is a vector that satisfies the Killing equation:

$$\nabla_\nu X_\mu + \nabla_\mu X_\nu = 0. \quad (6.1)$$

The above equation in contravariant form can be written as

$$\nabla_\nu X^\mu + \nabla_\mu X^\nu = 0. \quad (6.2)$$

Since we are working on commutative $\mathcal{M}_{R\times S^3}$, covariant derivative is thus defined as

$$\nabla_\nu X^\mu = L_\nu X^\mu + \Gamma^\mu_{\rho\nu} X^\rho. \quad (6.3)$$

If a set of points is displaced by $X^\mu d\Theta_\mu$ on the vector field X and all distance relation remain same then X is a Killing vector. This thus concludes that if you move along the distance of Killing vector then the metric tensor does not changes. Thus, the particle moving along Killing vector will not experience any force.

We shall now mathematically show that if the lie derivative of metric tensor $\pounds_X g$ vanishes then that implies Killing equation for the vector field X. The lie derivative of metric tensor on commutative $R \times S^3$ topology is given by

$$\pounds_X g_{\mu\nu} = X^\rho L_\rho g_{\mu\nu} + g_{\rho\nu} L_\mu X^\rho + g_{\mu\rho} L_\nu X^\rho. \quad (6.4)$$

Since covariant derivative of metric tensor vanishes, we have

$$\nabla_\rho g_{\mu\nu} = L_\rho g_{\mu\nu} - \Gamma^\sigma_{\mu\rho} g_{\sigma\nu} - \Gamma^\sigma_{\nu\rho} g_{\mu\sigma} = 0 \quad (6.5)$$

$$L_\rho g_{\mu\nu} = \Gamma^\sigma_{\mu\rho} g_{\sigma\nu} + \Gamma^\sigma_{\nu\rho} g_{\mu\sigma}. \quad (6.6)$$

Using this, we can rewrite the definition of lie derivative of metric tensor as

$$L_X g_{\mu\nu} = X^\rho L_\rho g_{\mu\nu} + g_{\rho\nu} L_\mu X^\rho + g_{\mu\rho} L_\nu X^\rho \quad (6.7)$$

$$= g_{\sigma\nu} \Gamma^\sigma_{\mu\rho} X^\rho + g_{\mu\sigma} \Gamma^\sigma_{\nu\rho} X^\rho + g_{\rho\nu} L_\mu X^\rho + g_{\mu\rho} L_\nu X^\rho \quad (6.8)$$

Consider the second term $g_{\mu\sigma}\Gamma^\sigma_{\nu\rho}X^\rho$. Since ρ and σ are dummy indices, we can rewrite the term as $g_{\mu\rho}\Gamma^\rho_{\nu\sigma}X^\sigma$. And secondly, using the definition of covariant derivative, we can rewrite the lie derivative of the metric tensor as

$$\pounds_X g_{\mu\nu} = g_{\mu\rho} L_\nu X^\rho + g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} X^\rho + g_{\rho\nu} L_\mu X^\rho + g_{\sigma\nu}\Gamma^\sigma_{\mu\rho} X^\rho \quad (6.9)$$

$$= g_{\mu\rho} L_\nu X^\rho + g_{\mu\rho}\Gamma^\rho_{\nu\sigma} X^\sigma + g_{\rho\nu} L_\mu X^\rho + g_{\sigma\nu}\Gamma^\sigma_{\mu\rho} X^\rho \quad (6.10)$$

$$= g_{\mu\rho}\nabla_\nu X^\rho + g_{\rho\nu} L_\mu X^\rho + g_{\sigma\nu}\Gamma^\sigma_{\mu\rho} X^\rho. \quad (6.11)$$

Now similarly, switching indices in the last term yields

$$\pounds_X g_{\mu\nu} = g_{\mu\rho}\nabla_\nu X^\rho + g_{\rho\nu} L_\mu X^\rho + g_{\rho\nu}\Gamma^\rho_{\mu\sigma} X^\sigma \quad (6.12)$$

$$= g_{\mu\rho}\nabla_\nu X^\rho + g_{\rho\nu}\nabla_\mu X^\rho \quad (6.13)$$

$$= \nabla_\nu (g_{\mu\rho} X^\rho) + \nabla_\mu (g_{\rho\nu} X^\rho). \quad (6.14)$$

$$\nabla_\nu X_\mu + \nabla_\mu X_\nu. \quad (6.15)$$

Since $\pounds_X g_{\mu\nu} = 0$, we get

$$\nabla_\nu X_\mu + \nabla_\mu X_\nu = 0 \quad (6.16)$$

6.2 Conserved current

Define the following quantity as a current

$$J^\mu = T^{\mu\nu} X_\nu. \quad (6.17)$$

where X is the Killing vector and $T^{\mu\nu}$ is the stress-energy tensor. If we take the covariant derivative of J^μ, we get

$$\nabla_\mu J^\mu = \nabla_\mu (T^{\mu\nu} X_\nu) \quad (6.18)$$

$$= (\nabla_\mu T^{\mu\nu}) X_\nu + T^{\mu\nu} (\nabla_\mu X_\nu) \quad (6.19)$$

$$= T^{\mu\nu} (\nabla_\mu X_\nu) \quad (6.20)$$

where we have dropped the covariant derivative of stress-energy tensor because it is a conserved quantity. Furthermore, due to the symmetric nature of stress-energy indices, we can rewrite $\nabla_\mu J^\mu$ as

$$\nabla_\mu J^\mu = \frac{1}{2}(T^{\mu\nu}\nabla_\mu X_\nu + T^{\nu\mu}\nabla_\nu X_\mu) \quad (6.21)$$

$$= \frac{1}{2} T^{\mu\nu} (\nabla_\mu X_\nu + \nabla_\nu X_\mu) = 0 \quad (6.22)$$

Thus proves that J^μ is a conserved quantity.

6.3 Killing equations on non commutative topology

Similarly as of in the previous case of commutative $R \times S^3$ topology, we can define the Killing equation on non commutative $R \times S^3$ topology as

$$\nabla_\nu X_\mu + \nabla_\mu X_\nu = 0. \quad (6.23)$$

Here, the covariant derivative is now defined by

$$\nabla_\nu X_\mu = \partial_\nu X_\mu + \Gamma^\mu_{\nu\rho} X_\rho \quad (6.24)$$

where

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\gamma\mu} \left(\frac{\partial g_{\rho\gamma}}{\partial \theta^{\nu}} + \frac{\partial g_{\nu\gamma}}{\partial \theta^{\rho}} - \frac{\partial g_{\nu\rho}}{\partial \theta^{\gamma}} \right)$$
$$+\frac{1}{2} g^{\gamma\mu} \left(C^{\sigma}_{\upsilon\rho} g_{\sigma\gamma} + C^{\sigma}_{\gamma\nu} g_{\sigma\rho} - C^{\sigma}_{\rho\gamma} g_{\sigma\nu} \right). \qquad (6.25)$$

and the derivatives are defined by definitions provided in sub section 3.2.

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