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A Rotated Similarity Reduction Approach with Half-Sweep Successive Over-Relaxation Iteration for Solving Two-Dimensional Unsteady Convection-Diffusion Problems

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Abstract In this paper, we transformed a two-dimensional unsteady convection-diffusion equation into a two-dimensional steady convection-diffusion equation using the similarity transformation technique. This technique can be easily applied to linear or nonlinear problems and is capable of reducing the size of computational works since the main idea of this technique is to reduce at least one independent variable. The corresponding similarity equation is then solved numerically using an effective numerical technique, namely a new five-point rotated similarity finite difference scheme via half-sweep successive over-relaxation iteration. This work compared the performance of the proposed method with Gauss-Seidel and successive over-relaxation with the full-sweep concept. Numerical tests were carried out to obtain the performance of the proposed method using C simulation. The results revealed that the combination of the five-point rotated similarity finite difference scheme via half-sweep successive over-relaxation iteration is the most superior method in terms of the iteration number and computational time compared to all these methods. Additionally, in terms of accuracy, all three iterative methods are also comparable.

Keywords Partial Differential Equation, Similarity Solution, Similarity Rotated Finite Difference Scheme, Half-Sweep Weighted Iteration

1. Introduction

Convection-diffusion equation (CDE) is used to describe many linear processes in the physical sciences where the first-order derivatives model the convection or transport processes and the second-order derivatives represent the diffusion. Convection has a relatively small impact on the analysis, compared to diffusion in the typically major term. The analysis is easier and typical numerical solution techniques work well when the diffusion is dominant. Meanwhile, standard approaches become unstable when convection dominates, and unique techniques are required to numerically approximate solutions. CDE appears in various applications of science and engineering such as gas transport [1-3], rotating disks [4-6], water pollution [7,8], and porous media [9-11]. Owing to CDE's wide applications, it has attracted significant attention and interest from mathematicians and researchers in recent years.

In practical situations, numerical approaches to convection-diffusion problems have grown to be increasingly popular in research since analytical approaches are fairly limited. Mostly, the numerical approaches for solving CDE include the finite difference method (FDM), finite volume method (FVM), and finite element method (FEM). In addition to these numerical approaches, Kuldeep and Kumar [12] solved the parabolic problems of the convection-diffusion type using the FDM. In addition, compact FDM has been applied to the two-dimensional CDE [13], while Fu et al. [14] applied generalized FDM to solve nonlinear CDE. Furthermore, Cheichan et al. [15] constructed weak Galerkin FEM to solve nonlinear CDE in two dimensions, while Zhang and Liu [16] used the FEM to solve a singularly perturbed CDE. Tan [17] also solved the CDE by using an upwind FVM. Later, Xu [18] solved the convection-diffusion-reaction problems by applying modified FVM. Overall, although these approaches are mostly used in practical situations, the computational burden still arises when the number of independent variables is large.

Generally, it is well known that a large number of independent variables lead to higher computational complexity and mathematical difficulties. To overcome these problems, a powerful tool is needed to alleviate this computational burden and enhance efficiency by reducing the independent variables. The similarity method is one of the most commonly used techniques for reducing independent variables and has been applied to many problems [19-23]. In fact, Blasius [24] originally proposed the similarity solution to solve an application problem of Prandtl's boundary layer theory. In article [25], Nguyen et al. developed a similarity solution for cavity expansion in thermoplastic soil where the partial differential equations (PDEs) are reduced to a system of the first-order ordinary differential equation (ODE). Next, Siavashi et al. [26] transformed the conservation equation in the form of PDE into nonlinear ODE. In [27], the authors obtained similarity solutions of Darcy-Forchheimer Magnetohydrodynamic (MHD) flow in which the governing PDEs of the nanofluid flow are changed into a third-order nonlinear quasi-ODE using the pseudo-similarity variable. Besides, Aziz [28] obtained a similarity solution for a laminar thermal boundary layer over a flat plate with a convective surface boundary condition. This technique has been shown to be very helpful in reducing the independent variables in governing equations and, thus, significantly reducing the size of computational work.

This paper is specialized to verify the performance of numerical solutions for CDE using an improved discretization scheme and an efficient computational algorithm. Although there have been plenty of discussions for numerical methods to solve these issues, here we are only focusing on the simplest numerical discretization of the CDE composed of the finite difference method due to its simplicity, time-saving, and ease of implementation. To reduce the size of computational complexity, we are interested to combine the similarity technique with the finite difference method as a new discretization scheme called similarity finite difference. Lately, an improved similarity finite difference scheme is called the half-sweep similarity finite difference scheme [29]. In this paper, we investigated the performance of this half-sweep similarity finite difference scheme combined iteratively in which this combination can be a potential tool to solve partial differential equations. In fact, the half-sweep concept was first introduced by Abdullah [30] to solve the two-dimensional Poisson equation. The main idea of the half-sweep concept is taking only half of the entire points in the solution domain of the observed problem. The applications of computational complexity reduction via the half-sweep concept are shown in [31-36]. Based on the literature, the half-sweep technique has demonstrated a good computation complexity reduction approach, which led to the reduction of the iteration number and execution time. In addition, the successive over-relaxation iterative (SOR) method is also used in this study to solve a linear system generated from the five-point rotated similarity finite difference approximation equation. Presently, this method has been dramatically considered and exploited in a variety of applications, see [37-41].

This paper contains four sections. In Section 2, the half-sweep similarity finite difference approximation equation is derived. Besides, the formulation of the successive over-relaxation iterative method is also presented. Next, the numerical experiments and discussion are carried out in Section 3. Lastly, a conclusion is included in Section 4.

Nomenclature

Ω	Rectangular domain
$\partial\Omega$	Boundary of Ω
f	Smooth function
r	Source term
D	Matrix $n \times n$ size
P	diagonal matrix
V	Lower triangular matrix of D
G	Upper triangular matrix of D
$u(x, y, t)$	Transported scalar variable
$\underline{u}^{(k)}$	Approximation value of u at k iteration
$\underline{u}^{(k+1)}$	Approximation value of u at $k+1$ iteration
m	Grid Size
k	Number of iterations
Greek Symbols	
ξ, η	Similarity variables
α_1, α_2	Diffusion coefficients
β_1, β_2	Convection coefficients
ω	Relaxation factor
ε	Tolerance error
Abbreviations	
CDE	Convection-Diffusion Equation
FS	Full-Sweep
HS	Half-Sweep
FSGS	Full-Sweep Gauss-Seidel
FSSOR	Full-Sweep Successive Over Relaxation
SOR	Successive Over-Relaxation
GS	Gauss-Seidel
HSSOR	Half-Sweep Successive Over-Relaxation
FVM	Finite Volume Method
FEM	Finite Element Method
FDM	Finite Difference Method
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
MHD	Magnetohydrodynamic

2. Methodology

This section provides the formulation of the five-point rotated similarity finite difference approximation equation and discusses the application of the half-sweep successive over-relaxation (HSSOR) method.

2.1. Five-Point Rotated Similarity Finite Difference Approximate Equation

In this paper, we concentrate on finding numerical solutions for parabolic partial differential equation problems in specific two-dimensional unsteady CDE, which is defined as follows:

$$\frac{\partial u}{\partial t} + \beta_1 \frac{\partial u}{\partial x} + \beta_2 \frac{\partial u}{\partial y} = \alpha_1 \frac{\partial^2 u}{\partial x^2} + \alpha_2 \frac{\partial^2 u}{\partial y^2} + r(x, y, t), \quad (1)$$

$$(x, y, t) \in \Omega \times (0, T]$$

with the initial condition

$$u(x, y, 0) = u_0(x, y), (x, y) \in \Omega,$$

and the boundary condition

$$u(x,y,t) = r(x,y,t), (x,y) \in \partial\Omega, t \in (0,T]$$

where β_1 and β_2 are the convection coefficients, α_1 and α_2 are the diffusivity coefficients, and r is a forcing function. Based on equation (1) without using the similarity technique, the resulting approximation equation clearly generates a sequence of linear systems at each time level, in which the computational works will affect the computational time of getting an approximate solution since the computational complexity is higher. Therefore, similarity transformation is used to ensure that this issue forms a single linear system, which is also known as a similarity linear system. Based on the advantages of similarity solutions described in section 1, the similarity transformation is used to reduce the independent variables of a two-dimensional unsteady CDE (1) into a two-dimensional steady CDE. To do this, we made a change to the independent variables. Let the new variables be $u(x,y,t) = u(\xi,\eta)$; to seek the similarity solution of (1), we introduced the wave variables transformation given as [42]

$$\xi = x - ct, \qquad \eta = y - dt, \tag{2}$$

where c and d are constants. Hence, a reduction is obtained by substituting the wave variables (2) into (1), yielding a steady convection-diffusion equation in two-dimensional form as follows:

$$\nu\left(\frac{du}{d\xi} + \frac{du}{d\eta}\right) + \alpha\left(\frac{d^2u}{d\xi^2} + \frac{d^2u}{d\eta^2}\right) = -r(\xi,\tau) \tag{3}$$

where $\nu = \frac{\mu}{2} - \beta,\ \mu = c = d.$

The finite difference method is used to discretize the first and second derivative terms of two-dimensional steady CDE (3). Let us commence, especially on the discussion of approximating the similarity solution of two-dimensional steady CDE, which is (3) by introducing a new scheme formula named five-point similarity finite difference scheme as follows:

$$\left.\frac{dU}{d\xi}\right|_{ij} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$
$$\left.\frac{dU}{d\eta}\right|_{ij} = \frac{u_{i,j+1} - u_{i,j-1}}{2h} \tag{4}$$
$$\left.\frac{d^2U}{d\xi^2}\right|_{ij} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$
$$\left.\frac{d^2U}{d\eta^2}\right|_{ij} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$$

The approximation of (3) can be approached using the standard five-point similarity finite difference scheme (4) yielded as

$$\begin{aligned}(2\alpha - \nu h)u_{i-1,j} &+ (2\alpha + \nu h)u_{i+1,j} \\ &+ (2\alpha - \nu h)u_{i,j-1} \\ &+ (2\alpha + \nu h)u_{i,j+1} - 8\alpha u_{i,j} \\ &= -2h^2 r_{i,j}\end{aligned} \tag{5}$$

Actually, (5) essentially depicts a full-sweep iteration, in which all nodes in the mesh points are taken into account. Another approximation based on the cross-orientation operator, often known as half-sweep iteration, is provided by rotating the axis $45°$ degrees clockwise [43]. Therefore, another new scheme formula is introduced, which is named a new five-point rotated similarity finite difference scheme, as follows:

$$\left.\frac{du}{d\xi}\right|_{i,j} = \frac{U_{i+1,j+1} - U_{i-1,j+1} + U_{i+1,j-1} - U_{i-1,j-1}}{4h}$$
$$\left.\frac{du}{d\eta}\right|_{i,j} = \frac{U_{i+1,j+1} - U_{i+1,j-1} + U_{i-1,j+1} - U_{i-1,j-1}}{4h} \tag{6}$$
$$\left.\frac{d^2u}{d\xi^2} + \frac{d^2u}{d\eta^2}\right|_{i,j} = \frac{U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1} - 4U_{i,j}}{2h^2}$$

If the axis of the rectangular grid is rotated by $45°$, we can obtain the five-point rotated similarity finite difference approximation written as

$$\begin{aligned}(\alpha - \nu h)u_{i-1,j-1} &+ \alpha u_{i-1,j+1} + \alpha u_{i+1,j-1} \\ &+ (\alpha + \nu h)u_{i+1,j+1} - 4\alpha u_{i,j} \\ &= -2h^2 r_{i,j}\end{aligned} \tag{7}$$

Figures 1 and 2 depict the computational molecules for the corresponding five-point approximation of full- and half-sweep iterations, respectively, to help the viewer grasp the idea of the finite scheme concept. Additionally, Figures 3 and 4 provide a thorough depiction of the computational nodes for these five-point approximations of both concepts.

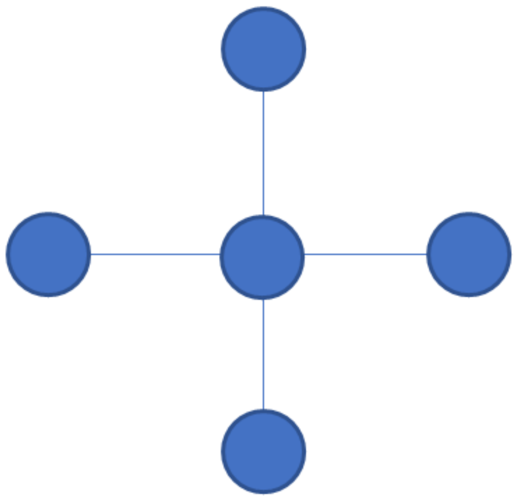

Figure 1. Computational molecules of five-point centered approximation

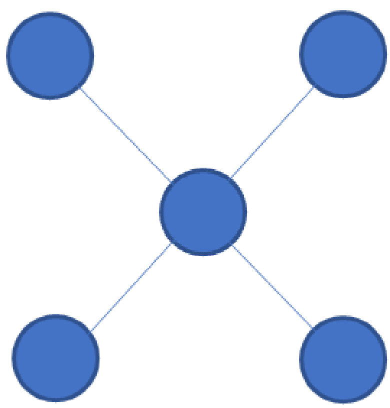

Figure 2. Computational molecules of five-point rotated approximation

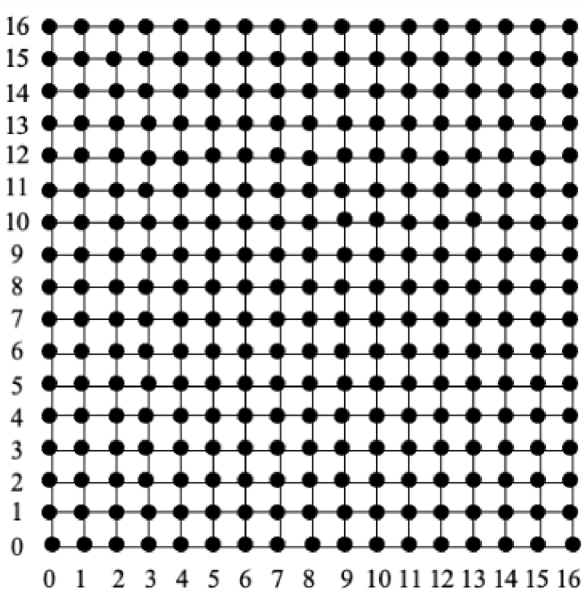

Figure 3. Computational nodes for the FS iteration in case $m = 16$

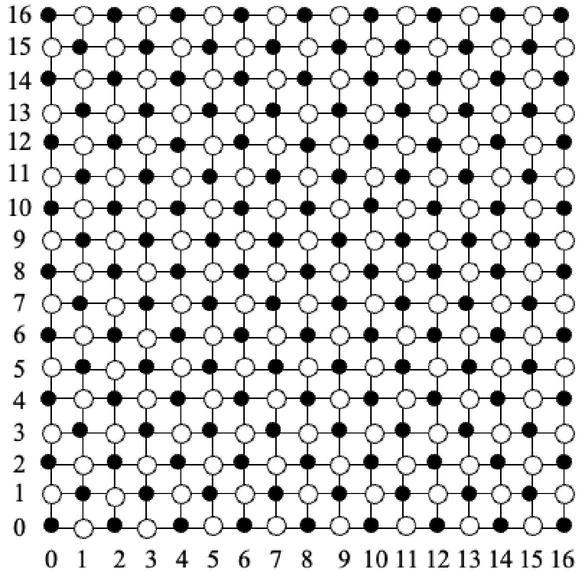

Figure 4. Computational nodes for the HS iteration in case $m = 16$

In general, the application of these five-point finite difference formulae (5) and (7) to approximate the problem (3) will generate a large and sparse linear system that can be stated in matrix form as

$$D\underline{u} = \underline{R} \tag{8}$$

where the matrix D and column vector $\underline{R}$ are both known and the column vector $\underline{u}$ is unknown. To solve the linear system (8) and obtain the potential values for each node, the half-sweep successive over-relaxation (HSSOR) iterative approach is implemented. For performance comparison purposes, the existing full-sweep successive over-relaxation (FSSOR) and full-sweep Gauss-Seidel (FSGS) are also executed.

2.2. Half-Sweep Successive Over-Relaxation Iterative Method

As mentioned in the previous section, we developed the half-sweep successive over-relaxation (HSSOR) iteration to boost the convergence speed to solve the large-scale and sparse linear system (8). To begin with, we decomposed the coefficient matrix D of the linear system (8) to obtain the formulation of the proposed iterative method that can be stated as [44-45]

$$D = P - V - G \tag{9}$$

where

P is the diagonal matrix of D

V is the lower triangular matrix of D

G is the upper triangular matrix of D

To derive the SOR iteration formulation, first consider the implementation of a parameter ω as a relaxation factor into (8) and rewrite it as

$$\omega D\underline{u} = \omega\underline{R}. \tag{10}$$

By substituting (9) with (10), we obtained

$$\omega(P-V-G)\underline{u}=\omega\underline{R}. \tag{11}$$

In general, the SOR iterative schemes can be defined in matrix form as follows:

$$\underline{u}^{(k+1)}=(P-\omega.V)^{-1}(\omega.G+(1-\omega).P)\underline{u}^{(k)}+\omega(P-\omega.V)^{-1}\underline{R} \tag{12}$$

Where $\underline{u}^{(k+1)}$ denotes an unknown vector at k iteration. Note that if $\omega=1$, (12) is turning to be the Gauss-Seidel (GS) method, which can be defined in matrix form as follows:

$$\underline{u}^{(k+1)}=(P-V)^{-1}G.\underline{u}^{(k)}+(P-V)^{-1}\underline{R} \tag{13}$$

According to (12), we can express the HSSOR iterative method as

$$u_{i,j}{}^{(k+1)} = (1-\omega)u_{i,j}{}^{(k)}+\omega(2h^2 r_{i,j}+(\alpha-\nu h)u_{i-1,j-1}+\alpha u_{i-1,j+1}+\alpha u_{i+1,j-1}+(\alpha+\nu h)u_{i+1,j+1})/4\alpha. \tag{14}$$

We complemented the HSSOR iterative method with the following algorithm to facilitate the implementation method to solve the two-dimensional unsteady convection-diffusion equation. The following algorithm is as follows:

Algorithm 1: HSSOR Method

i). Set $\underline{u}^{(0)}\leftarrow 0$ and $\varepsilon\leftarrow 10^{-10}$

ii). For $k=1,2,3,\dots,$ perform

(a) Compute (14)

(b) Convergence test: $\left\|\underline{u}^{(k)}-\underline{u}^{(k-1)}\right\|<\varepsilon$

iii). Display the numerical solution.

iv). Stop.

3. Numerical Test

For the numerical tests, three cases of the two-dimensional unsteady convection-diffusion equations are considered to evaluate the performance of the proposed HSSOR iterative method against the previously developed iterative methods in our research, namely FSSOR and FSGS. To seek the performance of the proposed methods, three parameters have been considered such as the iteration number, the time taken (in seconds), and the maximum absolute error. The iteration number for the running program based on Algorithm 1 is fixed by the tolerance error, $\varepsilon=10^{-10}$. To illustrate the performance of the HSSOR iterative method, let us consider the two-dimensional unsteady convection-diffusion equations as follows:

Case 1: Consider the second-order two-dimensional unsteady convection-diffusion equation as given in [46]

$$\frac{\partial u}{\partial t}+0.8\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)=0.05\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{15}$$

with an analytic solution of (16) as

$$u(x,y,t)=\frac{1}{1+4t}exp\left(-\frac{(x-\beta_1 t-0.5)^2}{\alpha_1(1+4t)}-\frac{(y-\beta_2 t-0.5)^2}{\alpha_2(1+4t)}\right)$$

Case 2: Consider the two-dimensional inhomogeneous unsteady convection-diffusion equation as given in [47]

$$\frac{\partial u}{\partial t}+1.0.\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)=0.5\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+r(x,y,t) \tag{16}$$

with an analytic solution of (18) as

$$u(x,y,t)=exp(-2t)\cos(x)\sin(2y)$$

and the source term is

$$r(x,y,t)=exp(-2t)\left[\cos(x+2y)+\cos(x)\left(\frac{1}{2}\sin(2y)+\cos(2y)\right)\right]$$

Case 3: Consider the two-dimensional unsteady convection-diffusion equation as given in [48]

$$\frac{\partial u}{\partial t}+64.0\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)=1.0\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+r(x,y,t) \tag{17}$$

with an analytic solution of (21) as

$$u(x,y,t)=e^{-(a^3+b^3)t}\sin(ax+by)$$

and the source term is

$$r(x,y,t)=(a\beta+b\beta)e^{-(a^3+b^3)t}\cos(ax+by)$$

Throughout the numerical experiments, we performed five numerical experiments for each test case with a different grid size, $m=64\times 64, 128\times 128, 256\times 256, 512\times 512,$ and 1024×1024. The results of the simulation execution for the considered methods are tabulated in Tables 1, 2, and 3 for Cases 1, 2, and 3, respectively. We reported the results of the simulations of HSSOR, FSSOR, and FSGS methods with all the values being rounded up to six decimal places. Based on the observation in Tables 1-3, it was found that HSSOR iteration requires a smaller iteration number compared to FSSOR and FSGS. In terms of the time taken, the HSSOR iterative method provides significantly faster performance than FSSOR and FSGS. In terms of accuracy, all three iterative methods are comparable. Additionally, it is evident that the percentage reduction of the iteration number for HSSOR has declined approximately by 93.38% - 99.71% and 26.87% - 35.43% compared to the FSGS and FSSOR methods, respectively. In terms of the time taken, the HSSOR method is faster, specifically at 95.43% - 99.86% and 55.05% - 76.91% than the FSGS and FSSOR methods, respectively. Clearly, the performance of the proposed HSSOR iterative method is more efficient than the FSSOR and FSGS iterative methods.

Table 1. Computational results for Case 1

Method	**FSGS**	**FSSOR**	**HSSOR**
Grid Size		**Iteration Number**	
64x64	1268	122	84
128x128	3919	238	164
256x256	11434	465	320
512x512	31146	902	621
1024x1024	78287	1737	1199
		Time	
64x64	0.087437	0.008889	0.003996
128x128	0.943376	0.069594	0.017104
256x256	10.404804	0.517506	0.131791
512x512	112.54839	3.923255	1.06645
1024x1024	1144.852174	30.129002	8.477936
		Error	
64x64	1.504651E-05	1.503587E-05	1.525480E-05
128x128	1.511897E-05	1.507977E-05	1.513482E-05
256x256	1.522844E-05	1.507538E-05	1.508881E-05
512x512	1.565958E-05	1.507556E-05	1.507813E-05
1024x1024	1.725004E-05	1.507871E-05	1.507726E-05

Table 2. Computational results for Case 2

Method	**FSGS**	**FSSOR**	**HSSOR**
Grid Size		**Iteration Number**	
64x64	5848	256	168
128x128	21101	512	332
256x256	75242	1024	662
512x512	264347	2038	1316
1024x1024	910963	3898	2608
-		**Time**	
64x64	0.419567	0.020054	0.004649
128x128	4.999732	0.151434	0.03496
256x256	68.369897	1.127477	0.263367
512x512	955.974294	8.848148	2.277498
1024x1024	13323.97359	67.561031	18.488508
		Error	
64x64	9.462852E-04	9.462567E-04	9.468046E-04
128x128	9.466455E-04	9.465295E-04	9.466652E-04
256x256	9.470473E-04	9.465792E-04	9.466123E-04
512x512	9.484788E-04	9.466008E-04	9.466065E-04
1024x1024	9.541349E-04	9.466064E-04	9.466055E-04

Table 3. Computational results for Case 3

Method	FSGS	FSSOR	HSSOR
Grid Size		**Iteration Number**	
64x64	4885	201	147
128x128	17632	398	291
256x256	62846	793	578
512x512	220570	1580	1146
1024x1024	758850	3113	2270
		Time	
64x64	0.36099	0.015474	0.004632
128x128	4.18093	0.117948	0.030555
256x256	57.137681	0.878014	0.229662
512x512	804.459473	6.866247	1.954414
1024x1024	11113.83991	54.005512	16.260097
		Error	
64x64	1.815294E-04	1.815049E-04	1.815205E-04
128x128	1.816218E-04	1.815226E-04	1.815264E-04
256x256	1.819314E-04	1.815273E-04	1.815281E-04
512x512	1.831673E-04	1.815444E-04	1.815444E-04
1024x1024	1.880993E-04	1.815465E-04	1.815460E-04

4. Conclusion

In this research work, the similarity transform has been successfully applied to reduce a two-dimensional unsteady convection-diffusion equation to a two-dimensional steady convection-diffusion equation, which is capable of reducing the size of computational work. The five-point rotated similarity finite difference scheme was used to formulate the approximation of the two-dimensional steady convection-diffusion equation. An effective iterative method known as HSSOR was derived, and the computational algorithm was shown. From the numerical results, it can be concluded that the computational complexity reduction owned by the five-point rotated similarity finite difference scheme and the HSSOR method is an excellent fusion as a numerical method for solving a two-dimensional convection-diffusion equation. To further increase the speed of the convergence rate of the iteration process, a study on quarter-sweep iteration [49–51] shall be taken into consideration as an extension to the half-sweep iteration approach.

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