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Convergence Analysis of Space Discretization of Time Fractional Telegraph Equation

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Abstract The role of fractional differential equations in the advancement of science and technology cannot be overemphasized. The time fractional telegraph equation (TFTE) is a hyperbolic partial differential equation (HPDE) with applications in frequency transmission lines such as the telegraph wire, radio frequency, wire radio antenna, telephone lines, and among others. Consequently, numerical procedures (such as finite element method, H^1 – Galerkin mixed finite element method, finite difference method, and among others) have become essential tools for obtaining approximate solutions for these HPDEs. It is also essential for these numerical techniques to converge to a given analytic solution to certain rate. The Ritz projection is often used in the analysis of stability, error estimation, convergence and superconvergence of many mathematical procedures. Hence, this paper offers a rigorous and comprehensive analysis of convergence of the space discretized time-fractional telegraph equation. To this effect, we define a temporal mesh on $[0, T]$ with a finite element space in Mamadu-Njoseh polynomial space, φ_{m-1}, of degree $\leq m-1$. An interpolation operator (also of a polynomial space) was introduced along the fractional Ritz projection to prove the convergence theorem. Basically, we have employed both the fractional Ritz projection and interpolation technique as superclose estimate in L_2 - norm between them to avoid a difficult Ritz operator construction to achieve the convergence of the method.

Keywords Time-Fractional Telegraph Equation, Caputo Derivative, Mamadu-Njoseh Basis Functions, Finite Element Method, Ritz Projection, Convergence

1. Introduction

In recent years, the concept of fractional calculus has been explored extensively by researchers due to its immerse contribution to science and engineering. Leibnitz in 1965 was first to propose the theory of fractional calculus [1]. Leibnitz theory was further developed by Riemann, Letnikov, Grunwald and Liouville in the 19th century [2-4]. Fractional derivatives have essential benefits in the hereditary and memory processes of various materials. Also, the electrical and mechanical properties of materials, theories of fractals and control, can be effectively described by fractional derivatives.

Due to the confounded property and nature of fractional derivatives, the existence of analytic techniques for resolving fractional differential equations seems not to exist. Accordingly, numerical procedures have become essential tools for obtaining approximate solutions for these equations. Popular approximate techniques for fractional differential equations include the variational iteration method [5], finite difference method [6], homotopy analysis method [7], etc.

It is essential for a numerical technique to converge to a given analytic solution to certain rate. Hence, the study of convergence analysis of fractional differential equations has not left untouched. For instance, Tang [8] studied the convergence and superconvergence for the time fractional optimal control problems via a fully discrete finite element scheme. Sontakke and Pandit [9] studied the convergence of nonlinear fractional partial differential equations via the fractional Adomian decomposition method. An [10] investigated the superconvergence of a time-space discretized scheme for a time-fractional diffusion problem. Zhao *et al*. [11] studied the superconvergence of nonconforming finite element method for two-dimensional time fractional diffusion equation.

In Mamadu *et al*. [12], a space discretization of time fractional telegraph equation with Mamadu-Njoseh basis functions was studied with detailed numerical illustrations for various values of the fractional order, where resulting numerical evidences were expressed in L_2 and L_∞ maximum errors. The aim of this paper is to investigate the convergence of the space discretization of time fractional telegraph equation. Here, the fractional Ritz projection and interpolation technique are superclose estimates in L_2- norm between them to avoid a difficult Ritz operator construction.

2. Materials and Methods

2.1. Basic Preliminaries and Properties

We study in this section some basic concepts and properties of fractional calculus.

Definition 2.1. The function y of fractional order α is said to possess a Caputo derivative if

$$D_t^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^x y^{(n)}(s)(t-s)^{(n-1-\alpha)}\, ds, \quad (2.1)$$

for $n-1<\alpha \le n,\ n \in \mathbb{N},\ y \in C_{-1}^n,\ x>0$.

Definition 2.2. A function y is a Riemann-Liouville integral of fractional order $\geq 0,\ y \in C_\beta,\ \beta+1 \geq 0$, if

$$\begin{cases} J^\alpha y(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x y(s)(t-s)^{\alpha-1}\, ds, & \alpha>0, \\ J^\alpha y(t) = y(t) \text{ for } \alpha = 0. \end{cases} \quad (2.2)$$

Property I:

The properties of fractional derivatives and integrals of y for $y \in C_{-1}^n,\ n \in \mathbb{N}$, essential to this research are stated as follows [12] :

(i). $D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha},\ \beta \in \mathbb{N}_b,\ \ \mathbb{N}_b = \{0,1,2,\ldots\}$.

(ii). $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}$.

(iii). $D^\alpha J^\beta y(t) = y(t)$.

Property II:

Let $\mathbb{R}$ (N) >0. The following exists [10];

If $\mathbb{R}\ (\alpha) \geq 0$, then $(D_-^\alpha e^{\lambda t})(x)= \lambda^\alpha e^{\lambda x}$, and $(I_+^\alpha e^{\lambda t})(x)=\lambda^{-\alpha} e^{\lambda x}$.

Lemma 2.1: Let $p \in [1,\alpha], q \in [1,\alpha]$ and $\alpha>0$ and $L_p(\mathbb{R}) \rightarrow L_q(\mathbb{R}^+)$, then the operators I_+^α and I_-^α are bounded if and only if [2]

$$(I_+^\alpha e^{\lambda t})(x) = \lambda^{-\alpha} e^{-\lambda x},\ \mathbb{R}(\lambda) > 0, \mathbb{R}(x) > 0.$$

Lemma 2.2: Let $p \in [1,\alpha], k>0$ and $\alpha>0$. In the space $L_{p,k}$,the operator I_+^α is bounded. That is,

i. $\| I_+^\alpha \mathrm{U}\|_{p,k} \leq w = \left(\frac{p}{k}\right)^\alpha$, p $\in [1,\alpha]$, w=$k^\alpha (p=\alpha)$.

ii. I_-^α is bounded in the space $L_{p,-k}$. That is,

$$\|I_-^\alpha \mathrm{U}\|_{p,-k} \leq w\|U\|_{p,-k},\ k = \left(\frac{p}{|k|}\right)^\alpha,\ \text{p} \in [1,\alpha],$$
$$w = k^\alpha (p=\alpha).$$

Lemma 2.3: Let $\alpha>0, \gamma>0$, and $\mathrm{p} \geq 1$ such that $\frac{1}{p} > \alpha+\gamma$. If $U \in L_p(\mathbb{R})$, then

$$(I_+^\alpha I_+^\gamma U)(x) = (I_+^{\alpha+\gamma} U)(x);\ (I_-^\alpha\ I_-^\gamma U)(x) = (I_-^{\alpha+\gamma} \mathrm{U})(\mathrm{x})$$

Lemma 2.4: Supposing $\alpha>0$, then for $U \in L_1(\mathbb{R})$,

$$(\mathrm{D}_+^\alpha\ I_+^\alpha \mathrm{U})(x) = \mathrm{U}(x);\ (\mathrm{D}_-^\alpha\ I_-^\alpha \mathrm{U})(x) = \mathrm{U}(x).$$

2.2. Finite Element Method

For a differential equation, the central idea for any numerical scheme is to convert or (discretize) a given continuous system to obtain a discrete system with many finite unknowns that may be solved via computer software.

In the finite element method (FEM), the process of discretization is quite different from the classical numerical method – *finite difference method*. In FEM, the given differential equation has to be reformulated as a variational problem. For example, let

$$(\mathrm{q}(\mathrm{x})u')' = g(x) \text{ with } u(0) = a \text{ and } u(1) = b, \quad (2.3)$$

be any boundary value problem. A variational equivalent of (2.3) is to find some $u(x)$ such that

$$\int_0^1 w((\mathrm{q}(\mathrm{x})u')' - q(x))dx = 0, \quad (2.4)$$

where w is the weight function.

However, for elliptic PDEs, the variational formulation is a typical minimization problem given as:

$$\text{Find } v \in U \text{ such that } (u) \leq f(v)\ \forall\ \ u \in U, \quad (2.5)$$

where F : $U \rightarrow \mathbb{R}$ is a functional, U denotes the set of all admissible functions. The functions $u \in U$ however represent a temperature, a displacement in an elastic body, etc, $f(u)$ denotes the total energy in reference to u. However, in totality the dimension U is infinite (that is, functions in U cannot be explicitly defined by a finite number of parameters) and thus, the problem (2.5) is unsolvable. To obtain a solution for the problem (2.5) the central idea in FEM is to replace U by U_h, which is a simple function that depends only on finite number of parameters. Thus, the minimization problem (2.5) can be reformulated as:

$$\text{Find } v_h \in U_h \text{ such that } f(v_h) \leq f(u)\ \forall\ \ u \in U_h. \quad (2.6)$$

The aim now is to ensure that the solution v_h of the Problem (3.1) yields a good approximation of the solution v of the optimal problem (2.5). Now, if we choose $U_h \in U$, i.e, if $u \in U_h$ then $u \in U$, then (2.6) corresponds to the Ritz-Galenkin method. A special feature of the FEM as a particular Ritz-Galenkin method is that the simple functions in U_h are piecewise polynomials [12].

To solve any integral or differential equation using FEM, the following steps are involved:

(i) equivalent variational formulation of the stated problem.
(ii) finite dimensional space construction, U_h. This is the discretization process.
(iii) seeking solution to the resultant discrete problem.
(iv) implementation through a computer programming.

Note:

i). There exist several kinds of variational formulation that may be used depending on the choice of dependent variables.
ii). The construction of finite dimensional spaces is influenced by the following:
  (a) Accuracy requirement.
  (b) Variational formulation.
  (c) Regularly property of the analytic solution.

2.3. Some Useful Hilbert Spaces $L_2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$

Before introducing these Hilbert spaces, let have a brief recall of few useful concepts in linear algebra that will be of immersed benefit to this present study. Now, if U is a linear space, we say that L is a linear on U if L: $U \rightarrow \mathbb{R}$, and for all $u_1,\ u_2 \in\ U$ and $\alpha,\ \beta \in \mathcal{R}$,

$$L(\alpha u_1 + \beta u_2\ ) = \alpha L(u_1) + \beta L(u_2).$$

Furthermore, U is a bilinear form on $U \times U$ if L: $U \times U \rightarrow \mathcal{R}$, that is, for all, $u_1,\ u_2 \in\ U$ and $\alpha,\ \beta \in \mathcal{R}$, we have,

$$L\ (u_1\ ,\ \alpha u_2 + \beta u_3\ ) = \beta\ L\ (u_{1\ ,} u_2) + \ \alpha\ L(u_1\ , u_3).$$

$$L\ (\beta u_1\ + \alpha u_2\ ,\ u_3\ ) = \beta\ \mathrm{L}\ (u_{1\ ,} u_3) + \ \alpha\ \mathrm{L}(u_2\ , u_3).$$

The bilinear form on $U \times U$ is symmetric if

$$\mathrm{L}\ (\ u_1\ , u_2\ ) = \ \mathrm{L}\ (u_1\ ,\ u_2)\ ,\ \forall\ u_1,\ u_2 \in\ U.$$

The norm $\|.\|_L$ associated with $L(\dots)$, a scalar product, is defined as

$$\|.\|_L = \sqrt{L(u\ , v)}\quad ,\ u, v\ \in U\ .$$

Now, if $(.,.)$ is an inner product with corresponding norm $\|.\|$, we have

$$|(u_1, u_2)| \leq \|u_1\|\ \cdot\ \|u_2\|\ , \qquad u_1, u_2 \in U,$$

called the Cauchy inequality.

Furthermore, if u is a linear space with an inner product corresponding to $\|.\|$, then u is said to be a Hilbert space if u is complete.

Now, we introduce some Hilbert spaces that are essential to FEM. Let us start with a one dimensional case. Let $\Omega := (a, b)$ be an interval. We define the space of "square integrable functions" in Ω as

$$L_2(\Omega) = \left\{u : u\ is\ defined\ in\ \Omega\ and\ \int_\Omega\ u^2 dx < \infty\right\}.$$

The space $L_2(\Omega)$ is a Hilbert space corresponding with the L_2-norm.

$$\|u\|_{L_2(\Omega)} = \left(\int_\Omega\ u^2 dx\right)^{1/2} = (u, v)^{1/2},$$

and the scalar product

$$(u_1,\ u_2) = \int_\Omega\ (u_1\ u_2) dx\ .$$

By Cauchy inequality, we have that $|(u_1, u_2)|\ \leq \|u_1\|_{L_2(\Omega)}\ \|u_2\|_{L_2(\Omega)}$.

Let's also introduce the space

$$H'^1(\Omega) = \{\ u : u\ \ and\ u'\ \ belong\ to\ L_2(\Omega)\ \}$$

with the scalar product [10 -12]

$$(u_1, u_2)\ H'^1(\Omega) = \int_\Omega\ (u_1\ u_2 + \ u_1^1\ u_2^1\ )\ dx\ , (u_1,\ u_2) \in H'^1(\Omega),$$

and the corresponding norm

$$\|u\|_{H'(\Omega)} = \left(\int_\Omega\ (u^2 + \ (u^1)^2 dx\right)^{1/2}.$$

Thus, the space $H'(\Omega)$ is made of functions of u defined in Ω together with their first derivatives.

In case of boundary value problem of the form (1.7), we use the space

$$H_0^1(\Omega) = \{\ u\ \in H'(\Omega) : u(0) = u(1) = 0\},$$

with the same scalar product and norm as for $H'(\Omega)$.

3. Finite Element Method for Time Fractional Telegraph Equation

Let the model equation be given as

$$\left.\begin{array}{l} {}_0^C D_t^\alpha y(x,t) + y_t(x,t) - y_{xx}(x,t) = f(x,t), \\ y(x,0)\ =\ a_0,\ \ y_t(x,0) = a_1, \\ y(0,t) = y(0,T) = 0, \end{array}\right\}, \qquad (3.1)$$

$$x \in [0, T], t > 0,$$

with usual definition of parameters as given in Mamadu *et al*. [12].

Let a linear and continuous finite element space defined in [a, b] be given as u_h. Let the space partitioning of $[a, b]$ be given as

$$a = \ x_0\ < \ x_1\ < \ x_2\ < \ \dots < \ x_n\ = \ b,$$

and u_h ={v_h (x) : v_h(x) is partioned in [a,b]}.

Thus, the weak FEM formulation for (3.1) is to find $y(t) \in H_0^1\ (a,b)$ such that

$$\left({}_0^RD_t^\alpha[y(x,t) - y_0], v\right) + (y_t\ , \text{y(x)}) - (y_x\ ,\ v) = (f\ (x,t)\ ,\ v \in H_0^1 \quad (3.2)$$

By FEM, we compute v_h(t) $\in u_h$, such that

$$({}_0^RD_t^\alpha[y(x,t) - y_0], \tau) + \left(\frac{\partial y}{\partial t}, \tau\right) - \left(\frac{\partial y}{\partial x}, \frac{\partial \tau}{\partial x}\right) = \left(\tau, \frac{\partial \tau}{\partial x}\right), \tau \in u_h, \quad (3.3)$$

Suppose $B_h = -\Delta_h : u_h \to u_h$ is defined by

$$(B_h y_h, \tau) = \left(\frac{\partial y}{\partial t}, \tau\right) - \left(\frac{\partial y}{\partial x}, \frac{\partial \tau}{\partial x}\right),\ \tau \in u_h, \quad (3.4)$$

and let $F_h:\ F \to u_h$ be a L_2 operator given by

$$(F_h v,\ ,\tau) = (v,\tau),\ \forall\ \tau \in u_h\ ,\ v \in L_2.$$

Then (3.3) can be written in the abstract sense as

$$\left({}_0^RD_t^\alpha[y(x,t) - y_0], v\right) + B_h\, y_h = F_h f,\ t > 0, \quad (3.5)$$

where

$${}_0^RD_t^\alpha\ (y(x,t)) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(s-x)}{(t-s)^{-\alpha}}\, \text{ds}\ ,\ \alpha \in \text{(a,b)}, \quad (3.6)$$

is the Riemann – Liouville fractional derivative, and Γ is the Gamma function.

Applying quadrative formula ($[11-12]$) on (3.5), we obtain

$${}_0^RD_t^\alpha\ y(t_j) = \sum_{k=0}^n w_{kn}[y(t_n - t_k) - y_0] + \frac{t_j^{-\alpha}}{\Delta t^\alpha\ \Gamma(-\alpha)} F_j(f), \quad (3.7)$$

where,

$$w_{kn} = \frac{-2k^{1-\alpha} + (k-1)^{1-\alpha} + (k+1)^{1-\alpha}}{\Gamma(2-\alpha)},\ k = 1,2,\dots,n,$$

and $F_n(f)$ satisfies

$$\|F_n(f)\| \le K_n{}^{\alpha-2} sup_{0\le t\le T}\left\|u''(t_n - t_{jw})\right\|,\quad w \in [0,T].$$

Now, let $y(x,t) = y_k \approx y_h\ (t_k) = \sum_{k=1}^{N-1} \beta_k \varphi_k(x,t_k)$, be the approximation to $u_h(t_j)$, where $\varphi_k(x)$, $k = 0\ (1)(N-1)$, are Mamadu – Njoseh Basis function of V_h, and $\beta_k's$ are constants.

Suppose f_k = f(t_k) defines the time discretization, then

$$\Delta t^{-\alpha} \sum_{k=0}^n w_{kn}\left(y_{n-k} - y_{0,}\, \tau\right) + \left({}^{\partial y_j}/_{\partial t}\ , \tau\right) - \left({}^{\partial y_j}/_{\partial x}\ ,\ \frac{\partial \tau}{\partial x}\right) = \left(f_j\ ,\ \frac{\partial \tau}{\partial x}\right),\ n = 0\ (1)N,\ \forall \tau \in u_h. \quad (3.8)$$

which can be implemented for $n = 0(1)N$.

4. Convergence Analysis

Here, we present the convergence analysis of the space discretization of the time fractional telegraph equation with Mamadu-Njoseh orthogonal basis functions (3.1).

Let have our temporal finite mesh as

$$0 \equiv t_0 < t_1 < \cdots < t_{M-1} < \cdots < t_M \equiv T,$$

and

$$\Delta t_m = t_m - t_{m-1},\ m = 1(2)M,$$

and define a finite element space $s^n \subset H_0^\alpha(\Omega)$ as

$$s^n = \left\{s \in H_0^\alpha(\Omega) : s|_{\Delta t_n} \in \varphi_{m-1}\right\}, \quad (4.1)$$

where φ_{m-1} is the Mamadu-Njoseh polynomial space [13 – 15] of degree $\le m-1$. Now, the space discretization is to find $y^n \in s^n$, $n = 1(2)M$, such that (3.1) can be re-written in its abstract sense as

$$({}_0^RD_t^\alpha[y(x,t) - \tilde{y}_0], s) + B(y^n, s) = G^n(f),\quad t > 0,\quad s \in s^n, \quad (4.2)$$

$$(y^0, s) = (a_k, s),\ \ s \in s^n. \quad (4.3)$$

Let us consider an interpolation operator of polynomial space of the form

$$\gamma^n : H^\alpha(\Omega) \to s^n, \quad (4.4)$$

such that

$$\|y - \gamma^n\|_{H^\alpha(\Omega)} \le \sqrt{\left(\sum_{r=1}^N (\Delta t_r)^{2(k-\alpha)} \|y\|_{H^k(\Omega)}\right)}, \quad (4.5)$$

is satisfied.

To implement the convergence analysis, we introduce the Ritz operator of fractional order [16]

$$\chi^n : H_0^\alpha(\Omega) \to s^n,$$

defined by

$$B(\chi^n y, s) = 0,\ s \in s^n,\ y \in H_0^\alpha(\Omega). \quad (4.6)$$

We consider the following lemmas.

Lemma 4.1. Let $y \in H^k(\Omega) \cap H_0^\alpha(\Omega)$ for $k \in [\alpha, m]$, then the Ritz operator of fractional order defined by (4.6) satisfies

$$\|y - \chi^n y\|_{L^2(\Omega)} \le \aleph^{-k}\|y\|_{H^k(\Omega)},\ k = 0(1)n. \quad (4.7)$$

Lemma 4.2. Let the space $H_0^\alpha(\Omega)$ possess continuous and coercive properties. Then $B(.,.)$, a bilinear form, satisfies the following:

$$\left.\begin{array}{l} i.\ \ \|y\|_{H^\alpha(\Omega)}^2 \le B(y,y) \\ ii.\ \ |B(y,u)| \le \|y\|_{H^\alpha(\Omega)}\|u\|_{H^\alpha(\Omega)} \end{array}\right\}\ y,u \in H_0^\alpha(\Omega)\ , \quad (4.8)$$

with the L_2 inner product denoted by (y,u). The statement of proof of Lemmas 4.1 and 4.2, respectively, can be found in Ervin and Roop [17].

We now proceed with the convergence analysis and its proof.

Theorem 4.1. Let $y \in H^k(\Omega) \cap H_0^\alpha(\Omega)$ for $k \in [\alpha, m]$. Then the space discretization of (3.1) has the error estimation for $n = 1(2)M$,

$$\|y(.,t_n) - y_n\|_{L^2(\Omega)} \le \aleph^{-k}\left(\int_0^{t_n}\|y_t\|_{H^k(\Omega)}dt + \sum_{m=1}^{n+1}\|y(.,t_{m-1})\|_{H^k(\Omega)}\right) + b\sum_{m=1}^n \Delta t_m \|y_{tt}(.,t_m)\|_{L^2(\Omega)},$$

where $b = \max_{0\leq m\leq n}\{\Delta t_m\}$ denotes the time step size, and $\aleph$ is the total enumeration of finite element mesh points.

Proof. Let the truncation error be given as

$$\vartheta^n(x) = {}_0^R D_t^\alpha\,[y\,(x,t_n) - y(x,t_{n-1})] - y_t(x,t_n),$$

where the analytic solution is $y(x,t)$. Thus, we have the identity,

$$({}_0^R D_t^\alpha[y(.,t_n) - \tilde{y}_0(.,t_{n-1})], s) + B(y^n, s) = (\vartheta^n(x), s) + G^n(f),\ t > 0,\ s \in s^n. \quad (4.9)$$

Let

$$E^n = y(.,t_n) - y^n$$

and

$$\tilde{E}^n = y(.,t_n) - \tilde{y}^n.$$

Then, the error equation by (4.9) becomes

$$\left({}_0^R D_t^\alpha[E^n - \tilde{E}^n], s\right) + B(y^n, s) = (\vartheta^n(x), s) + G^n(f),\ \ s \in s^n. \quad (4.10)$$

Define

$$\omega^n \coloneqq \chi^n y(.,t_n) - y^n,\ \ e^n \coloneqq \chi^n y(.,t_n) - y(.,t_n) \quad (4.11)$$

where the Ritz fractional projection operator is given by (4.6). Thus,

$$E^n = \omega^n - e^n \Longrightarrow\ \ \omega^n = e^n + E^n. \quad (4.12)$$

From Mamadu *et al*. [12], it was deduced that

$$ {}_0^R D_t^\alpha[u(x,t) - u_0(x,t)]|_{t=t_n} = \sum_{k=0}^n w_{kn}[u\,(t_n - t_k) - \mathrm{u}(0)] + \frac{{t_n}^{-\alpha}}{\Delta t^{-\alpha}\Gamma(-\alpha)} G_n(\mathrm{g}), \quad (4.13)$$

Using (4.13), (4.10) can be re-written as

$$\sum_{k=0}^n w_{kn}[({E_k}^n - {E_n}^n) - \tilde{E}^n, \mathrm{s})] + \frac{{t_n}^{-\alpha}}{\Delta t^{-\alpha}\Gamma(-\alpha)}\mathrm{G}^n(\mathrm{f}) + B(y^n, s) = (\vartheta^n(x), s),\ \ s \in s^n. \quad (4.14)$$

Applying (4.11) and (4.12), the error equation (4.14) can be re-written as

$$\sum_{k=0}^n w_{kn}(\omega^n, \mathrm{s}) + \frac{{t_n}^{-\alpha}}{\Delta t^{-\alpha}\Gamma(-\alpha)}\mathrm{G}^n(\mathrm{f}) + B(y^n, s) = (\vartheta^n(x), s),\ \ s \in s^n.$$

$$\Longrightarrow \sum_{k=0}^n w_{kn}(\omega^n, \mathrm{s}) + B(y^n, s) = \sum_{k=0}^n w_{kn}({e_k}^n - {e_n}^{n-1}, \mathrm{s}) + \sum_{k=0}^n w_{kn}(\omega^{n-1}, \mathrm{s}) - \sum_{k=0}^n w_{kn}({E_k}^{n-1} - {\tilde{E}_n}^{\ n-1}, \mathrm{s}) - \frac{{t_n}^{-\alpha}}{\Delta t^{-\alpha}\Gamma(-\alpha)}\mathrm{G}^n(\mathrm{f}) + (\vartheta^n(x), s),\ \ s \in s^n. \quad (4.15)$$

Choosing $s = \omega^n$ in (4.15) and using Lemma 3.2(i) and the Cauchy-Schwartz inequality (see [18], [19]), we obtain

$$\sum_{k=0}^n w_{kn}\|\omega^n\|_{L^2(\Omega)} \leq \sum_{k=0}^n w_{kn}(\|{e_k}^n - {e_n}^{n-1}\|_{L^2(\Omega)} + \|\omega^{n-1}\|_{L^2(\Omega)}) + \sum_{k=0}^n w_{kn}\left(\sup_{s\in s^n, s\neq 0}\frac{\left({E_k}^{n-1} - {\tilde{E}_n}^{\ n-1}, s\right)}{\|s\|_{L^2(\Omega)}}\right) + \|(\vartheta^n\|_{L^2(\Omega)} + \frac{{t_n}^{-\alpha}}{\Delta t^{-\alpha}\Gamma(-\alpha)}\mathrm{G}^n(\mathrm{f}). \quad (4.16)$$

Using the Gronwall inequality (see [20], [21]) on (4.16), we get

$$\sum_{k=0}^n w_{kn}\|\omega^n\|_{L^2(\Omega)} \leq \sum_{r=1}^j \sum_{k=0}^{n-r} w_{k(n-r)}(\left\|{e_k}^{(n-r)} - {e_{(n-r)}}^{(n-r-1)}\right\|_{L^2(\Omega)} + \|e^0\|_{L^2(\Omega)}) + \sum_{\mathrm{r}=1}^{\mathrm{j}} \sum_{k=0}^n w_{k(n-r)}\left(\sup_{s\in s^n, s\neq 0}\frac{\left({E_k}^{n-r-1} - {\tilde{E}_{n-r}}^{\ n-r-1}, s\right)}{\|s\|_{L^2(\Omega)}}\right) \sum_{\mathrm{r}=1}^{\mathrm{j}}\|\vartheta^r\|_{L^2(\Omega)} + \frac{{t_n}^{-\alpha}}{\Delta t^{-\alpha}\Gamma(-\alpha)}\sum_{\mathrm{r}=1}^{\mathrm{j}}\|\mathrm{G}^{\mathrm{r}}(\mathrm{f})\|_{L^2(\Omega)}, \quad (4.17)$$

here $\|\omega^0\|_{L^2(\Omega)} \leq \|e^0\|_{L^2(\Omega)}$

Applying the triangle inequality on (4.17), we get

$$\sum_{k=0}^n w_{kn}\|E^n\|_{L^2(\Omega)} \leq \sum_{k=0}^n w_{kn}\|\omega^n\|_{L^2(\Omega)} + \sum_{k=0}^n w_{kn}\|e^n\|_{L^2(\Omega)}$$

$$\leq \sum_{k=0}^n w_{kn}\|e^n\|_{L^2(\Omega)} + \sum_{r=1}^j \sum_{k=0}^{n-r} w_{k(n-r)}(\left\|{e_k}^{(n-r)} - {e_{(n-r)}}^{(n-r-1)}\right\|_{L^2(\Omega)} + \|e^0\|_{L^2(\Omega)}) + \sum_{\mathrm{r}=1}^{\mathrm{j}} \sum_{k=0}^n w_{k(n-r)}\left(\sup_{s\in s^n, s\neq 0}\frac{\left({E_k}^{n-r-1} - {\tilde{E}_{n-r}}^{\ n-r-1}, s\right)}{\|s\|_{L^2(\Omega)}}\right) + \sum_{\mathrm{r}=1}^{\mathrm{j}}\|\vartheta^r\|_{L^2(\Omega)} + \frac{{t_n}^{-\alpha}}{\Delta t^{-\alpha}\Gamma(-\alpha)}\sum_{\mathrm{r}=1}^{\mathrm{j}}\|\mathrm{G}^{\mathrm{r}}(\mathrm{f})\|_{L^2(\Omega)}. \quad (4.18)$$

Using lemma 3.1, it is obvious that

$$\|e^n\|_{L^2(\Omega)} = \|\chi^n y(.,t_n) - y(.,t_n)\|_{L^2(\Omega)} \leq \aleph^{-k}\|y(.,t_n)\|_{H^k(\Omega)}, k = 0(1)n. \quad (4.19)$$

Now,

$$ {e_k}^{(n-r)} - {e_{(n-r)}}^{(n-r-1)} = \left(\chi^{(n-r)}\left(y(.,t_n) - y(.,t_{n-r})\right) - \left(y(.,t_n) - y(.,t_{n-r-1})\right)\right) + (\chi^n - \chi^{n-r-1})y(.,t_{n-r-1}),$$

$$j = 1(2)n.$$

Thus,

$$\left\|{e_k}^{(n-r)} - {e_{(n-r)}}^{(n-r-1)}\right\|_{L^2(\Omega)} \leq \left\|\left(\chi^{(n-r)}\left(y(.,t_n) - y(.,t_{n-r})\right) - \left(y(.,t_n) - y(.,t_{n-r-1})\right)\right)\right\|_{L^2(\Omega)} + \|(\chi^n - \chi^{n-r-1})y(.,t_{n-r-1})\|_{L^2(\Omega)} \quad (4.20)$$

Using (4.19) on (3.21), we arrive at

$$\left\|\left(\chi^{(n-r}\left(y(.,t_n)-y(.,t_{n-r})\right)-\left(y(.,t_n)-y(.,t_{n-r-1})\right)\right)\right\|_{L^2(\Omega)} \leq \aleph^{-k}\|T\|_{H^k(\Omega)}$$

$$\leq \aleph^{-k}\int_{t_{n-r-1}}^{t_{n-r}}\|y_t\|_{H^k(\Omega)}dt,$$

where

$$T=\int_{t_{n-r-1}}^{t_{n-r}} y_t dt.$$

Thus, it is obvious that

$$\|(\chi^n-\chi^{n-r-1})y(.,t_{n-r-1})\|_{L^2(\Omega)} \leq \aleph^{-k}\|y(.,t_{n-r-1})\|_{H^k(\Omega)}. \tag{4.21}$$

Suppose that the L^2 Ritz projection is defined from $y^{(n-1)}$ to $\tilde{y}^{(n-1)}$. Then

$$(E^{n-1}-\tilde{E}^{n-1},s)=\left(y^{(n-1)}-\tilde{y}^{(n-1)},s\right)=0, s \in s^n, n=1(2)n. \tag{4.22}$$

Also, applying Taylor's theorem, we can deduce that ([22])

$$\|\vartheta^r\|_{L^2(\Omega)} \leq \frac{t_n{}^{-\alpha}}{\Delta t^{-\alpha}\Gamma(-\alpha)}\|\mathrm{y}_{\mathrm{tt}}(.,\mathrm{t}_{\mathrm{n-r}})\|_{L^2(\Omega)}\|\mathrm{G}^{\mathrm{r}}(\mathrm{f})\|_{L^2(\Omega)}. \tag{4.23}$$

Incorporating (4.19) – (4.23) into (4.18), we arrive at our result.

5. Conclusion

In this paper, we have carried out a rigorous analysis of the convergence rate of space discretization of time fractional telegraph equation. Basically, we proved the convergence by combining the fractional Ritz projection and interpolation technique as superclose estimate in L_2- norm between them to avoid a difficult Ritz operator construction. It is pertinent to note that numerical illustrations of the space discretization scheme for the time fractional telegraph equation have reported and enclosed in Mamadu *et al.* [12].

Conflict of Interest

The authors declare no conflict of interest.

Acknowledgment

Not applicable.

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