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Toeplitz Determinant For Error Starlike & Error Convex Function

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Abstract Normalised Error function has been coined and analyzed in 2018 [13].The concept of normalised error function discussed in [13], motivated us to find the new results of Toeplitz determinant for the subclasses of analytic univalent functions concurrent with error function. By seeing the history of error function in Geometric functions theory, Ramachandran et. al [13] derived the coefficient estimates followed by the Fekete-Szegö problem for the normalised subclasses of starlike and convex functions associated with error function. Finding coefficient estimates is one of the most provoking concepts in geometric function theory. In current scenario scientists are concentrating on special functions which are connected with univalent functions. Based on these concepts, the present paper deals with supremum and infimum of Toeplitz determinant for starlike and convex in terms of error function with convolution product using the concept of subordination. Also, we derive the sharp bounds for probability distribution associated with error starlike and error convex functions.

Keywords Analytic Functions, Error Function, Toeplitz Determinant, Subordination and Fekete-Szegö Inequality

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1 Introduction and preliminaries

Many of the special function has been involved in applied mathematics and sciences. One of such function is called Error functions or Gaussian error function. It has a wide range of applications in the theory of optical sciences. A sigmoid-type function that occurs as a non-elementary function is treated as the error function erf. Error functions and numerical approximations can be applied in all domains of applied mathematics [5].

The Error function, which is also an entire function can be defined as,

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z exp(-u^2)du = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)k!}. \tag{1.1}$$

In statistics, random variable can be derived as the error function. Whereas in complex analysis, error function will be analytic everywhere in regard to Taylor series. It has no singular points except from infinity. The Taylor expansion of error function is always convergent.

Let $\mathcal{A}$ be the class of functions f which is analytic in the open unit disc $\mathbb{D}=\{z : z\in\mathbb{C} : |z|<1\}$ and is of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \qquad (z\in\mathbb{D}). \tag{1.2}$$

Further, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$.

In the year 2016, Ramachandran et al.[13] introduced and defined the following function will be of the form

$$\mathbb{E}_rf(z)=\frac{\sqrt{\pi z}}{2}erf(\sqrt{z})=z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{(2k-1)(k-1)!}z^k. \tag{1.3}$$

By the concept of hadamard product, a family of analytic function can be defined as:

$$E=\mathcal{A}*\mathbb{E}_rf=\left\{\mathbb{F}:\mathbb{F}(z)=(f*\mathbb{E}_rf)(z)=z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}a_k}{(2k-1)(k-1)!}z^k, f\in\mathcal{A}\right\}, \tag{1.4}$$

and we denote by $\mathbb{E}_rf$ which is the class consists of a single function $\mathbb{E}_rf$.

A Toeplitz determinant can be thought of as an *upside-down* Hankel determinant. It is observed that, there is a close relation between Hankel determinant and Toeplitz determinant. The main diagonal elements are constant in Toeplitz determinant whereas the reverse diagonal elements are constant in Hankel determinant. The Hankel determinant of the function $f(z)$ for $r\geq 1$ and $n\geq 1$ was introduced and studied by Pommerenke [11, 12] as follows:

$$\mathcal{H}_r(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+r}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2}\end{vmatrix}.$$

The symmetric Toeplitz determinant $\mathcal{T}_r(n)$ is

$$\mathcal{T}_r(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1}\\ a_{n+1} & a_{n} & \cdots & a_{n+r}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ a_{n+r-1} & a_{n+r} & \cdots & a_{n}\end{vmatrix}.$$

Recall that an analytic function f in $\mathbb{D}$ is subordinate to an analytic function g in $\mathbb{D}$, written $f\prec g$, if there exists a Schwarz function ω, analytic in $\mathbb{D}$ with

$$\omega(0)=0 \quad and \quad |\omega(z)|<1 \quad (z\in\mathbb{D})$$

such that $f(z)=g(\omega(z)), z\in\mathbb{D}$. In particular, if the function g is univalent in $\mathbb{D}$, the above subordination is equivalent to

$$f(0)=g(0) \quad and \quad f(\mathbb{D})\subset g(\mathbb{D}).$$

Inspired by the concept discussed in [13], we define the following two generalized subclasses of $\mathcal{A}$.

Definition 1. *Let ϕ be an analytic univalent function with positive real part in $\mathbb{D}$ with $\phi(0)=1$ and $\phi'(0)>0$, starlike with respect to 1 and symmetric about the real axis, the classes $\mathcal{ES}^*(\phi)$ and $\mathcal{EC}(\phi)$ satisfying the conditions:*

$$\frac{z\mathbb{F}'(z)}{\mathbb{F}(z)}\prec\phi(z) \qquad (z\in\mathbb{D}) \tag{1.5}$$

and

$$1+\frac{z\mathbb{F}''(z)}{\mathbb{F}'(z)}\prec\phi(z) \qquad (z\in\mathbb{D}). \tag{1.6}$$

By seeing the history of Error function in geometric function theory, in the year 2016, Ramachandran et al. [13] introduced and studied the normalised error starlike and error convex function in the conical region. Immediately, Ramachandran et al. [15] studied the Hankel determinant for the starlike and convex function associated with error function. Then, Ramachandran et al. [16] introduced and found the sharp bounds for error close to convex function in 2019. Recently, Altinkaya et al. [1] studied the concept of error function.

The major aim of this paper is to explore the Toeplitz determinents $T_2(2)$ and $T_3(1)$ for the classes $\mathcal{ES}^*(\phi)$ and $\mathcal{EC}(\phi)$. In particular our results yield the corresponding results of the function $\mathcal{K}_\phi$ and $\mathcal{J}_\phi$ defined as,

$$\frac{z\mathcal{K}_\phi^{'}(z)}{\mathcal{K}_\phi(z)} = \phi(iz), \quad \mathcal{K}_\phi(0) = \mathcal{K}_\phi^{'}(0) - 1 = 0 \tag{1.7}$$

and

$$1 + \frac{z\mathcal{J}_\phi^{''}(z)}{\mathcal{J}_\phi{}'(z)} = \phi(iz), \quad \mathcal{J}_\phi(0) = \mathcal{J}_\phi^{'}(0) - 1 = 0. \tag{1.8}$$

Using these functions, we can illustrate the sharp bounds for the subclasses of $\mathcal{ES}^*(\phi)$ and $\mathcal{EC}(\phi)$.

2 Preliminary Results

The main results have been proved with the help of following results associate with the coefficient bounds.

Lemma 1. *[8] Let $p \in \mathcal{P}$ be the class of analytic function of the form*

$$1 + p_1 z + p_2 z^2 + \ldots, \tag{2.1}$$

for which $\mathcal{R}(p(z)) > 0$. It is well known that $|p_n| \leq 2$.

Lemma 2. *[10] Let $p \in \mathcal{P}$ be given by the power series in* (2.1) *with positive real part in $\mathbb{D}$, then*

$$|p_2 - \nu p_1^2| \leq \begin{cases} -4\nu + 2 & if \quad \nu \leq 0 \\ 2 & if \quad 0 \leq \nu \leq 1 \\ 4\nu - 2 & if \quad \nu \geq 1 \end{cases}$$

when $\nu < 0$ or $\nu > 1$, the quality holds if and only if $p(z) = \dfrac{1+z}{1-z}$ or one of its rotations.

3 Main Results

In this section, we give sharp estimates for $T_2(2) = a_3^2 - a_2^2$ for the class of functions $\mathcal{ES}^*(\phi)$ and $\mathcal{EC}(\phi)$.

Theorem 1. *If f is a function belongs to the class $\mathcal{ES}^*(\phi)$ and let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $0 < B_1 \leq |B_2 + B_1^2|$, then the following ineqality holds:*

$$T_2(2) \leq 25(B_2 + B_1^2)^2 + 9B_1^2$$

is sharp.

Proof. If $f \in \mathcal{ES}^*(\phi)$ then there exists a Schwarz function $\omega \in \mathbb{D}$ of the form

$$\omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \ldots, \tag{3.1}$$

such that

$$\frac{z\mathbb{F}^{'}(z)}{\mathbb{F}(z)} = \phi(\omega(z)). \tag{3.2}$$

With the concept of subordination, we see that the function

$$P_1(z) = \frac{1 - \phi^{-1}(p(z))}{1 + \phi^{-1}(p(z))} = 1 + p_1 z + p_2 z^2 + \ldots.$$

We note that,

$$\frac{z\mathbb{F}'(z)}{\mathbb{F}(z)} = 1 - \frac{1}{3}a_2 z + \left(\frac{1}{5}a_3 - \frac{1}{9}a_2^2\right) - \left(\frac{3}{42}a_4 - \frac{1}{10}a_2 a_3 + \frac{1}{27}a_2^3\right) \ldots . \tag{3.3}$$

and

$$\phi(\omega(z)) = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2\right) z^2 + \ldots . \tag{3.4}$$

From (3.2), (3.3) and (3.4), we obtain

$$a_2 = -\frac{3}{2}B_1 p_1 \tag{3.5}$$

and

$$a_3 = \frac{5}{4}\left(2B_1 p_2 + \left(B_1^2 - B_1 + B_2\right) p_1^2\right). \tag{3.6}$$

Using above two equations, we have

$$a_3 - \mu a_2^2 = \frac{5}{2}B_1\left(p_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - B_1 + \frac{9}{5}\mu B_1\right)p_1^2\right)$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - B_1 + \frac{9}{5}\mu B_1\right).$$

Also from Lemma 2,

$$|a_3 - \nu a_2^2| \leq \begin{cases} 5(B_2 + B_1^2 - \frac{9}{5}\mu B_1^2) & if \quad \frac{9}{5}\mu B_1^2 \leq B_2 + B_1^2 - B_1; \\ 5B_1 & if \quad B_2 + B_1^2 - B_1 \leq \frac{9}{5}\mu B_1^2 \leq B_2 + B_1^2 - B_1; \\ -5(B_2 + B_1^2 - \frac{9}{5}\mu B_1^2) & if \quad \frac{9}{5}\mu B_1^2 \geq B_2 + B_1^2 - B_1. \end{cases} \tag{3.7}$$

Since $|p_n| \leq 2$,

$$|a_2| \leq 3|B_1| \tag{3.8}$$

and

$$|a_3| \leq 5|(B_1^2 + B_2)|. \tag{3.9}$$

Using these coefficient estimates,

$$|T_2(2)| = |a_3^2| + |a_2^2| \leq 25(B_1^2 + B_2)^2 + 9B_1^2. \tag{3.10}$$

The function $\mathcal{K}_\phi$ has the power series expansion corresponding to the function $f \in \mathbb{D}$ when $\omega(z) = iz$.

$$\mathcal{K}_\phi = z - iB_1 z^2 - \frac{1}{2}(B_1^2 + B_2)z^3 + \ldots .$$

Our result is sharp for the function $\mathcal{K}_\phi$ defined in (1.7).

In particular, $P_1(z) = 1 + 2iz - 2z^2 + \ldots$ with $p_1 = 2i$ and $p_2 = -2$, so that

$$a_2 = iB_1$$

and

$$a_3 = -\frac{(B_1^2 + B_2)}{2}.$$

Now, for the function $\mathcal{K}_\phi$, we have the desired result (3.10). □

Theorem 2. *If $f \in \mathcal{ES}^*(\phi)$ and $B_1 - B_1^2 \leq B_2 \leq 3B_1^2 - B_1$, then*

$$T_3(1) \leq 1 + 18B_1^2 + 25(B_2 + B_1^2)\left(\frac{13}{5}B_1^2 - B_2\right). \tag{3.11}$$

Which is sharp for the function $\mathcal{K}_\phi$.

Proof. Since $T_3(1) = 1 - 2a_2^2 - a_3(a_3 - 2a_2^2)$ is obtained from the determinant

$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}$$

Now, from (3.8) and (3.9), we obtained (3.11). □

Theorem 3. *If f is said to be in the class $\mathcal{EC}(\phi)$ and let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ with $0 < B_1 \leq |B_2 + B_1^2|$, then the following inequality holds:*

$$T_2(2) \leq \frac{25}{9}(B_2 + B_1^2)^2 + \frac{9}{4}B_1^2$$

is sharp.

Proof. If $f \in \mathcal{EC}(\phi)$ then

$$a_3 - \mu a_2^2 = \frac{5}{6}B_1\left(p_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - B_1 + \frac{27}{5}\mu B_1\right)p_1^2\right)$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - B_1 + \frac{27}{5}\mu B_1\right).$$

From Lemma (2),

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{5}{3}(B_2 + B_1^2 - \frac{27}{5}\mu B_1^2) & if \quad \frac{27}{5}\mu B_1^2 \leq B_2 + B_1^2 - B_1; \\ \frac{5}{3}B_1 & if \quad B_2 + B_1^2 - B_1 \leq \frac{27}{5}\mu B_1^2 \leq B_2 + B_1^2 + B_1; \\ -\frac{5}{3}(B_2 + B_1^2 - \frac{27}{5}\mu B_1^2) & if \quad \frac{27}{5}\mu B_1^2 \geq B_2 + B_1^2 + B_1. \end{cases} \tag{3.12}$$

Also,

$$|T_2(2)| = |a_3^2| + |a_2^2| \leq \frac{25}{9}(B_1^2 + B_2)^2 + 9B_1^2, \tag{3.13}$$

which is sharp for the function $\mathcal{J}_\phi$. when $\omega(z) = iz$,

$$a_2 = -\frac{3}{2}iB_1$$

and

$$a_3 = -\frac{5}{3}(B_1^2 + B_2).$$

Hence, for the function $\mathcal{J}_\phi$, we have the desired result (3.13). □

Theorem 4. *If $f \in \mathcal{EC}(\phi)$ and $B_1 - B_1^2 \leq B_2 \leq 2B_1^2 - B_1$, then*

$$T_3(1) \leq 1 + \frac{9}{2}B_1^2 + \frac{25}{9}(B_2 + B_1^2)\left(\frac{49}{5}B_1^2 - B_2\right)$$

is sharp for the function $\mathcal{J}_\phi$.

4 Generalised distribution function

This section deals with the generalised distribution function for error starlike and error convex function subordinate to $\mathcal{Q}(z)$. Where,

$$\mathcal{Q}(z) = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = e^{e^z - 1} \qquad (z \in \mathbb{D}). \tag{4.1}$$

is starlike with respect to 1 and its efficients generate Bell numbers [14]. Where, B_n called the Bell numbers which satisfy the recurrence relation involving Binomial coefficients [2, 3, 4, 17].

Altikaya et al.[1] introduced the Taylor's series expansion for the generalized distribution functions

$$\mathcal{V}_\phi(z) = z + \sum_{k=2}^{\infty} \frac{a_k - 1}{\mathcal{S}} z^k, \tag{4.2}$$

where $\mathcal{S} = \sum_{n=0}^{\infty} a_n, a_n \geq 0$ for $n \in N$.

Using convolution or Hadamord product, (1.3) and (4.2):

$$\mathcal{EV}(z) = \mathcal{V}_\phi * E_r f = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} a_{n-1}}{(2n-1)(n-1)!\mathcal{S}} z^n, \quad f \in \mathcal{A}. \tag{4.3}$$

For brief introduction about the method of estimation for discrete probability distribution function, readers can refere [1].

Now, defining the new subclasses of starlike and convex functions associated with error function which are subordination to $\mathcal{Q}(z)$.

Definition 2. *If $\mathcal{F} \in \mathcal{A}$ is said to be in the class $\phi\mathcal{ES}^*(\mathcal{Q})$, then*

$$\frac{z\mathcal{EV}'(z)}{\mathcal{EV}(z)} \prec \mathcal{Q}(z) \qquad (z \in \mathbb{D}) \tag{4.4}$$

and if $\mathcal{F} \in \phi\mathcal{EC}(\mathcal{Q})$, then

$$1 + \frac{z\mathcal{EV}''(z)}{\mathcal{EV}'(z)} \prec \mathcal{Q}(z) \qquad (z \in \mathbb{D}). \tag{4.5}$$

In this particular section, we are going to find the sharp estimates for the initial coefficients of subclasses $\phi\mathcal{ES}^*(\mathcal{Q})$ and $\phi\mathcal{EC}(\mathcal{Q})$. Further, the authors give sharp bounds for the toeplitz determinants $T_2(2)$ and $T_3(1)$ for the class of functions $\phi\mathcal{ES}^*(\mathcal{Q})$ and $\phi\mathcal{EC}(\mathcal{Q})$.

Lemma 3. *Let ω be the class of all analytic functions of the form* (3.1)*, then*

$$|\omega_2 + t\omega_1^2| \leq max\{1, |t|\}.$$

The above inequality is sharp for $\omega(z) = z$ and $\omega(z) = z^2$.

Theorem 5. *Let $\mathcal{EV}(z)$ be a function defined in* (4.3) *belongs to $\phi\mathcal{ES}^*(\mathcal{Q})$, then*

$$|T_2(2)| = \left|\frac{a_2^2}{\mathcal{S}^2} - \frac{a_1^2}{\mathcal{S}^2}\right| \leq 109$$

and

$$|T_3(1)| = \left|\frac{a_2}{\mathcal{S}} - 2\frac{a_1^2}{\mathcal{S}^2}\right| \leq 99.$$

Proof. Since $\mathcal{EV}(z) \in \phi\mathcal{ES}^*(\mathcal{Q})$

$$\frac{z\mathcal{EV}'(z)}{\mathcal{EV}(z)} = \mathcal{Q}(\omega(z)). \tag{4.6}$$

With the simple computation, we have

$$\left|\frac{a_1}{\mathcal{S}}\right| \leq 3, \tag{4.7}$$

$$\left|\frac{a_2}{\mathcal{S}}\right| \leq 10 \tag{4.8}$$

and

$$\left|\frac{a_2}{\mathcal{S}} - \mu\frac{a_1^2}{\mathcal{S}^2}\right| \leq 5max\left\{1, \left|2 - \frac{9}{5}\mu\right|\right\}. \tag{4.9}$$

From (4.7) and (4.8), we can obtain our desired results of $|T_2(2)|$ and $|T_3(1)|$. □

In this similar way we can acquire the following result:

Theorem 6. *Let $\mathcal{EV}(z)$ be a function defined in* (4.3) *belongs to $\phi\mathcal{EC}(\mathcal{Q})$, then*

$$\left|\frac{a_1}{\mathcal{S}}\right| \leq \frac{3}{2},$$

$$\left|\frac{a_2}{\mathcal{S}}\right| \leq 5,$$

$$\left|\frac{a_2}{\mathcal{S}} - \mu\frac{a_1^2}{\mathcal{S}^2}\right| \leq \frac{5}{2}max\left\{1, \left|2 - \frac{27}{10}\mu\right|\right\},$$

$$|T_2(2)| = \left|\frac{a_2^2}{\mathcal{S}^2} - \frac{a_1^2}{\mathcal{S}^2}\right| \leq \frac{109}{4}, \tag{4.10}$$

and

$$|T_3(1)| = \left|\frac{a_2}{\mathcal{S}} - 2\frac{a_1^2}{\mathcal{S}^2}\right| \leq 48. \tag{4.11}$$

5 Conclusions

In this paper, we obtained the sharp estimates of Toeplitz determinant for the subclass of analytic stalike and convex functions associated with error function. Also, we found the sharp result using the concept of probability theory.

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