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Flows Local Control in Resource Networks with A Low Resource

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Abstract The flow control problem in resource networks consists in finding such a set of vertices and capacities of arcs, which go out from these vertices, such that the limit state of the resource network Q^* is the closest to the given state Q'. This problem is naturally divided into two subproblems. The first of them is the "local" subproblem, which consists in determining the capacities of arcs which go out from the vertices of a given subset S (hereinafter, the set S will be called the set of controlled vertices). The second subproblem is the "global" subproblem, which consists in finding the optimal set of controlled vertices S, consisting of at most s elements. The paper is devoted to the study of the possibility of flows local control in resource networks. Methods for solving a local subproblem for regular resource networks with a low resource allocation are proposed. The conditions for the unreachability of the limit state Q^*, which coincides with the given state Q' are obtained. Three cases are considered for the distribution of controlled vertices on a resource network. In each of the considered cases, it is shown that if the condition of unreachability of the limit state is not satisfied, then there is a set of the capacities values of the arcs that go out the controlled vertices, for which the limit state Q^* coincides with the state Q'.

Keywords Flows in Networks, Resource Networks, Limit State, Limit Flow, Graph Algorithms, Flow Control

1 Introduction

Research on dynamic flow models has been going on for quite some time. So, for example, in papers [7], [9]-[11] dynamic periodic networks and flow problems in such networks are investigated. In [5] it is shown that such networks are similar to graphs with non-standard reachability and an approach for solving problems on flows in such networks is proposed. Resource networks, which are special case of dynamic flow models on graphs, have been proposed in the papers of O.P. Kuznetsov and L.Yu. Zhilyakova (see, for example, [10]-[12], [17]).

A resource network is a connected directed graph, for each arc of which the capacity (throughput) and for each vertex, the amount of resource at this vertex are assigned. At each moment of discrete time, the resource of each vertex is reallocated among adjacent vertices according to certain rules that satisfy two conditions. The first condition (closeness condition): the resource at any vertex of the network is not added from the outside and does not disappear. The second condition (continuity condition): the resource leaving the vertex is subtracted, and the one entering the vertex is added to the resource of the given vertex. Thus, between each successive moments of time, a flow in its classical sense passes along the arcs of the network.

Since the resource is reallocated among the network vertices in a certain proportion, the problem of finding the limit resource distribution in the network is similar both to the problem of finding a balanced flow that considered in the article [5], and to the problem of flow distribution considered in the article [14].

For resource networks, the question of finding the next in time states, as well as the limit state, if it exists for a given initial resource distribution, has been well studied. Also, in the article [18], the direct and inverse problems of limit state control in the case of absorbing networks are considered. This paper is devoted to the development of methods for solving the flow control problem in resource networks (hereinafter referred to as the flow control problem), which consists in finding such a set of vertices and capacities of arcs, which go out from these vertices, such that the limit state of the resource network Q^* is the closest to the given state Q'. This problem is naturally divided into two subproblems:

1. the "local" subproblem, which consists in determining the capacities of arcs which go out from the vertices of a given subset S (hereinafter, the set S will be called the set of controlled vertices);

2. the "global" subproblem, which consists in finding the optimal set of controlled vertices S, consisting of at most s elements.

Within the framework of this article, methods for solving a local subproblem for regular resource networks with a low resource allocation are proposed. A condition for the unreachability of the limit state Q^*, which coincides with the given state Q', is obtained. Three cases regarding the distribution of controlled vertices on a resource network are considered. In each of the considered cases, it is shown that if the condition of unreachability of the limit state is not satisfied, then there is a values collection of the capacities of the arcs going out from controlled vertices, for which the limit state Q^* coincides with the state Q'.

2 Basic concepts

Here are the basic concepts, definitions and statements (see [10]-[12], [17]-[20]).

Definition 1. *A resource network is a directed network $G(X,U)$ (where $X = \{x_1,\dots,x_n\}$) without sources and sinks, for each arc (x_i,x_j) of which the capacity r_{ij} is assigned and the vector-function $Q(t) = (q_1(t),\dots,q_n(t))$, where $q_i(t) \geq 0$ for all $i \in [1;n]_Z$, is given.*

An element $q_i(t)$ is a quantity of resource at the vertex x_i at the moment of time t.

In order to determine the vector-function $Q(t)$, the vector $Q(0)$ of the initial resource allocation in the network G is given and the rules for resource reallocation (the rules of the network functioning) are specified:

$$q_i(t+1) = q_i(t) - \sum_{j=1}^{n} F_{ij}(t) + \sum_{j=1}^{n} F_{ji}(t) \quad \forall\, i \in [1;n]_Z, \quad (1)$$

where the value $F_{ij}(t)$ of the resource flow that goes along the arc (x_i,x_j) at the time t is determined as follows:

$$F_{ij}(t) = \begin{cases} r_{ij}, & q_i(t) > \sum\limits_{k=1}^{n} r_{ik}; \\ \dfrac{r_{ij}}{\sum\limits_{k=1}^{n} r_{ik}} \cdot q_i(t), & q_i(t) \leq \sum\limits_{k=1}^{n} r_{ik} \end{cases}. \quad (2)$$

Expressions (1)-(2) can be rewritten in a shorter form (3) (see [15])

$$Q(t) = \mathcal{A}(Q(t+1)). \quad (3)$$

Denote the value of total resource in network as W, that is $W = \sum\limits_{i=1}^{n} q_i(0)$.

Definition 2. *A state $Q(t)$ is called a stable state, if $Q(t) = Q(t+1)$.*

According to the rules of resource reallocation, if $Q(t)$ is stable, then for all natural numbers i the equality $Q(t) = Q(t+i)$ holds.

Definition 3. *The state $Q^* = (q_1^*,\dots,q_n^*)$ is called asymptotically reachable from the state $Q(0)$ if for each $i \in [1;n]_Z$ and every $\varepsilon > 0$ there exists t_ε such that for all $t > t_\varepsilon$ the inequality $|q^* - q_i(t)| < \varepsilon$ holds.*

Definition 4. *A state Q^* is called limit if it is either stable and there exists a moment of time t such that $Q^* = Q(t)$, or it is asymptotically reachable from the state $Q(0)$.*

Definition 5. *The distance between the states of the resource network $Q_1 = (q_1^1,\dots,q_n^1)$ and $Q_2 = (q_1^2,\dots,q_n^2)$ is the quantity $\rho(Q_1,Q_2) = \sum\limits_{i=1}^{n} |q_i^1 - q_i^2|$.*

Definition 6. *A resource network is called an ergodic if it is strongly connected.*

Definition 7. *An ergodic resource network is called a regular if there are at least two cycles that lengths are co-prime numbers.*

Within the framework of this article, we consider regular resource networks only.

Define the sets of vertices $Z^+(t)$ and $Z^-(t)$ as follows. We say that for all $i \in [1;n]_Z$ at moment of time t the vertex x_i belongs to $Z^-(t)$ if $q_i(t) \leq \sum\limits_{j=1}^{n} r_{ij}$, otherwise we say that $x_i \in Z^+(t)$.

In other words, the set $Z^-(t)$ consists of vertices x_i of the resource network that at moment of time t transmit their entire current resource along outgoing arcs, i.e. each arc that go out from the vertex $x_i \in Z^-(t)$ is saturated with the resource flow according to the second rule (second line) in (2). The set $Z^+(t)$ is formed by all such a vertices, for which all their outgoing arcs are completely saturated with the resource flow according to rule 1 (first line) in (2). The last means that such vertices do not transmit their entire current resource, but only part of it.

Definition 8. *We say that vertex x enters the Z^- zone, if there exists a moment of time t' such that $x \in Z^-(t)\ \forall\, t \geq t'$.*

Definition 9. *A threshold value for the resource network G is such a value T for which if $W \leq T$, then all vertices of the resource network G go to the Z^- zone . Otherwise, for each moment of time t the set $Z^+(t) \neq \emptyset$.*

Definition 10. *We will say that a low resource is reallocated in the resource network if $W \leq T$.*

In addition, we introduce the following notation: $[x]^+$ is the set of arcs that go out from the vertex x; $[x]^-$ is the set of arcs that go to the vertex x; $V^+ = \bigcup\limits_{x\in V} [x]^+$ is the set of all arcs that go out from the vertices of the set V; $V^- = \bigcup\limits_{x\in V} [x]^-$ is the set of all arcs that go to the vertices of the set V.

3 The problem of low resource flows local control in regular resource networks

Let $G(X,U)$ be a resource network for which the state $Q' = (q'_1, \ldots, q'_n)$ and the set of controlled vertices S are given. Consider the problem of determining the capacities of the arcs of set S^+ in the network G to minimize the value $\rho(Q^*, Q')$.

First, consider the question of reachability of such a limit state for which the distance $\rho(Q^*, Q') = 0$. In this case, the equality $Q^* = Q'$ takes place, which means that the state Q' must be the limit state for the network G. The last implies that $\mathcal{A}(Q') = Q'$. Since the capacities of the arcs of the set S^+ have not yet been determined, therefore, the following theorem holds.

Theorem 1. *Let G be a resource network and S be a set of controlled vertices. If there exists a vertex $x_i \in X \setminus (S \cup \Gamma(S))$ for which $q'_i \neq (\mathcal{A}(Q'))_i$ for a certain collection of capacities values of arcs of the set S^+, then for any collection of capacities values of arcs of the set S^+, the inequality $\rho(Q^*, Q') > 0$ takes place.*

Thus, the first condition for the unreachability of a limit state that equal to a given one has been obtained.

Let, for a given state Q', for a certain set of values of the capacities of arcs of the set S^+, the values of the local flows F'_{xy} are determined at one moment of the time between each pair of vertices x and y in (2). Then the second condition of unreachability of the limit state that equal to a given one takes place:

Theorem 2. *Let G be a resource network and S be a set of controlled vertices. If for any vertex $x_i \in X \setminus (S \cup \Gamma(S))$ for a certain collection of capacities values of arcs of the set S^+ the equality $q'_i \neq (\mathcal{A}(Q'))_i$ takes place, but at least one of the following inequalities (4) is not satisfied*

$$\begin{cases} q'_x - \sum\limits_{y \in \Gamma^-(x) \setminus S} F'_{yx} > 0 \quad \forall\, x \in \Gamma(S); \\ q'_x \le \sum\limits_{y \in \Gamma(x)} \left(q'_y - \sum\limits_{z \in \Gamma^-(y) \setminus S} F'_{zy} \right) \quad \forall\, x \in S; \\ q'_x \le C(x) \quad \forall\, x \in S \cup \Gamma(S). \end{cases}, \qquad (4)$$

where

$$C(x) = \sum_{y \in \Gamma^-(x) \setminus S} \left(q'_y - \sum_{z \in \Gamma(y) \setminus \{x\}} F'_{yz} \right) + \sum_{y \in \Gamma(x) \cap S} q'_y,$$

then for any collection of capacities values of arcs of the set S^+, the inequality $\rho(Q^*, Q') > 0$ takes place.

Proof. Consider the inequality which is described by the first line of (4). In the its leftside part there is the difference between the resource value at the vertex x and the sum of flows which go from the vertices that do not belong to the set of controlled vertices S. It is clear that if such a difference is less than or equal to zero, and since there exists at least one arc from the vertices of the set S to the vertex x, then there is an excess of resource for the element $\mathcal{A}(Q')_x$. This means $q'_x < \mathcal{A}(Q')_x$. Hence, the state Q' is not stable, and therefore, not a limit state.

Consider the inequality which is described by the second line of (4). In the its right-side part, in brackets, there are the resource values, which must be reached using flows from controlled vertices only. It is clear that if the sum of such values is strictly less than the resource value of the vertex $x \in S$, then at least for one of the vertices $y \in \Gamma(x)$, there is a lack of resource for the element $\mathcal{A}(Q')_y$. The last means $q'_y > \mathcal{A}(Q')_y$. Hence, the state of Q' is not stable, and therefore, not a limit state.

The analogical case is for the inequality which is described by the third line of (ref eq: mainIneq). In the its right-side part, there are the total sum of the resource values, such that with only the values of the second sum, it is necessary to reach the value of q'_x. It is clear that if the sum of such values is strictly less than the resource value of the vertex x, then at the vertex x, there is a lack of resource for the element $\mathcal{A}(Q')_x$. The last means $q'_x < \mathcal{A}(Q')_x$. Hence, the state Q' is not stable, and therefore, not a limit state.

Theorem is proved.

Remark. *If we consider large flows in resource networks, then in the case when for each vertex $x_i \in X \setminus (S \cup \Gamma(S))$ the equality $q'_i = (\mathcal{A}(Q'))_i$ and all inequalities in (4) hold for a certain collection of capacities values of arcs of the set S^+, it is not always possible to choose a collection such that the limit state Q^* to be equal the given state Q'.*

Consider now the problem of determining the capacities of the arcs of the network G to minimize the value $\rho(Q^*, Q')$. Hereinafter assume that for a certain collection of values of the capacities of the arcs of the set S^+ for each vertex $x_i \in X \setminus (S \cup \Gamma(S))$ the equality $q'_i = (\mathcal{A}(Q'))_i$ holds.

Denote the subgraph of the graph G which is generated by the set $Y \subset X$ by $\widehat{G}_Y$. For convenience, consider three cases regarding the distribution of the set of controlled vertices on the network G:

a) the subgraph $\widehat{G}_S$ does not contain arcs, and the subgraph $\widehat{G}_{S \cup \Gamma(S)}$ does not contain any pair of arcs of the form (x, y) and (z, y), where $x, z \in S$;

b) the condition of item a) is not satisfied, but the subgraph $\widehat{G}_{S \cup \Gamma(S)}$ does not contain cycles;

c) the subgraph $\widehat{G}_{S \cup \Gamma(S)}$ contains at least one cycle.

Note that in case a) the problem of finding the capacities of arcs of the set S^+ is reduced to a similar problem for the one-element set S.

Let $G(X,U)$ be a regular resource network for which a state Q' and a set of controlled vertices $S \subset X$ are given. And assume that for a certain collection of capacities values of arcs of the set S^+, for each vertex $x_i \in X \setminus (S \cup \Gamma(S))$ the equality $q'_i = (\mathcal{A}(Q'))_i$ and all inequalities (4) hold. Then the following Lemma takes place.

Lemma 1. *If the given set of controlled vertices S in resource network G satisfies the following condition*

$$\forall\, x, y \in S \quad (x \neq y) \;\Rightarrow\; \Rightarrow \; ((\{x\} \cup \Gamma(x)) \cap (\{y\} \cup \Gamma(y)) = \emptyset)\,, \qquad (5)$$

then there exists a collection of capacities values of arcs of the set S^+ for which in network G there exists a limit state $Q^* = Q'$.

Proof. Consider schematically two adjacent layers (see [9]) of the network expander-in-time G' relatively to some vertices $x, y \in S$ in the figure 1. Here, the oblique arcs correspond to the arcs of the original network, and the vertical arcs correspond to possible transfers of the resource remnant between moments of time. Thus, the condition of proportionality of flows values is specified for oblique arcs only.

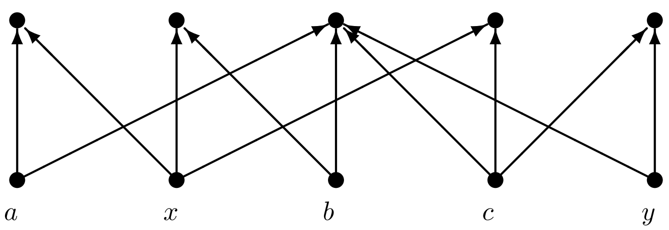

Figure 1. Schema of the network expander-in-time relatively to vertices $x, y \in S$.

In order to prove, it is necessary to show that in such a network there exists a limit state Q^* which is equal to the state Q'. Since the state Q' must be stable (due to the stability of the limit state), which means that the resource quantities at all vertices are known at each of the two considered here moments of time t and $t+1$, therefore, the required capacities of the arcs, which go out from vertex x, can be found from the relations (6) for the graph G'

$$\forall\, y \in \Gamma(x) \quad r_{xy} = q'_y(t+1) - \sum_{z \in \Gamma^-(y) \setminus \{x\}} F_{zy}(t). \qquad (6)$$

Note that all values $F_{yz}(t)$ for $y \notin S$ are known, since the capacities are given for all such vertices. Then since the condition (5) means, that for the case under consideration the subgraph, which is generated by the set $S \cup \Gamma(S)$, does not contain any pair of arcs of the form (x, y) and (z, y), where $x, z \in S$, therefore, in the right-hand sides of of the relations (6), all values F_{yz} and F_{zy} are known. This also allows to consider in this case only the tree structure, which is represented in the figure 2, for the scheme in the figure 1.

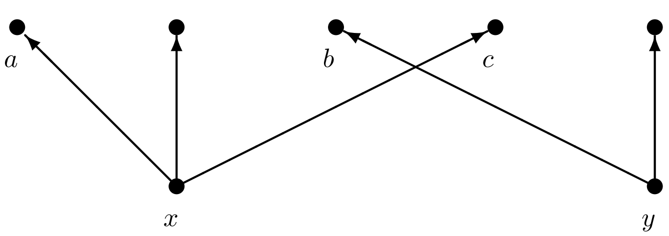

Figure 2. The tree structure of the network expander-in-time relatively to vertices $x, y \in S$.

Thus, the relations (6) for a vertex $x \in S$ gives us all the capacities of the arcs of the set $[x]^+$, and hence all the capacities of arcs of the set S^+ are obtained. The state Q' is stable for obtained capacities, hence Q' is a limit state. Wherein, since the resource network is regular, then the limit state Q^* exists in the network with the found capacities. Moreover, since only the case of a small resource is considered, therefore, by Theorem 2 from [17], such a limit state Q^* is unique. It follows that $Q^* = Q'$.

The lemma is proved.

Consider the case b). The set S is defined in such a way that the subgraph $\widehat{G}_S$ contains at least one arc or on the subgraph $\widehat{G}_{S \cup \Gamma(S)}$ there is at least one pair of arcs of the form (x, y) and (z, y), where $x, z \in S$, but the subgraph $\widehat{G}_{S \cup \Gamma(S)}$ does not contain cycles.

This case, for a bipartite expander-in-time G', only a tree structure can also be considered. However, some connectivity components of such a structure contain several "roots", which are the vertices of the bottom layer and correspond to the vertices of the set S. Consider in the figure 3 a scheme of such a tree structure relatively to vertices $x, y \in S$. For a larger number of root vertices, all reasoning is similar.

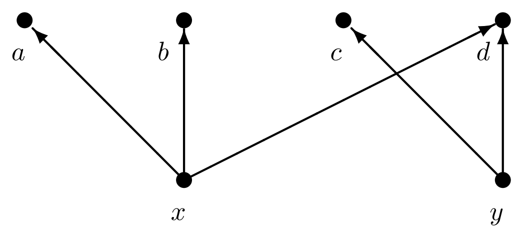

Figure 3. The tree structure of the network expander-in-time relatively to vertices $x, y \in S$ in the case b).

Note that since the reallocation of the resource occurs according to the rules(1)-(2), the total resource quantity is constant and the inequalities (4) take place, then the equality

$$q_a(t+1) + q_b(t+1) + q_c(t+1) + q_d(t+1) = q'_x(t) + q'_y(t)$$

and inequalities

$$\begin{cases} q'_x(t) \le q_a(t+1) + q_b(t+1) + q_d(t+1); \\ q'_y(t) \le q_c(t+1) + q_d(t+1). \end{cases}$$

hold. The values $q_z(t+1)$ $(z \in \{a, b, c, d\})$ are defined as follows (similar to the right-hand sides of the relations (6)):

$$q_z(t+1) = q'_z(t+1) - \sum_{v \in \Gamma^-(z) \setminus S} F_{vz}(t).$$

Based on this, the capacities of the arcs (x, a), (x, b) and (y, c) are assumed to be equal $r_{xa} = q_a(t+1)$ $r_{xb} = q_a(t+1)$ and $r_{yc} = q_c(t+1)$ respectively. Note that flow values along these arcs are equal to capacities. Then, we obtain the capacities of the remaining arcs: $r_{xd} = q'_x(t) - F_{xa}(t) - F_{xb}(t)$ and $r_{yd} = q'_y(t) - F_{yc}(t)$.

Thus, the following statement holds.

Lemma 2. *Let G be a regular resource network, S be a set of controlled vertices for which the subgraph $\widehat{G}_{S \cup \Gamma(S)}$ does not contain cycles. Then there exists such a collection of capacities values of arcs of the set S^+ for which in network G there exists a limit state $Q^* = Q'$.*

Remark. *Lemma 2 is a generalization of Lemma 1.*

Now consider case c). The subgraph $\widehat{G}_{S\cup\Gamma(S)}$ contains at least one cycle. Similarly to the previous case, one can step by step find the capacities of arcs of the set S^+ that are incident to vertices of degree 1 on the subgraph $\widehat{G}_{S\cup\Gamma(S)}$. Therefore, the main problem is reduced to finding the capacities of the arcs of a bipartite graph without vertices of degree 1, similar to the scheme in the figure 1.

Since the considered graph is divided into connectivity components, and since for each such component the reasoning is similar, we can consider the case when the considered bipartite graph which consists of one connectivity component only. For definiteness, denote the set of vertices of the bottom part as $A = \{a_1, \ldots, a_s\} \subset S$, the set of vertices of the top part as $B = \{b_1, \ldots, b_k\}$. Such a graph is schematically shown in the figure 4

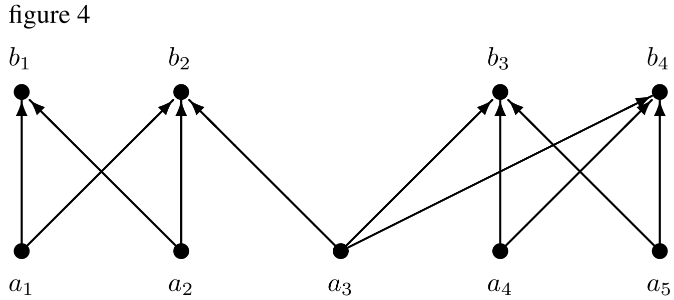

Figure 4. Scheme of the network expander-in-time in the case c).

Values q_{a_j} ($j \in [1;s]_N$) and q_{b_i} ($i \in [1;k]_N$) are determined according to rules that are similar to relations (6):

$$\begin{array}{l} q_{a_j} = q'_{a_j}(t) - \sum\limits_{x\in\Gamma(a_j)\setminus B} F_{a_j\, x}(t) \quad \forall j \in [1;s]_N, \\ q_{b_i} = q'_{b_i}(t+1) - \sum\limits_{x\in\Gamma^-(b_i)\setminus A} F_{x\, b_i}(t) \quad \forall i \in [1;k]_N. \end{array} \quad . \tag{7}$$

At the same time, note that the following relations hold:

$$\begin{array}{l} \sum\limits_{j=1}^{s} q_{a_j} = \sum\limits_{i=1}^{k} q_{b_i}; \\ q_{a_j} \le \sum\limits_{b\in\Gamma a_j} q_b \quad \forall j \in [1;s]_N; \\ q_{b_i} \le \sum\limits_{a\in\Gamma b_i} q_a \quad \forall j \in [1;k]_N. \end{array} \quad , \tag{8}$$

Since networks with a small resource distribution are only considered, the required capacities can be assumed to be equal to the values of the limit flows values along the corresponding arcs for which the state Q' is stable. Consequently, the solution to the problem of finding the required capacities can be obtained as a solution of the system of equations (9) for unknown variables F^*_{ij}:

$$\begin{cases} \sum\limits_{y\in\Gamma(a_j)\cap B} F^*_{a_j\, y} = q_{a_j}, & \forall j \in [1;s]_N; \\ \sum\limits_{y\in\Gamma^-(b_i)\cap A} F^*_{y\, b_i} = q_{b_i}, & \forall i \in [1;k]_N; \end{cases} \quad . \tag{9}$$

The system of equations (9) is joint because it is equivalent to the subsystem of joint system (5) from [14] (see Theorem 1 in [14]), and has an infinite number of solutions. Select a particular solution such that each value F^*_{xy} belongs to the interval $(0; \min\{q_x, q_y\})$. The last is possible since the relations (8) hold.

Thus, we have obtained that the following statement holds.

Lemma 3. *Let G be a regular resource network, S be a set of controlled vertices for which the subgraph $\widehat{G}_{S\cup\Gamma(S)}$ contains at least one cycle. Then there exists such a collection of capacities values of arcs of the set S^+ for which in network G there exists a limit state $Q^* = Q'$.*

The following Theorem is a direct consequence of Lemmas 1-3.

Theorem 3. *Let G be a regular resource network with low resource, for which the state Q' and the set of controlled vertices S are given. If for a certain collection of capacities values of arcs of the set S^+, for each vertex $x_i \in X \setminus (S \cup \Gamma(S))$ the equality $q'_i = (\mathcal{A}(Q'))_i$ takes place and all inequalities (4) hold, then there exists such a collection of capacities values of arcs of the set S^+ for which in network G there exists a limit state $Q^* = Q'$.*

Example 1. Consider resource network G_1 with low resource in the figure 5. The state $Q' = \{3, 3, 4, 6, 8, 6, 6, 4, 8, 9\}$ and the set $S = \{2, 5\}$ are given for network G_1. In the figure 5 the capacities is shown for arcs of the set $U \setminus S^+$. Consider the capacities finding problem for the arcs of the set S^+ in such a way that the limit state Q^* in the network G_1 coincides with the given state Q'.

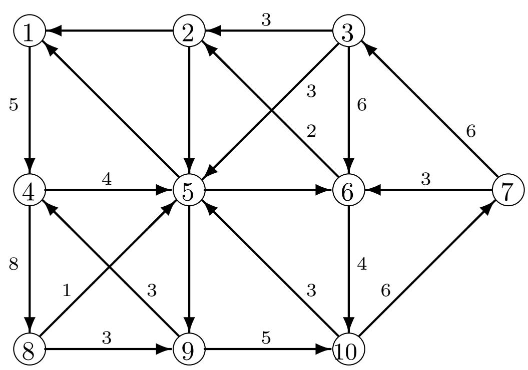

Figure 5. Resource network G_1 with given set $S = \{2, 5\}$.

Check first whether the condition of Theorem 3 is satisfied. For this, set the capacities of the arcs of the set S^+ equal to $r = \max\limits_{x\in S}\{q_x\} = 8$. By the hypothesis of Theorem 3, one can take arbitrary values, therefore, we select the value r so that a knowingly low flow is distributed in the network G_1. Next, find the state $Q'' = \mathcal{A}(Q')$. For the selected capacities of the arcs of the set S^+, we have $Q'' = \left\{\frac{25}{6}, 3, 4, 6, \frac{17}{2}, \frac{20}{3}, 6, 4, \frac{17}{3}, 9\right\}$. Thus, for each vertex x of the set $X \setminus (S \cup \Gamma(S)) = \{3, 4, 7, 8, 10\}$ the equality $Q''_x = Q'_x$ holds. In a similar way, one can check that all the inequalities of the system (4) hold. Consequently, according to Theorem 3, there exists a collection of the capacities of the arcs of the set S^+ for which the limit state $Q^* = Q'$.

In the figure 6 the bipartite graph that corresponds to subgraph G_1 of resource network, which is generated by set $S \cup \Gamma(S)$, relatively to vertices 2 and 5 is shown. Values q_x

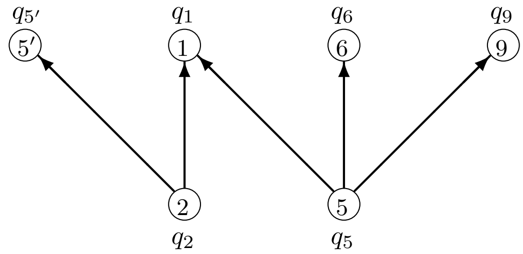

Figure 6. Bipartite graph that corresponds to subgraph G_1 of resource network relatively to vertices 2 and 5.

($x \in \{1, 2, 5, 5', 6, 9\}$) are determined according to the relations (7) as follows:

$$\begin{aligned}
&q_2 = q_2'(t) = 3;\\
&q_5 = q_5'(t) = 8;\\
&q_1 = q_1'(t+1) = 3;\\
&q_{5'} = q_5'(t+1) - F_{35}(t) - F_{45}(t) - F_{85}(t) - F_{10\,5}(t) =\\
&\quad = 8 - 1 - 2 - 1 - 3 = 1;\\
&q_6 = q'(t+1) - F_{36}(t) - F_{76}(t) = 6 - 2 - 2 = 2;\\
&q_9 = q_9'(t+1) - F_{89}(t) = 8 - 3 = 5;
\end{aligned}$$

Therefore, the capacities of arcs of the set S^+ can be obtained as follows:

$$\begin{aligned}
&r_{25} = q_{5'} = 1; \;\; r_{56} = q_6 = 2; \;\; r_{59} = q_9 = 5;\\
&r_{21} = q_2 - r_{25} = 3 - 1 = 2;\\
&r_{51} = q_5 - r_{56} - r_{59} = 8 - 2 - 5 = 1.
\end{aligned}$$

Solving the problem of the limit state finding in the network G_2 with new capacities, we obtain $Q^* = \{3, 3, 4, 6, 8, 6, 6, 4, 8, 9\} = Q'$.

Example 2. Consider resource network G_2 in the figure 7. The state $Q' = \{5, 7, 9, 8, 13, 5\}$ and the set $S = \{1, 4\}$ are given for network G_2. In the figure 7 the capacities is shown for arcs of the set $U \setminus S^+$. Consider the capacities finding problem for the arcs of the set S^+ in such a way that the limit state Q^* in the network G_2 coincides with the given state Q'.

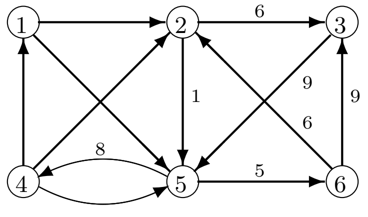

Figure 7. Resource network G_2 with given set $S = \{1, 4\}$.

Check first whether the condition of Theorem 3 is satisfied. For this, set the capacities of the arcs of the set S^+ equal to $r = \max_{x \in S}\{q_x\} = 7$. By the hypothesis of Theorem 3, one can take arbitrary values, therefore, we select the value r so that a knowingly low flow is distributed in the network G_2. Next, find the state $Q'' = \mathcal{A}(Q')$. For the selected capacities of the arcs of the set S^+, we have $Q'' = \left\{\frac{8}{3}, \frac{43}{6}, 9, 8, \frac{91}{6}, 5\right\}$. Thus, for each vertex x of the set $X \setminus (S \cup \Gamma(S)) = \{3, 6\}$ the equality $Q''_x = Q'_x$ holds. In a similar way, one can check that all the inequalities of the system (4) hold. Consequently, according to Theorem 3, there exists a collection of the capacities of the arcs of the set S^+ for which the limit state $Q^* = Q'$.

In the figure 8 the bipartite graph that corresponds to subgraph G_2 of resource network, which is generated by set $S \cup \Gamma(S)$, relatively to vertices 1 and 4 is shown. Values q_x

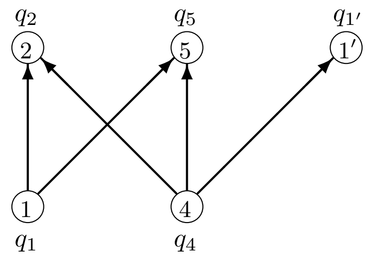

Figure 8. Bipartite graph that corresponds to subgraph G_2 of resource network relatively to vertices 1 and 4.

($x \in \{1, 2, 5, 5', 6, 9\}$) are determined according to the relations (7) as follows:

$$\begin{aligned}
&q_1 = q_1'(t) = 5;\\
&q_4 = q_4'(t) = 8;\\
&q_2 = q_1'(t+1) - F_{62}(t) = 7 - 2 = 5;\\
&q_5 = q_5'(t+1) - F_{25}(t) - F_{35}(t) = 13 - 1 - 9 = 3;\\
&q_{1'} = q_1'(t+1) = 5.
\end{aligned}$$

Therefore, the capacities of arcs of the set S^+ can be obtained as follows: $r_{41} = q_{1'} = 5$, the other capacities are determined as a solution of following system of equations

$$\begin{cases} r_{12} + r_{42} = q_2;\\ r_{15} + r_{45} = q_5;\\ r_{12} + r_{15} = q_1;\\ r_{42} + r_{45} = q_4 - r_{41}. \end{cases} \sim \begin{cases} r_{12} = 5 - r_{42};\\ r_{15} = r_{42};\\ r_{45} = 3 - r_{42}. \end{cases}.$$

As the capacities, we take a particular solution: $r_{12} = 4$, $r_{15} = 1$, $r_{42} = 1$ and $r_{45} = 2$. Solving the problem of the limit state finding in the network G_2 with new capacities, we obtain $Q^* = \{5, 7, 9, 8, 13, 5\} = Q'$.

Consider now the question of finding the capacities of the arcs which go out from controlled vertices, in the case when the limit state $Q^* = Q'$ cannot be reached.

Let $G(X, U)$ be a resource network for which the state $Q' = (q_1', \ldots, q_n')$ and the set of controlled vertices S are given. Moreover, hereinafter we assume that the condition of Theorem 1 or Theorem 2 is satisfied, that is, for any capacities of the arcs of the set S^+, the inequality $\rho(Q^*, Q') > 0$ holds. Consider the problem of finding capacities of the arcs of the set S^+ such that the distance $\rho(Q^*, Q')$ is minimal.

In the classical case, for a regular resource network with a low resource allocation, the problem of finding the limit state is reduced to solving the following system (10) (system (6) in [13]) of linear equations for unknown variables $F_{x\,y}$ and Q_x:

$$\begin{cases} F_{x\,y} - \dfrac{r_{x\,y}}{\sum\limits_{z\in\Gamma(x)} r_{x\,z}} \cdot Q_x = 0, \quad \forall\, x \in X,\ \forall\, y \in \Gamma(x); \\ Q_x - \sum\limits_{z\in\Gamma^-(x)} F_{z\,x} = 0, \qquad \forall\, x \in X; \\ F_{x_0\,y_0} = w. \end{cases}, \tag{10}$$

where (x_0, y_0) is some arc, and the value w is selected so that the total resource of the limit state Q^* is equal to the total resource of the initial state $Q(0)$. In the article [13], it is shown that a solution of such a system exists and is unique for any value w. Thus, as a method for the approximate calculation of capacities, one can use well-known numerical methods to calculate the capacities r_{xy} for all vertices $x \in S$.

Example 3. Consider the resource network G_2 with the given set $S = \{1, 4\}$ from Example 2, however, assume $Q' = \{5, 7, 6, 8, 13, 5\}$. Thus, since in this case $Q''_3 = (\mathcal{A}(Q'))_3 \neq Q'_3$, hence, by Theorem 1, for any collection of the capacities values of the arcs of the set S^+ the distance $\rho(Q^*, Q') > 0$. Then, applying the gradient descent method, we obtain the capacities $r_{12} = 1.945$, $r_{15} = 5.2$, $r_{41} = 19.718$, $r_{42} = 8$ and $r_{45} = 3.82$ (values are specified up to the third decimal place), and the distance $\rho(Q^*, Q') = 3.234$.

4 Conclusions

This paper is devoted to the development of methods for solving the flow control problem in resource networks (hereinafter referred to as the flow control problem), which consists in finding such a set of vertices and capacities of arcs, which go out from these vertices, such that the limit state of the resource network Q^* is the closest to the given state Q'.

Methods for solving a local subproblem for regular resource networks with a low resource allocation are proposed. A condition for the unreachability of the limit state Q^*, which coincides with the given state Q', is obtained. Three cases regarding the distribution of controlled vertices on a resource network are considered. In each of the considered cases, it is shown that if the condition of unreachability of the limit state is not satisfied, then there is a values collection of the capacities of the arcs going out from controlled vertices, for which the limit state Q^* coincides with the state Q'.

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