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m-Continuity and Fixed Points in $\mathcal{N}$-Complete G-Metric Spaces

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Abstract Fixed point technique can be considered as one of the most powerful tools to solve problems which occur in several fields like Physics, Chemistry, Computer Science, Economics and other subbranches of Mathematics etc. Banach [3] gave the first result in the field of metric fixed point theory which guarantees the existence and uniqueness of a fixed point in a complete metric space. Thereafter, many Mathematicians replace the notion of metric space and Banach contractive condition with various generalized metric spaces and different contractions to prove fixed point theorems. One such generalized metric space, called G-metric space, was proposed in [6]. Abhijit Pant, R.P.Pant [1] introduced a new type of contraction and obtained some results in metric spaces in the year 2017. The purpose of this paper is to define $\mathcal{N}$-complete G-metric space and study three metric fixed point results for such spaces. In the first two fixed point results, we use weaker form of continuity, called m-continuity and new type contractive conditions while in the third result simulation function is used. The results which we obtained will improve, extend and generalize some results in [1] and [2] in the existing literature. In addition to this, we give examples to validate our results.

Keywords G-metric Space, m-continuity, $\mathcal{N}$-set, $\mathcal{N}$-complete, Simulation Function

1 Introduction

The study of various metric spaces has been attracting the interest of several mathematicians. Banach contraction principle, which states that every contraction mapping defined on a complete metric space has a unique fixed point, was given by S . Banach [3] in the year 1922 . Since then several authors improved and generalized this theorem for different contractions in various generalizations of metric spaces . In this direction , Gahler [4] attempted to generalize the concept of metric space and introduced new concept called 2-metric space . Later on, Dhage [5] changed some axioms of 2-metric space and gave the concept of a D-metric space . In the year 2006, Mustafa and Sims [6] found that the topological structure of D-metric space was false. This forces them to introduce a new generalized metric space, called G-metric space and study its topological properties. Afterwards, several mathematicians studied many fixed and common point theorems for various contractions for such spaces, for example, see [7-13] .
In 2015, A.H.Ansari [14] introduced a new function called C-class function and obtained some results for such functions . Very recently, F. Khojasteh, S. Shukla, S. Raden [15] generalized this C-class function and introduced simulation function .They also proved some intresting results using simulation functions .
The main objective of this article is to prove three fixed point theorems in the setting of $\mathcal{N}$-Complete G-Metric Space which are analogous to the theorems given in metric spaces by R.P.Pant etal . [1] and Stojan N Randenovic, Sumit Chandok [2] .

2 Preliminaries

2.1 Definition [7]

Let $\mathcal{X}$ be a nonempty set and $G : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying the following properties:
(i) $G(\zeta, \mu, \nu) = 0$ if $\zeta = \mu = \nu$

(ii) $0 < G(\zeta,\zeta,\mu)$ for all $\zeta,\mu \in \mathcal{X}$ with $\zeta \neq \mu$
(iii) $G(\zeta,\zeta,\mu) \leq G(\zeta,\mu,\nu)$ for all $\zeta,\mu,\nu \in \mathcal{X}$ with $\nu \neq \mu$
(iv) $G(\zeta,\mu,\nu) = G(\zeta,\nu,\mu) = G(\mu,\nu,\zeta) = ...$ (symmetry in all three variables)
(v) $G(\zeta,\mu,\nu) \leq G(\zeta,\eta,\eta) + G(\eta,\mu,\nu)$ for all $\zeta,\mu,\nu,\eta \in \mathcal{X}$ (rectangle inequality).

Then the function G is called a generalization metric or more specially G-metric on $\mathcal{X}$ and the pair ($\mathcal{X}$,G) is called a G-metric space.

2.2 Definition [1]

A self map $\mathscr{F} : \mathcal{X} \to \mathcal{X}$ is said to be m-continuous at a point $\theta \in \mathcal{X}$ iff $\mathscr{F}^{n+m}\zeta_0 \to \mathscr{F}\theta$ in $\mathcal{X}$ whenever $\mathscr{F}^{n+(m-1)}\zeta_0 \to \theta$ for some $\zeta_0 \in \mathcal{X}$ as $n \to \infty$.

2.3 Remark

It is observed that every continuous is m-continuous, but its converse need not be true.
For this, consider

2.4 Example

Let $\mathcal{X}$=[0,2], $G(\zeta,\mu,\nu)=\begin{cases} 0, \text{if } \zeta=\mu=\nu \\ max\{\zeta,\mu,\nu\}\text{otherwise}\end{cases}$ and $\mathscr{F}\zeta=\begin{cases} 1, \ \zeta \in [0,1] \\ 0, \zeta \in (1,2]\end{cases}$. Now, let $\mathscr{F}^n\zeta_0 \to \theta$ for some $\zeta_0, \theta \in \mathcal{X}$. So $\theta = \lim_{n\to\infty} \mathscr{F}^n(\zeta_0) = \begin{cases} 1, \ \zeta_0 \in [0,1] \\ 0, \zeta_0 \in (1,2]\end{cases}$ and therefore $\theta=\begin{cases} 1, \ \zeta_0 \in [0,1] \\ 0, \zeta_0 \in (1,2].\end{cases}$ Let θ=1 and assume that $\mathscr{F}^{n+1}\zeta_0 \to 1$. This implies $\zeta_0 \in [0,1]$ and hence $\mathscr{F}^{n+2}\zeta_0 = 1 \to 1 = \mathscr{F}(1)$. Thus, $\mathscr{F}$ is 2-continuous at 1, but $\mathscr{F}$ is discontinuous at 1.Throughout the paper, we assume that $\mathscr{F} : \mathcal{X} \to \mathcal{X}$ is a function and fix a point ζ_0.

2.5 Definition

Let $\mathcal{X}$ be a G-metric space. The set defined by $\mathcal{N}(\zeta_0) := \{\mathscr{F}^n\zeta_0 : n \in \mathbb{N}\}$ is called a $\mathcal{N}$-set w.r.t. $\mathscr{F}$ and $\zeta_0 \in \mathcal{X}$.

2.6 Definition

A sequence $(\mathscr{F}^n\zeta_0)$ in $\mathcal{N}(\zeta_0)$ is called $\mathcal{N}$-Cauchy sequence w.r.t. $\mathscr{F}$ and $\zeta_0 \in \mathcal{X}$ iff
$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^m\zeta_0) \to 0$ as $n,m \to \infty$.

2.7 Definition

A G-metric space $\mathcal{X}$ is called $\mathcal{N}$-complete w.r.t. some $\mathscr{F}$ and $\zeta_0 \in \mathcal{X}$ iff there exists $\theta \in X$ such that $G(\mathscr{F}^n\theta_0, \mathscr{F}^n\theta_0, \theta) \to 0$ as $n \to \infty$, whenever $(\mathscr{F}^n\zeta_0)$ is any $\mathcal{N}$-Cauchy in $\mathcal{X}$ with respect to $\mathscr{F}$ and $\zeta_0 \in \mathcal{X}$.

2.8 Remark

It can be easily seen that every complete G-metric space is $\mathcal{N}$-complete w.r.t. some $\mathscr{F}$ and $\zeta_0 \in \mathcal{X}$. But, its converse need not be true.
The following example illustrates it.

2.9 Example

Let $\mathcal{X}$=[0,1), $G(\zeta,\mu,\nu)=\begin{cases} 0, \text{if } \zeta=\mu=\nu \\ max\{\zeta,\mu,\nu\} \text{ otherwise}\end{cases}$ and we define a map $\mathscr{F} : \mathcal{X} \to \mathcal{X}$ by$\mathscr{F}\zeta = \zeta^2$ for $\zeta \in \mathcal{X}$. Take $\zeta_0 = \frac{1}{2} \in \mathcal{X}$.Then, we have
$\mathcal{N}(\zeta_0) = \{\mathscr{F}^n\zeta_0 / n \in \mathbb{N}\} = \{(\frac{1}{2})^{2^n} / n \in N\}$. Now,
$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^m\zeta_0)$=max$\{(\frac{1}{2})^{2^n}, (\frac{1}{2})^{2^n}, (\frac{1}{2})^{2^m}\}$

$\leq (\frac{1}{2})^{2^n} + (\frac{1}{2})^{2^m} \to 0$ as n,m $\to \infty$.

This tells us that $(\mathscr{F}^n\zeta_0)$ is $\mathcal{N}$-Cauchy sequence with respect to $\mathscr{F}$ and $\zeta_0 \in \mathcal{X}$. Also, we obtain that $G(\mathscr{F}^n x_0, \mathscr{F}^n x_0, 0) = max\{(\frac{1}{2})^{2^n}, (\frac{1}{2})^{2^n}, 0\}$
$= (\frac{1}{2})^{2^n} \to 0$, as $n \to \infty$ $\mathscr{F}^n\zeta_0 \to 0$, as n $\to \infty$. Therefore, $\mathcal{X}$ is $\mathcal{N}$- complete w.r.t. $\mathscr{F}$ and ζ_0 but $\mathcal{X}$ is not G-complete.

3 Main Results

We begin with the following theorem.

3.1 Theorem

Let $\mathcal{X}$ be $\mathcal{N}$-complete w.r.t. some $\mathscr{F}$ and ζ_0 such that
(i) $G(\mathscr{F}\zeta, \mathscr{F}\zeta, \mathscr{F}\mu) < \max\{G(\zeta,\zeta,\mathscr{F}\zeta), G(\mu,\mu,\mathscr{F}\mu)\}$ if $\max\{G(\zeta,\zeta,\mathscr{F}\zeta), G(\mu,\mu,\mathscr{F}\mu)\} > 0$
(ii) For given $\epsilon > 0$, there is a $\delta > 0$ so that if $\epsilon \leq \max\{G(\zeta,\zeta,\mathscr{F}\zeta), G(\mu,\mu,\mathscr{F}\mu)\} < \epsilon + \delta$, then $G(\mathscr{F}\zeta, \mathscr{F}\zeta, \mathscr{F}\mu) < \epsilon$. If $\mathscr{F}$ is m-continuous for some positive integer m, then there is a point in $\mathcal{X}$ which is a fixed point of $\mathscr{F}$.

Proof:

Fix $\zeta_0 \in \mathcal{X}$. If $\mathscr{F}^{n+1}\zeta_0 = \mathscr{F}^n\zeta_0$ for some n, then $\mathscr{F}(\mathscr{F}^n\zeta_0) = \mathscr{F}^n\zeta_0$ and hence $\mathscr{F}^n\zeta_0$ is a fixed point of $\mathscr{F}$. Thus,let us suppose that $\mathscr{F}^{n+1}\zeta_0 \neq \mathscr{F}^n\zeta_0$ for all $n \in \mathbb{N}$. Firstly, we show that $(\mathscr{F}^n\zeta_0)$ is $\mathcal{N}$-Cauchy in $\mathcal{X}$ w.r.t. some $\mathscr{F}$ and ζ_0 . By (i), we have

$$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) < \max\{G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0), G(\mathscr{F}^{n+1}\zeta_0, \mathscr{F}^{n+1}\zeta_0, \mathscr{F}^n\zeta_0)\} = G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0))$$

$$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) \leq G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0)$$

for all n$\in$ $\mathbb{N}$. Thus, $\{G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0)\}_{n=1}^{\infty}$ is a decreasing sequence of reals and bounded below by 0. Therefore, $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) \to r$ for some r, where $r = \inf_{n\in N}\{G(\mathscr{F}^n x_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0)\}$. This implies that $r \geq 0$. Let us claim that r=0 . Suppose that $r > 0$. Since $G(\mathscr{F}^n\zeta_0\mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) \to r$, for such $r > 0$, there exists n$\in$ $\mathbb{N}$ such that

$$|G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) - r| < 2r, \forall n \geq N,$$
which yields
$$0 \leq G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) - r < 2r, \forall n \geq N$$
so that
$$r \leq G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) < r + 2r$$
Thus, $r \leq G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) < r + \delta(r)$, for all $n \geq \mathbb{N}$, where $\delta(r) = 2r$. It follows from (ii) that $G(\mathscr{F}(\mathscr{F}^n\zeta_0), \mathscr{F}(\mathscr{F}^n\zeta_0), \mathscr{F}(\mathscr{F}^{n+1}\zeta_0)) < r$ for each n$\geq N$. Consequently,
$$G(\mathscr{F}^{n+1}\zeta_0, \mathscr{F}^{n+1}\zeta_0, \mathscr{F}^{n+2}\zeta_0) < r, \forall n \geq N.$$
In particular,
$G(\mathscr{F}^{N+1}\zeta_0, \mathscr{F}^{N+1}\zeta_0, \mathscr{F}^{N+2}\zeta_0) < r$, so that $r \leq G(\mathscr{F}^{N+1}\zeta_0, \mathscr{F}^{N+1}\zeta_0, \mathscr{F}^{N+2}\zeta_0) < r$, which leads to a contradiction . Thus r=0 and we conclude that $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) \to 0$ as $n \to \infty$.
For $s \in \mathbb{N}$,
$$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+s}\zeta_0) < \max\{G(\mathscr{F}^{n+1}\zeta_0, \mathscr{F}^{n+1}\zeta_0, \mathscr{F}^n\zeta_0) \; G(\mathscr{F}^{n+s-1}\zeta_0, \mathscr{F}^{n+s-1}x_0, \mathscr{F}^{n+s}x_0)\} = G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0)$$
for all n$\in \mathbb{N}$. Since $G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0) \to 0$ as $n \to \infty$, we have $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+s}\zeta_0) \to 0$ as $n \to \infty$ for each fixed $s \in \mathbb{N}$. This shows that $(\mathscr{F}^n\zeta_0)$ is $\mathcal{N}$-Cauchy sequence w.r.t. some $\mathscr{F}$ and ζ_0 . Since $\mathcal{X}$ is $\mathcal{N}$-complete w.r.t. some $\mathscr{F}$ and ζ_0, there exists $\theta \in \mathcal{X}$ such that $\mathscr{F}^n\zeta_0 \to \theta$. This implies that $\mathscr{F}^{n+m}\zeta_0 \to \theta$ and $\mathscr{F}^{n+(m-1)}\zeta_0 \to \theta$ by using subsequence. As $\mathscr{F}$ is m-continuous and $\mathscr{F}^{n+(m-1)}\zeta_0 \to \theta$, we get that $\mathscr{F}^{n+m}\zeta_0 \to \mathscr{F}\theta$ and hence $\mathscr{F}\theta = \theta$.

3.2 Example

Let $\mathcal{X} = [0, 2)$, $G(\zeta, \mu, \nu) = \begin{cases} 0 \text{ if } \zeta = \mu = \nu \\ max\{\zeta, \mu, \nu\} \text{ otherwise} \end{cases}$

and let $\mathscr{F} : \mathcal{X} \to \mathcal{X}$ be defined as $\mathscr{F}\zeta = \begin{cases} 1 \text{ if } 0 \leq \zeta \leq 1 \\ 0 \text{ if } 1 < \zeta < 2 \end{cases}$

and
fix $\zeta_0 = \frac{1}{2} \in \mathcal{X}$, we have $\mathscr{F}^n\zeta_0 = 0$ for n= 1,2,3,...
Therefore $\mathcal{X}$ is $\mathcal{N}$-complete w.r.t some $\mathscr{F}$ and ζ_0 . Let $\zeta, \mu \in \mathcal{X}$ and $\epsilon > 0$. Then, we have
$$\begin{aligned} \epsilon &\leq max\{G(\zeta, \zeta, \mathscr{F}\zeta), G(\mu, \mu, \mathscr{F}\mu)\} \\ &= max\{G(\zeta, \zeta, 1), G(\mu, \mu, 1)\} \text{ if } \zeta, \mu \leq 1 \\ &= max\{1, 1\} \\ &= 1 \\ &\begin{cases} < (2 - \epsilon) + \epsilon, \text{ if } \epsilon < 1 \\ < 1 + \epsilon, \text{ if } \epsilon = 1 \end{cases} \end{aligned}$$
$G(\mathscr{F}\zeta, \mathscr{F}\zeta, \mathscr{F}\mu) = G(1, 1, 1) = 0 < \epsilon$.
Now, $\epsilon \leq max\{G(\zeta, \zeta, \mathscr{F}\zeta)G(\mu, \mu, \mathscr{F}\mu)\}$
$$\begin{aligned} &= max\{G(\zeta, \zeta, 0), G(\mu, \mu, 0)\} \\ &= max\{\zeta, \mu\} < 2 \\ &= (2 - \epsilon) + \epsilon \text{ if } \epsilon < 1 \\ &< (1 + \epsilon) \text{ if } \epsilon \geq 1 \end{aligned}$$
$G(\mathscr{F}\zeta, \mathscr{F}\zeta, \mathscr{F}\mu) = G(0, 0, 0) = 0 < \epsilon$
$$\begin{aligned} \epsilon &\leq max\{G(\zeta, \zeta, \mathscr{F}\zeta), G(\mu, \mu, \mathscr{F}\mu)\} \\ &= max\{G(\zeta, \zeta, 1), G(\mu, \mu, 1)\}, \text{ if } \zeta \leq 1, \mu > 1 \\ &= max\{1, \mu\} \\ &= \mu \leq 2 = 1 + 1 < \epsilon + 1 \\ &= \epsilon + \delta, \text{ where } \delta = 1. \end{aligned}$$
$G(\mathscr{F}\zeta, \mathscr{F}\zeta, \mathscr{F}\mu) = 1 < \epsilon$.
Therefore, all conditions of Theorem 3.1 are fulfilled and 0 is a fixed point of $\mathscr{F}$.

3.3 Theorem

Let $\mathcal{X}$ be $\mathcal{N}$-complete w.r.t. some $\mathscr{F}$ and ζ_0 with $G(\mathscr{F}\zeta, \mathscr{F}\zeta, \mathscr{F}\mu) \leq \phi(\max\{G(\zeta, \zeta, \mathscr{F}\zeta), G(\mu, \mu, \mathscr{F}\mu)\})$, $\phi : [0, \infty) \to [0, \infty)$ is continuous function with $\phi(p) < p$ for every $p > 0$. Then there is a point in $\mathcal{X}$ which is a fixed point of $\mathscr{F}$.

Proof:

Fix $\zeta_0 \in \mathcal{X}$. If $f^{n+1}\zeta_0 = f^n\zeta_0$ for some n $\in \mathbb{N}$ and hence $\mathscr{F}^n\zeta_0$ itself is a fixed point of $\mathscr{F}$. Thus, let us suppose that $\mathscr{F}^{n+1}\zeta_0 \neq \mathscr{F}^n\zeta_0$ for each positive integer n. Firstly, we show that $(\mathscr{F}^n\zeta_0)$ is $\mathcal{N}$-Cauchy sequence in $\mathcal{X}$ w.r.t. some $\mathscr{F}$ and ζ_0. For, consider
$$\begin{aligned} G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) &= G(\mathscr{F}(\mathscr{F}^{n-1}\zeta_0), \mathscr{F}(\mathscr{F}^{n-1}\zeta_0), \mathscr{F}(\mathscr{F}^n\zeta_0)) \\ &\leq \phi(max\{G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n x_0), \\ &\qquad G((\mathscr{F}^n\zeta_0), \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0)\}) \\ &= \phi(G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0)) \\ &< G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0) \end{aligned}$$
$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) < G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0)$ for all n$\in \mathbb{N}$. This says that $\{G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0)\}$ is a decreasing sequence of numbers and hence there is a $r \geq 0$ so that $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0\mathscr{F}^{n+1}\zeta_0) \to r$ as $n \to \infty$. Now, let us claim that r=0. On contrary, let us assume that $r > 0$. By using this inequality
$$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0\mathscr{F}^{n+1}\zeta_0) \leq \phi(G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0))$$
for all n, we get by applying $n \to \infty$ that
$$r \leq \phi(r) < r\text{-contradiction.}$$
Thus r=0. Hence $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) \to 0$ as $n \to \infty$. For $s \in \mathbb{N}$,
$$\begin{aligned} G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+s}\zeta_0) &= G(\mathscr{F}(\mathscr{F}^{n-1}\zeta_0), \mathscr{F}(\mathscr{F}^{n-1}\zeta_0), \mathscr{F}(\mathscr{F}^{n+s-1}\zeta_0)) \\ &\leq \phi(\max\{G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0), \\ &\qquad G(\mathscr{F}^{n+s-1}\zeta_0, \mathscr{F}^{n+s-1}\zeta_0, \mathscr{F}^{n+s}\zeta_0)\}) \\ &= \phi(G\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0)). \end{aligned}$$
$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+s}\zeta_0) \leq \phi(G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0))$ for all positive integers n.
As $n \to \infty$,we obtain $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+s}\zeta_0) \to 0$ as $n \to \infty, \forall s \in \mathbb{N}$, which tells that $(\mathscr{F}^n\zeta_0)$ is $\mathcal{N}$-Cauchy w.r.t. some $\mathscr{F}$ and ζ_0 . Since $\mathcal{X}$ is $\mathcal{N}$-complete w.r.t. some $\mathscr{F}$ and ζ_0, there is a $\theta \in \mathcal{X}$ such that $\mathscr{F}^n\zeta_0 \to \theta$ as n$\to \infty$. We calim that $\mathscr{F}\theta = \theta$. If possible suppose that $\mathscr{F}\theta \neq \theta$, then
$$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}z) \leq \phi(\max\{G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0), G(\theta, \theta, \mathscr{F}\theta)\})$$
Applying $n \to \infty$, we get
$$\begin{aligned} G(\theta, \theta, \mathscr{F}\theta) &\leq \phi(\max\{0, g(\theta, \theta, \mathscr{F}\theta)\}) \\ &= \phi(G(\theta, \theta, \mathscr{F}\theta)), \\ &< G(\theta, \theta, \mathscr{F}\theta), \end{aligned}$$
which gives $G(\theta, \theta, \mathscr{F}\theta) < G(\theta, \theta, \mathscr{F}\theta)$-contradiction Thus $\mathscr{F}\theta = \theta$.

3.4 Example

Let $\mathcal{X} = [0,2)$, $G(\zeta,\mu,\nu)= \begin{cases} 0, \zeta=\mu=\nu \\ max\{\zeta,\mu,\nu\} \text{ otherwise} \end{cases}$

Fix $\zeta_0 = 1 \in X$. Let $\mathscr{F}: \mathcal{X} \to \mathcal{X}$ be defined by $\mathscr{F}\zeta = \frac{\zeta}{4}$ for $\zeta \in \mathcal{X}$. Clearly, $\mathscr{F}^n\zeta_0 = (\frac{1}{4})^n$ for n=1,2,3,...

$$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^m\zeta_0) = G((\tfrac{1}{4})^n, (\tfrac{1}{4})^n, (\tfrac{1}{4})^m) = \max\{(\tfrac{1}{2})^{2n}, (\tfrac{1}{2})^{2n}, (\tfrac{1}{2})^{2m}\} \leq (\tfrac{1}{2})^{2n} + (\tfrac{1}{2})^{2m}.$$

Thus, $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^m\zeta_0) \leq (\frac{1}{2})^{2n} + (\frac{1}{2})^{2m}$ for all n,m$\in \mathbb{N}$. It follows that $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^m\zeta_0) \to 0$ as $n,m \to \infty$.Therefore, $(\mathscr{F}^n\zeta_0)$ is $\mathcal{N}$-Cauchy w.r.t. some $\mathscr{F}$ and ζ_0. Also, we have

$$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, 0) = \max\{(\tfrac{1}{4})^n, (\tfrac{1}{4})^n, 0\} = (\tfrac{1}{2})^{2n} \to 0 \text{ as } n \to \infty.$$

Thus $\mathscr{F}^n\zeta_0 \to 0$. This concludes that $\mathcal{X}$ is $\mathcal{N}$-complete w.r.t. some $\mathscr{F}$ and ζ_0.

Define $\phi: [0,\infty) \to [0,\infty)$ by $\phi(p) = \frac{p}{2}$ for $p \in [0,\infty)$.For $\zeta \leq \mu$,

$$G(f\zeta, f\zeta, f\zeta) = max\{\tfrac{\zeta}{4}, \tfrac{\zeta}{4}, \tfrac{\mu}{4}\} = \tfrac{\mu}{4} < \tfrac{\mu}{2} = \phi(max\{G(\zeta,\zeta,f\zeta), G(\mu,\mu,f\mu)\})$$

Therefore, all conditions of Theorem 3.3 are fulfilled and 0 is a fixed point of $\mathscr{F}$.

3.5 Definition [15]

A function $\phi: [0,\infty)^2 \to \mathbb{R}$ is said to be a simulation function if

(i) $\phi(0,0) = 0$

(ii) $\phi(p,q) < q - p$ for all $p,q > 0$

(iii) if (p_n) and (q_n) are two sequences in $(0,\infty)$ with $p_n < q_n$ such that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 0$,then $\limsup \phi(p_n, q_n) < 0$.

Now, we denote class of such functions by Φ.

3.6 Example

A function ϕ defined by $\phi(p,q) = \frac{q}{6} - p$ for $p,q \in [0\ \ \infty)$ is a simulation function.

3.7 Theorem

Let $\mathcal{X}$ be $\mathcal{N}$-complete w.r.t. some $\mathscr{F}$ and ζ_0 with $\phi(G(\mathscr{F}\zeta, \mathscr{F}\zeta, \mathscr{F}\mu), G(\zeta,\zeta,\mu)) \geq 0$ for all $\zeta,\mu \in \mathcal{X}$ and for some $\phi \in \Phi$. Then there is a point in $\mathcal{X}$ which is a fixed point of $\mathscr{F}$.

Proof:

Fix $\zeta_0 \in \mathcal{X}$. If $f^{n+1}\zeta_0 = f^n\zeta_0$ for some n $\in \mathbb{N}$ and hence $\mathscr{F}^n\zeta_0$ itself is a fixed point of $\mathscr{F}$. Thus, let us suppose that $\mathscr{F}^{n+1}\zeta_0 \neq \mathscr{F}^n\zeta_0$ for each n $\in \mathbb{N}$. Firstly, we prove that $(\mathscr{F}^n\zeta_0)$ is $\mathcal{N}$- Cauchy sequence in $\mathcal{X}$ w.r.t. some $\mathscr{F}$ and ζ_0. For, consider

$$0 \leq \phi(G(\mathscr{F}^{n+1}\zeta_0, \mathscr{F}^{n+1}\zeta_0, \mathscr{F}^n\zeta_0), G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n-1}\zeta_0)) < G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n-1}\zeta_0) - G(\mathscr{F}^{n+1}\zeta_0, \mathscr{F}^{n+1}\zeta_0, \mathscr{F}^n\zeta_0),$$

which implies $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) < G(\mathscr{F}^{n-1}\zeta_0, \mathscr{F}^{n-1}\zeta_0, \mathscr{F}^n\zeta_0)$ for all n$\in \mathbb{N}$. This says that $\{G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0)\}$ is a decreasing sequence of numbers and hence there is a $r \geq 0$ so that $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0\mathscr{F}^{n+1}\zeta_0) \to r$ as $n \to \infty$. Now, let us claim that r=0. On contrary, let us assume that $r > 0$. Since $G(\mathscr{F}^{n+1}\zeta_0, \mathscr{F}^{n+1}\zeta_0\mathscr{F}^n\zeta_0) \to r$ and $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0\mathscr{F}^{n-1}\zeta_0) \to r$ as $n \to \infty$, we get by definition of $\phi \in \Phi$ that

$$0 \leq \limsup \phi(G(\mathscr{F}^{n+1}\zeta_0, \mathscr{F}^{n+1}\zeta_0, \mathscr{F}^n\zeta_0), G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n-1}\zeta_0)) < 0.$$

This gives a contradiction. Therefore, $r = 0$ and hence $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n+1}\zeta_0) \to 0$ as $n \to \infty$. Let us assume that $(\mathscr{F}^n\zeta_0)$ is not a Cauchy sequence w.r.t. $\mathscr{F}$ and ζ_0. Then is an $\epsilon > 0$ such that

$$G(\mathscr{F}^{m_k+1}\zeta_0, \mathscr{F}^{m_k+1}\zeta_0\mathscr{F}^{n_k+1}\zeta_0) \to \epsilon$$

and

$$G(\mathscr{F}^{m_k}\zeta_0, \mathscr{F}^{m_k}\zeta_0\mathscr{F}^{n_k}\zeta_0) \to \epsilon, \text{ as } k \to \infty.$$

From the definition of ϕ, we obtain

$$0 \leq \limsup \phi(G(\mathscr{F}^{m_k+1}\zeta_0, \mathscr{F}^{m_k+1}\zeta_0, \mathscr{F}^{n_k+1}\zeta_0), G(\mathscr{F}^{m_k}\zeta_0, \mathscr{F}^{m_k}\zeta_0, \mathscr{F}^{n_k}\zeta_0)) < 0,$$

which yields a contradiction. This tells that $(\mathscr{F}^n\zeta_0)$ is $\mathcal{N}$-Cauchy w.r.t. some $\mathscr{F}$ and ζ_0.Since $\mathcal{X}$ is $\mathcal{N}$-complete w.r.t. some $\mathscr{F}$ and ζ_0, there is a $\theta \in X$ such that $\mathscr{F}^n\zeta_0 \to \theta$ as n$\to \infty$. We claim that $\mathscr{F}\theta = \theta$. If possible suppose that $\mathscr{F}\theta \neq \theta$, then we have

$$0 \leq \limsup \phi(G(\mathscr{F}^{n+1}\zeta_0, \mathscr{F}^{n+1}\zeta_0, \mathscr{F}^n\zeta_0), G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n-1}\zeta_0)) \leq \limsup[G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^{n-1}\zeta_0) - G(\mathscr{F}^{n+1}\zeta_0, \mathscr{F}^{n+1}\zeta_0, \mathscr{F}^n\zeta_0)], \leq -G(\theta,\theta,\mathscr{F}\theta).$$

Consequently, $G(\theta,\theta,\mathscr{F}\theta) \leq 0$ and so $\mathscr{F}(\theta) = \theta$.

3.8 Example

Let $\mathcal{X} = [0,4)$, $G(\zeta,\mu,\nu)= \begin{cases} 0, \zeta=\mu=\nu \\ max\{\zeta,\mu,\nu\}, \text{otherwise} \end{cases}$

Fix $\zeta_0 = 1 \in X$.Define $\mathscr{F}: \mathcal{X} \to \mathcal{X}$ by $\mathscr{F}\zeta = \frac{\zeta}{8}$ for $\zeta \in \mathcal{X}$. Clearly, $\mathscr{F}^n\zeta_0 = (\frac{1}{8})^n$ for n=1,2,3,...

$$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^m\zeta_0) = G((\tfrac{1}{8})^n, (\tfrac{1}{8})^n, (\tfrac{1}{8})^m) = \max\{(\tfrac{1}{2})^{3n}, (\tfrac{1}{2})^{3n}, (\tfrac{1}{2})^{3m}\} \leq (\tfrac{1}{2})^{3n} + (\tfrac{1}{2})^{3m}.$$

Thus, $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^m\zeta_0) \leq (\frac{1}{2})^{3n} + (\frac{1}{2})^{3m}$ for all n,m$\in \mathbb{N}$. It follows that $G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, \mathscr{F}^m\zeta_0) \to 0$ as $n,m \to \infty$. Therefore, $(\mathscr{F}^n\zeta_0)$ is $\mathcal{N}$-Cauchy w.r.t. some $\mathscr{F}$ and ζ_0. Also, we have

$$G(\mathscr{F}^n\zeta_0, \mathscr{F}^n\zeta_0, 0) = \max\{(\tfrac{1}{8})^n, (\tfrac{1}{8})^n, 0\} = (\tfrac{1}{2})^{3n} \to 0 \text{ as } n \to \infty.$$

Thus $\mathscr{F}^n\zeta_0 \to 0$. This concludes that $\mathcal{X}$ is $\mathcal{N}$-complete w.r.t. some $\mathscr{F}$ and ζ_0. Let us take a function

$$\phi: [0\ \ \infty)^2 \to \mathbb{R} \text{ by } \phi(p,q) = \tfrac{q}{4+q} - p$$

for $p,q \in [0\ \ \infty)$. Clearly, ϕ is a simulation function. Now, for $\zeta,\mu \in \mathcal{X}$, we have

$$\phi(G(f\zeta, f\zeta, f\mu), G(\zeta,\zeta,\mu)) = \frac{G(\zeta,\zeta,\mu)}{4 + G(\zeta,\zeta,\mu)} - G(\mathscr{F}\zeta, \mathscr{F}\zeta, \mathscr{F}\mu)$$

$$
\begin{aligned}
&= \frac{\max\{\zeta,\zeta,\mu\}}{4+\max\{\zeta,\zeta,\mu\}} - \frac{\max\{\zeta,\zeta,\mu\}}{8} \\
&\geq \frac{\max\{\zeta,\zeta,\mu\}}{8} - \frac{\max\{\zeta,\zeta,\mu\}}{8} \\
&= 0.
\end{aligned}
$$

Therefore, all conditions of Theorem 3.7 are fulfilled and 0 is a fixed point of $\mathscr{F}$.

4 Conclusions

We obtained three fixed point results of self-mappings in $\mathcal{N}$-complete G-Metric spaces which generalize several existing results in the literature. Further, these results can be extended to several other spaces.

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