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Performance Analysis of A Single Server Queue Operating in A Random Environment - A Novel Approach

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Abstract In this paper, we consider a single server queueing system operating in a random environment subject to disaster, repair and customer impatience. The random environment resides in any one of $N + 1$ phases $0, 1, 2, \cdots, N + 1$. The queueing system resides in phase $k, k = 1, 2, \cdots, N$ for a random interval of time and the sojourn period ends at the occurrence of a disaster. The sojourn period is exponentially distributed with mean $1/\eta_k$. At the end of the sojourn period, all customers in the system are washed out, the server goes for repair/set up and the system moves to phase 0. During the repair time, customers join the system, become impatient and leave the system. The impatience time is exponentially distributed with mean $\frac{1}{\xi}$. Immediately after the repair, the server is ready for offering service in phase i with probability $q_k, k = 1, 2, \cdots, N$. In the $k-$level of the environment, customers arrive according to a Poisson process with rate λ_k and the service time is exponential with mean $1/\mu_k$. Explicit expressions for time-dependent state probabilities are found and the corresponding steady-state probabilities are deduced. Some new performance measures are also obtained. Choosing arbitrary values of the parameters subject to the stability condition, the behaviour of the system is examined. For the chosen values of the parameters, the performance measures indicated that the system did not exhibit much deviation by the presence of several phases of the environment.

Keywords Single Server Queue, Random Environment, Disasters, Repair/set up, Impatience

1 Introduction

Queueing systems subject to randomly occurring disasters have been studied by several authors (see, Sengupta[11], Yechiali[15], Chakravarthy[4], Krishna Kumar et al.[7], Sudhesh[12], Paz and Yechiali[8], Udayabaskaran and Dora Pravina[13], Kim and Kim[6], Ammar et al.[2]). Queueing systems operating in random environment arise in telecommunication systems and industrial engineering. In these type of systems, the servers may be compulsorily recalled and assigned to serve with a different rate. For example, in a public transport corporation which operates long route buses, a driver in one route bus is given rest after driving for several hours and is assigned to drive in another route bus after some time without causing inconvenience to the passengers. As another example, in mobile communication network, a mobile moving in a complex of connecting towers may have to switch over from one tower to another tower and in the process, it is forced to get disconnected from communication for a positive duration of time. Paz and Yechiali[8] have analysed the steady-state behaviour of an $M/M/1$ queue operating in random environment subject to disasters where the underlying environment is described by a $n-$phase continuous-time Markov chain. However, it has been observed (see van Doorn[14] and Whitt[16]) that time-dependent analysis is needed in several applications of queueing theory in communication systems and storage processes. Udayabaskaran and Dora Pravina[13] have obtained time-dependent probabilities for the queueing model of Paz and Yechiali[8]. Recently, Ammar et al.[2] have extended the queueing model of Paz and Yechiali[8] by incorporating customer impatience during repair time, and obtained transient solution for the state probabilities of the system by using generating function technique and continued fraction approach. Queueing systems with set up times for servers have been studied recently by Phung-Duc[9] and Karunakaran and Maragatha Sundari[5]. At the end of the maintenance, the server may have to work with different service rates. In the present paper, we obtain the time-dependent behaviour of of a single server queueing system with set up time operating in a random environment by a new approach without using continued fractions and also obtain several measures of system performance.

The paper is organized as follows: Section 2 describes the model. Section 3 derives the integral equations for the time-dependent probabilities of the system. In section 4, explicit expressions for the transient probabilities are obtained. Section 5 deduces the steady-state probabilities. In section 6, we obtain performance measures. Section 7 provides a numerical illustration for the performance measures. Section 8 contains conclusion of the paper.

2 Model description

We consider a single server queueing system operating in a random environment. We assume that the environment is in any one of the $N+1$ states $0, 1, 2, \cdots, N$. The environmental state 0 corresponds to the state that the server is undergoing repair. The repair time is random which is exponentially distributed with mean $1/\eta_0$. During the repair time, customers join the system according to Poisson process with rate λ_0. Immediately after the repair, the system moves to phase $k, k \geq 1$ with probability q_k, and $\sum_{k=1}^{N} q_k = 1$. The system resides in phase k for a random interval of time which is exponentially distributed with mean $1/\eta_k$ and at the end of the residing period (that is, at the occurrence of a disaster in phase k), all customers in the system are washed out and the system moves to phase 0. When the environment is in phase $k \geq 1$, the system behaves like an $M(\lambda_k)/M(\mu_k)/1$ queue with arrival rate λ_k and service rate μ_k. During repair time, waiting customers become impatient independently and leave the system after a random time which follows exponential distribution with mean $\frac{1}{\xi}$.

3 Governing equations

At time $t = 0$, we assume that a catastrophe has just occurred. Let $E(t)$ be the phase of the environment at time t and let $X(t)$ denote the number of customers in the queueing system at time t. Then the two-dimensional stochastic process $\{(X(t), E(t)), t \geq 0\}$ is Markov. We define the state probabilities as follows:

$$p(j, k, t) = P[X(t) = j, E(t) = k | X(0) = 0, E(0) = 0],$$

$$j = 0, 1, \cdots; k = 0, 1, \cdots, N. \tag{3.1}$$

Denoting the convolution $\int_0^t f(u)g(t-u)du$ as $f(t)©g(t)$, and using the imbedding technique of Bellman et al. [3], we obtain the following governing equations:

$$p(0,0,t) = e^{-(\lambda_0+\eta_0)t} + \sum_{k=1}^{N}\sum_{j=0}^{\infty} \eta_k p(j,k,t)©e^{-(\lambda_0+\eta_0)t}$$

$$+\xi p(1,0,t)©e^{-(\lambda_0+\eta_0)t}; \tag{3.2}$$

$$p(j,0,t) = p(j-1,0,t)\lambda_0©e^{-(\lambda_0+\eta_0+j\xi)t}$$
$$+(j+1)\xi p(j+1,0,t)©e^{-(\lambda_0+\eta_0+j\xi)t}, j = 1,2,\cdots; \tag{3.3}$$

$$p(0,k,t) = \eta_0 q_k p(0,0,t)©e^{-(\lambda_k+\eta_k)t}$$
$$+\mu_k p(1,k,t)©e^{-(\lambda_k+\eta_k)t}, k = 1,2,\cdots,N; \tag{3.4}$$

$$p(j,k,t) = \eta_0 q_k p(j,0,t)©e^{-(\lambda_k+\mu_k+\eta_k)t}$$
$$+\lambda_k p(j-1,k,t)©e^{-(\lambda_k+\mu_k+\eta_k)t}$$
$$+\mu_k p(j+1,k,t)©e^{-(\lambda_k+\mu_k+\eta_k)t},$$
$$j = 1,2,\cdots; k = 1,2,\cdots,N. \tag{3.5}$$

To derive (3.2), consider the mutually exclusive and exhaustive cases:

(i) No event has occurred up to time t. For this, the probability is

$$e^{-(\lambda_0+\eta_0)t}.$$

(ii) At an intermediate time $u, 0 < u < t$, the system occupied the phase $k, k = 1, 2, \cdots, N$ and a disaster occurred in $(u, u + du)$ and thereafter no event occurred up to time t. For this, the probability is

$$\sum_{k=1}^{N}\sum_{j=0}^{\infty} \eta_k p(j,k,t)©e^{-(\lambda_0+\eta_0)t}.$$

(iii) At an intermediate time $u, 0 < u < t$, the server is in repair phase 0 and there was only one customer in the queue and that customer's impatience timer clicked to leave in the elementary interval $(u, u+du)$ and thereafter no event occurred up to time t. For this, the probability is

$$\xi p(1,0,t)©e^{-(\lambda_0+\eta_0)t}.$$

Like-wise, other governing equations $(3.3) - (3.5)$ are obtained.

4 Transient probabilities

In this section, we obtain explicit expressions for the time-dependent probabilities $p(j, k, t), j = 0, 1, 2, \cdots; k = 0, 1, 2, \cdots, N$. Denoting the Laplace transform of $p(j, k, t)$ by $p^*(j, k, s)$, and taking Laplace transform on both sides of $(3.2) - (3.5)$, we obtain

$$(s+\lambda_0+\eta_0)p^*(0,0,s) = 1 + \xi p^*(1,0,s)$$
$$+\sum_{k=1}^{N}\sum_{j=0}^{\infty} p^*(j,k,s)\eta_k, \quad .(4.1)$$

$$(s+\lambda_0+\eta_0+j\xi)p^*(j,0,s)=p^*(j-1,0,s)\lambda_0$$
$$+(j+1)\xi p^*(j+1,0,s), j=1,2,\cdots; \qquad .(4.2)$$

$$(s+\lambda_k+\eta_k)p^*(0,k,s)=p^*(0,0,s)\eta_0 q_k+p^*(1,k,s)\mu_k,$$
$$k=1,2,\cdots,N. \qquad .(4.3)$$
$$(s+\lambda_k+\mu_k+\eta_k)p^*(j,k,s)=p^*(j,0,s)\eta_0 q_k$$
$$+p^*(j-1,k,s)\lambda_k+p^*(j+1,k,s)\mu_k,$$
$$j=1,2,\cdots; k=1,2,\cdots,N. \qquad .(4.4)$$

To solve $(4.1)-(4.4)$, we define

$$G_k^*(u,s)=\sum_{j=0}^{\infty}p^*(j,k,s)u^j, k=0,1,2,\cdots,N.$$

Using (4.2), we get

$$\frac{\partial}{\partial u}G_0^*(u,s)-\left[\frac{s+\eta_0}{\xi(1-u)}+\frac{\lambda_0}{\xi}\right]G_0^*(u,s)=-\frac{K(s)}{\xi(1-u)}, \qquad (4.5)$$

where

$$K(s)=(s+\lambda_0+\eta_0)p^*(0,0,s)-\xi p^*(1,0,s). \qquad (4.6)$$

Solving (4.5), we obtain

$$G_0^*(u,s)=e^{\frac{\lambda_0}{\xi}u}(1-u)^{-(s+\eta_0)/\xi}\times$$
$$\left[p^*(0,0,s)-\frac{K(s)}{\xi}\int_0^u e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv\right]. \qquad (4.7)$$

As $G_0^*(1,s)$ is analytic in $|s|<1$, we get

$$K(s)=\frac{\xi p^*(0,0,s)}{\int_0^1 e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv}. \qquad (4.8)$$

Substituting (4.8) into (4.7), we get

$$G_0^*(u,s)=\frac{e^{\frac{\lambda_0}{\xi}u}}{(1-u)^{(s+\eta_0)/\xi}}\times$$
$$\left[1-\frac{\int_0^u e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv}{\int_0^1 e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv}\right]p^*(0,0,s), u\neq 1. \qquad (4.9)$$

Taking limit as $u\to 1$ on both sides of (4.9), and applying L'Hospital rule, we get

$$G_0^*(1,s)=\frac{\xi}{(s+\eta_0)\int_0^1 e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv}p^*(0,0,s). \qquad (4.10)$$

Using (4.4) along with (4.3), we get

$$G_k^*(u,s)=\frac{\mu_k(u-1)p^*(0,k,s)+\eta_0 q_k u G_0^*(u,s)}{(s+\lambda_k+\mu_k+\eta_k)u-\lambda_k u^2-\mu_k}, \quad (4.11)$$

where $k=1,2,\cdots,N$. As $G_k^*(u,s), k=1,2,\cdots,N$ is analytic in $|u|<1$, the numerator of $G_k^*(u,s)$ must vanish wherever the denominator of $G_k^*(u,s)$ vanishes in the region $|u|<1$. As a quadratic in u, the denominator of $G_k^*(u,s)$ has two roots

$$u_{k1}(s)=\frac{1}{2\lambda_k}\left[(s+\lambda_k+\eta_k+\mu_k)\right.$$
$$\left.-\sqrt{(s+\lambda_k+\eta_k+\mu_k)^2-4\lambda_k\mu_k}\right], \qquad (4.12)$$
$$u_{k2}(s)=\frac{1}{2\lambda_k}\left[(s+\lambda_k+\eta_k+\mu_k)\right.$$
$$\left.+\sqrt{(s+\lambda_k+\eta_k+\mu_k)^2-4\lambda_k\mu_k}\right]. \qquad (4.13)$$

For brevity, we shall denote these two roots simply as u_{k1} and u_{k2}. Assuming the condition $\mu_k+\eta_k>\lambda_k, k=1,2,\cdots,N$, it is seen that $|u_{k1}|<1$ and $|u_{k2}|>1$. So, the numerator of $G_k*(u,s)$ must vanish at u_{k1}. This leads to the equation

$$p^*(0,k,s)=\frac{\eta_0 q_k u_{k1}}{\mu_k(1-u_{k1})}G_0^*(u_{k1},s), \qquad (4.14)$$

where $k=1,2,\cdots,N$. Substituting (4.14) into (4.11) and expanding, we get

$$G_k^*(u,s)=\frac{\eta_0 q_k}{\mu_k}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}p^*(m,0,s)u_{k1}^{l+m+1}$$
$$+\sum_{j=1}^{\infty}\left[\frac{\eta_0\lambda_k^j q_k}{\mu_k^{j+1}}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}p^*(m,0,s)u_{k1}^{j+l+m+1}\right.$$
$$\left.+\frac{\eta_0 q_k}{\mu_k}\sum_{n=1}^{j}\sum_{r=0}^{\infty}p^*(n+r,0,s)\frac{\lambda_k^{j-n}u_{k1}^{j-n+r+1}}{\mu_k^{j-n}}\right]u^j, \quad (4.15)$$

where $k=1,2,\cdots,N$. Comparing u^0 in (4.15), we get

$$p^*(0,k,s)=\frac{\eta_0 q_k}{\mu_k}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}p^*(m,0,s)u_{k1}^{l+m+1}, \qquad (4.16)$$

where $k=1,2,\cdots,N$. Comparing $u^j, j\geq 1$ in (4.15), we get

$$p^*(j,k,s)=\frac{\eta_0 q_k}{\mu_k}\left[\frac{\lambda_k^j}{\mu_k^j}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}p^*(m,0,s)u_{k1}^{j+l+m+1}\right.$$
$$\left.+\sum_{n=1}^{j}\sum_{r=0}^{\infty}p^*(n+r,0,s)\frac{\lambda_k^{j-n}u_{k1}^{j-n+r+1}}{\mu_k^{j-n}}\right], \qquad (4.17)$$

where $j\geq 1, k=1,2,\cdots,N$. By total probability axiom, we get

$$\sum_{k=0}^{N}\sum_{j=0}^{\infty}p^*(j,k,s)=\frac{1}{s}. \qquad (4.18)$$

Equation (4.18) is rewritten in the following form:

$$G_0^*(1,s)+\sum_{k=1}^{N}G_k^*(1,s)=\frac{1}{s}. \qquad (4.19)$$

Replacing u by 1 in equation (4.11), we get

$$G_k^*(1,s) = \frac{\eta_0 q_k G_0^*(1,s)}{(s+\eta_k)}, k = 1, 2, \cdots, N. \quad (4.20)$$

Substituting (4.20) in (4.19), we get

$$G_0^*(1,s) = \frac{1}{s\left(1+\eta_0 \sum_{k=1}^{N} \frac{q_k}{s+\eta_k}\right)}.$$

Equating (4.10) and (4.21), we get

$$p^*(0,0,s) = \frac{(s+\eta_0)\int_0^1 e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv}{\xi s\left(1+\eta_0 \sum_{k=1}^{N} \frac{q_k}{s+\eta_k}\right)}. \quad (4.22)$$

Taking inverse Laplace transform on both sides of (4.22), we can obtain $p(0,0,t)$. We adopt a novel method to invert (4.22). We put

$$I_\theta = \int_0^1 e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\theta-1}dv, \quad (4.23)$$

where $\theta = \frac{(s+\eta_0)}{\xi}$. From (4.23), we obtain

$$I_\theta = \frac{1}{\theta}\left[1 - \frac{\lambda_0}{\xi} I_{\theta+1}\right]. \quad (4.24)$$

By Iteration, equation (4.24) yields

$$I_\theta = \frac{1}{\theta}\left[1 + \sum_{i=1}^{r}\left(\frac{\lambda_0}{\xi}\right)^i \frac{(-1)^i}{\prod_{j=1}^{i}(\theta+j)} + (-1)^{r+1}\frac{\left(\frac{\lambda_0}{\xi}\right)^{r+1}}{\prod_{j=1}^{i}(\theta+j)} I_{\theta+r+1}\right].$$

Taking limit as $r \to \infty$ and assuming $\lambda_0 < \xi$ for convergence, we obtain

$$I_\theta = \frac{\xi}{(s+\eta_0)}\left[1 + \sum_{i=1}^{\infty}\frac{(-1)^i \lambda_0^i}{\prod_{j=1}^{i}(s+\eta_0+j\xi)}\right]. \quad (4.26)$$

Substituting (4.26) into (4.22), we get

$$p^*(0,0,s) = \frac{1}{s\left(1+\eta_0 \sum_{k=1}^{N} \frac{q_k}{s+\eta_k}\right)} \times \left[1 + \sum_{i=1}^{\infty}\frac{(-1)^i \lambda_0^i}{\prod_{j=1}^{i}(s+\eta_0+j\xi)}\right]. \quad (4.27)$$

By expansion, (4.27) yields

$$p^*(0,0,s) = F_1^*(s) F_2^*(s), \quad (4.28)$$

where

$$F_1^*(s) = \sum_{l=0}^{\infty}(-\eta_0)^l \left(\sum_{k=1}^{N}\frac{q_k}{s+\eta_k}\right)^l, \quad (4.29)$$

$$F_2^*(s) = \frac{1}{s} + \frac{1}{s}\sum_{i=1}^{\infty}\frac{(-1)^i \lambda_0^i}{\prod_{j=1}^{i}(s+\eta_0+j\xi)}. \quad (4.30)$$

Inverting (4.29), we get

$$F_1(t) = L^{-1}\left[F_1^*(s)\right] = \sum_{l=0}^{\infty}(-\eta_0)^l \phi^{(l)}(t), \quad (4.31)$$

where $\phi^{(l)}(t)$ is the $l-$fold convolution of

$$\phi(t) = \sum_{k=1}^{N} q_k e^{-\eta_k t}. \quad (4.32)$$

Inverting (4.30), we get

$$F_2(t) = L^{-1}\left[F_2^*(s)\right] = 1 + \sum_{i=1}^{\infty}(-1)^i \lambda_0^i \psi_i(t), \quad (4.33)$$

where

$$\psi_i(t) = 1 © e^{-(\eta_0+\xi)t} © e^{-(\eta_0+2\xi)t} © \cdots © e^{-(\eta_0+i\xi)t}. \quad (4.34)$$

Now, inverting equation (4.28), we get explicitly

$$p(0,0,t) = F_1(t) © F_2(t). \quad (4.35)$$

We next proceed to compute $p(j,0,t), j = 1, 2, \cdots$. Defining $\Phi(u,s) = \frac{G_0^*(u,s)}{p^*(0,0,s)}$, equation (4.9) gives

$$\Phi(u,s) = \frac{e^{\frac{\lambda_0}{\xi}u}}{(1-u)^{(s+\eta_0)/\xi}} \times \left[1 - \frac{\int_0^u e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv}{\int_0^1 e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv}\right], u \neq 1. \quad (4.36)$$

By (4.10), we get

$$\Phi(1,s) = \frac{\xi}{(s+\eta_0)\int_0^1 e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv}. \quad (4.37)$$

Taking logarithm on both sides of (4.36), and differentiating with respect to u, we get

$$(1-u)\Phi'(u,s) = \left[\frac{\lambda_0}{\xi}(1-u) + \left(\frac{s+\eta_0}{\xi}\right)\right]\Phi(u,s) - \frac{1}{\int_0^1 e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv}. \quad (4.38)$$

Differentiating (4.38) again with respect to u, we get

$$(1-u)\Phi''(u,s) - \left[1 + \frac{\lambda_0}{\xi}(1-u) + \left(\frac{s+\eta_0}{\xi}\right)\right]\Phi'(u,s) + \frac{\lambda_0}{\xi}\Phi(u,s) = 0. \quad (4.39)$$

The point $u = 1$ is a regular singular point of the differential equation (4.39). We obtain a series solution for $\Phi(u,s)$ in the neighbourhood of $u = 1$. For this, we denote the $n-$th order derivative of $\Phi(u,s)$ with respect to u as $\Phi_n(u,s)$. Equations (4.38) and (4.39) yield

$$\Phi(1,s)=\frac{\xi}{(s+\eta_0)\int_0^1 e^{-\frac{\lambda_0}{\xi}v}(1-v)^{\frac{(s+\eta_0)}{\xi}-1}dv}, \quad (4.40)$$

$$\Phi_1(1,s)=\frac{\lambda_0}{(s+\eta_0+\xi)}\Phi(1,s). \quad (4.41)$$

Differentiating both sides of (4.39) n times with respect to u and applying Leibnitz's rule of successive differentiation, we get

$$(1-u)\Phi_{n+2}(u,s)-\left[(n+1)+\frac{\lambda_0}{\xi}(1-u)+\left(\frac{s+\eta_0}{\xi}\right)\right]\Phi_{n+1}(u,s)+\frac{\lambda_0}{\xi}(n+1)\Phi_n(u,s)=0, n=1,2,\cdots. \quad (4.42)$$

In the limit $u\to 1$, (4.42) yields

$$\Phi_{n+1}(1,s)=\frac{(n+1)\lambda_0}{s+\eta_0+(n+1)\xi}\Phi_n(1,s), n=1,2,\cdots. \quad (4.43)$$

By iteration, equation (4.43) leads to

$$\Phi_n(1,s)=\frac{n!\lambda_0^n}{\Pi_{j=1}^n(s+\eta_0+j\xi)}\Phi(1,s), n=2,3,\cdots. \quad (4.44)$$

Expanding $\Phi(u,s)$ about $u=1$, we get

$$\Phi(u,s)=\Phi(1,s)+\sum_{n=1}^{\infty}\frac{\Phi_n(1,s)}{n!}(u-1)^n. \quad (4.45)$$

Substituting (4.41) and (4.44) into (4.45), we get

$$\Phi(u,s)=\left[1+\sum_{n=1}^{\infty}\frac{\lambda_0^n(u-1)^n}{\Pi_{j=1}^n(s+\eta_0+j\xi)}\right]\Phi(1,s). \quad (4.46)$$

Using (4.40) in (4.46) and simplifying, we get

$$\sum_{j=0}^{\infty}p^*(j,0,s)u^j=\frac{1}{s\left(1+\eta_0\sum_{k=1}^N\frac{q_k}{s+\eta_k}\right)}\times \left[1+\sum_{n=1}^{\infty}(-\lambda_0)^n\Pi_{r=1}^n\left(\frac{1}{s+\eta_0+r\xi}\right)+\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\Pi_{r=1}^n\left(\frac{\lambda_0}{s+\eta_0+r\xi}\right)\binom{n}{j}(-1)^{n-j}u^j\right]. \quad (4.47)$$

Equating the coefficient of u^0 on both sides of (4.47), we get

$$p^*(0,0,s)=\frac{1}{s\left(1+\eta_0\sum_{k=1}^N\frac{q_k}{s+\eta_k}\right)}\times\left[1+\sum_{n=1}^{\infty}(-\lambda_0)^n\Pi_{r=1}^n\left(\frac{1}{s+\eta_0+r\xi}\right)\right]. \quad (4.48)$$

Equating the coefficient of $u^j, j=1,2,\cdots$ on both sides of (4.47), we get

$$p^*(j,0,s)=\frac{1}{s\left(1+\eta_0\sum_{k=1}^N\frac{q_k}{s+\eta_k}\right)}\times \sum_{m=0}^{\infty}(-1)^m\binom{m+j}{j}\Pi_{r=1}^{m+j}\left(\frac{\lambda_0}{s+\eta_0+r\xi}\right). \quad (4.49)$$

Equation (4.48) is same as (4.28). We have already obtained $p(0,0,t)$ in (4.35). Now, inverting (4.49), we get

$$p(j,0,t)=F_1(t)©\sum_{m=0}^{\infty}(-1)^m\binom{m+j}{j}\lambda_0^{m+j}\psi_{m+j}(t), \quad j=1,2,\cdots. \quad (4.50)$$

It remains to evaluate the probabilities $p(j,k,t), j=0,1,2,\cdots; k=1,2,\cdots,N$. By referring to Abramowitz and Stegun [1] and Schiff [10], we obtain

$$L^{-1}\left[u_{k1}^{\nu}(s)\right]=\frac{e^{-\alpha_k t}}{2^{\nu}\lambda_k^{\nu}}\frac{\nu a_k^{\nu}}{t}I_{\nu}(a_k t), \quad (4.51)$$

where $\alpha_k=\lambda_k+\mu_k+\eta_k$, $a_k^2=4\lambda_k\mu_k$, and $I_\nu(x)$ is the modified Bessel function of the I kind. For notational simplicity, we denote $L^{-1}\left[u_{k1}^{\nu}(s)\right]$ by $K_{k,\nu}(t)$. Consequently, taking inverse Laplace transform on both sides of equations (4.16) and (4.17), we get

$$p(0,k,t)=\frac{\eta_0 q_k}{\mu_k}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}p(m,0,t)©K_{k,l+m+1}(t), \quad k=1,2,\cdots,N; \quad (4.52)$$

$$p(j,k,t)=\frac{\eta_0 q_k}{\mu_k}\left[\frac{\lambda_k^j}{\mu_k^j}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}p(m,0,t)©K_{k,j+l+m+1}(t)+\sum_{n=1}^{j}\sum_{r=0}^{\infty}p(n+r,0,t)©\frac{\lambda_k^{j-n}K_{k,j-n+r+1}(t)}{\mu_k^{j-n}}\right], \quad j\geq 1, k=1,2,\cdots,N. \quad (4.53)$$

5 Steady-state probabilities

We define the steady-state probabilities as follows:

$$\pi(j,k)=\lim_{t\to\infty}p(j,k,t), \quad j=0,1,2,\cdots; k=0,1,2,\cdots,N. \quad (5.1)$$

By the final value theorem of Laplace transform, we have

$$\pi(j,k)=\lim_{s\to 0}s\times p^*(j,k,s), \quad j=0,1,2,\cdots; k=0,1,2,\cdots,N. \quad (5.2)$$

We assume that $\lambda_k < \mu_k, k = 1, 2, \cdots, N$. Then, it can be seen that steady-state distribution exists. Consequently, by using equations (4.35), (4.50), (4.52) and (4.53), we obtain

$$\pi(0,0) = \frac{1}{\left(1+\eta_0 \sum_{k=1}^{N} \frac{q_k}{\eta_k}\right)} \left[1 + \sum_{i=1}^{\infty} \frac{(-1)^i \lambda_0^i}{\prod_{r=1}^{i}(\eta_0 + r\xi)}\right], \tag{5.3}$$

$$\pi(j,0) = \frac{1}{\left(1+\eta_0 \sum_{k=1}^{N} \frac{q_k}{\eta_k}\right)} \times \sum_{m=0}^{\infty} (-1)^m \binom{m+j}{m} \Pi_{r=1}^{m+j} \left(\frac{\lambda_0}{\eta_0 + r\xi}\right), \quad j = 1, 2, \cdots, \tag{5.4}$$

$$\pi(0,k) = \frac{\eta_0 q_k}{\mu_k(1-\beta_k)} \sum_{m=0}^{\infty} \beta_k^{m+1} \pi(m,0), k = 1, 2, \cdots, N, \tag{5.5}$$

$$\pi(j,k) = \frac{\eta_0 q_k}{\mu_k} \left[\frac{\lambda_k^j}{\mu_k^j (1-\beta_k)} \sum_{m=0}^{\infty} \pi(m,0) \beta_k^{j+m+1} + \sum_{n=1}^{j} \sum_{r=0}^{\infty} \pi(n+r,0) \frac{\lambda_k^{j-n} \beta_k^{j-n+r+1}}{\mu_k^{j-n}}\right], \quad j = 1, 2, \cdots; k = 1, 2, \cdots, N. \tag{5.6}$$

where

$$\beta_k = \frac{(\lambda_k + \eta_k + \mu_k) - \sqrt{(\lambda_k + \eta_k + \mu_k)^2 - 4\lambda_k \mu_k}}{2\lambda_k}. \tag{5.7}$$

6 Performance measures

6.1 Mean queue size in $k-$th phase

Let $E[L_k]$ denote the stationary mean number of customers in the $k-$phase. Then, we have

$$E[L_k] = \sum_{j=1}^{\infty} j\pi(j,k), k = 0, 1, 2, \cdots, N. \tag{6.1}$$

Using (5.4), we obtain

$$E[L_0] = \frac{1}{\left(1+\eta_0 \sum_{k=1}^{N} \frac{q_k}{\eta_k}\right)} \times \sum_{m=0}^{\infty} (-1)^m \sum_{j=1}^{\infty} j \binom{m+j}{m} \Pi_{r=1}^{m+j} \left(\frac{\lambda_0}{\eta_0 + r\xi}\right). \tag{6.2}$$

Using (5.6), we get

$$E[L_k] = \frac{\eta_0 q_k \beta_k}{1-\beta_k} \sum_{j=1}^{\infty} j \times \left[\sum_{r=0}^{j} \sum_{m=0}^{\infty} \frac{\lambda^{j-r}}{\mu_k^{j-k+1}} \beta_k^{j-r+m} \pi(m+r,0) - \sum_{r=1}^{j} \sum_{m=1}^{\infty} \frac{\lambda^{j-r}}{\mu_k^{j-k+1}} \beta_k^{j-r+m} \pi(m+r-1,0)\right], \quad k = 1, 2, \cdots, N. \tag{6.3}$$

Using (6.2) and (6.3), we get the stationary mean of total number $L_T = \sum_{k=0}^{N} L_k$ of customers in the system

$$E[L_T] = \frac{1}{\left(1+\eta_0 \sum_{k=1}^{N} \frac{q_k}{\eta_k}\right)} \times \sum_{m=0}^{\infty} (-1)^m \sum_{j=1}^{\infty} j \binom{m+j}{m} \Pi_{r=1}^{m+j} \left(\frac{\lambda_0}{\eta_0 + r\xi}\right) + \sum_{k=1}^{N} \frac{\eta_0 q_k \beta_k}{1-\beta_k} \sum_{j=1}^{\infty} j \times \left[\sum_{r=0}^{j} \sum_{m=0}^{\infty} \frac{\lambda^{j-r}}{\mu_k^{j-k+1}} \beta_k^{j-r+m} \pi(m+r,0) - \sum_{r=1}^{j} \sum_{m=1}^{\infty} \frac{\lambda^{j-r}}{\mu_k^{j-k+1}} \beta_k^{j-r+m} \pi(m+r-1,0)\right]. \tag{6.4}$$

6.2 Mean number of customers washed out by disasters per unit time from the system

If C denotes the number of customers cleared from the system per unit time, then

$$E[C] = \sum_{k=1}^{N} \eta_k \sum_{j=1}^{\infty} j\pi(j,k) = \sum_{k=1}^{N} \eta_k E[L_k]. \tag{6.5}$$

6.3 Mean number of customers left by impatience per unit time from the system

If D denote the number of customers left from the system due to impatience per unit time, then

$$E[D] = \xi \sum_{j=1}^{\infty} j\pi(j,0) = \xi E[L_0]. \tag{6.6}$$

6.4 Fraction of customers receiving full service

The mean number of customers cleared from the system per unit time due to impatience or disaster is given by $E[C]+E[D]$. So, the fraction of customers receiving full service is given by

$$\frac{\lambda_0 - E[D]}{\lambda_0} + \sum_{k=1}^{N} \frac{\lambda_k - \eta_k E[L_k]}{\lambda_k} = (N+1) - \frac{E[D]}{\lambda_0} - \sum_{k=1}^{N} \frac{E[\eta_k L_k]}{\lambda_k}. \tag{6.8}$$

6.5 Mean Sojourn time of a virtual customer arriving when the system is in the state $(0,0)$

Following Paz and Yechiali[8], let $W_{0,0}$ denote the sojourn time of a customer that arrives to the system when there are no customers in the system and server is under repair. Two mutually exclusive and exhaustive cases arise, namely (i) the repair is not completed and the virtual customer leaves the system due to impatience; or (ii) the virtual customer does not become impatient till the repair completion and the server enters into phase $k, k=1,2,\cdots,N$ and in this phase the virtual customer either gets service completed or washed out by the occurrence of a disaster. Consequently, we get

$$E\left[W_{0,0}\right]=\int_0^{\infty} u e^{-(\eta_0+\xi)u}\xi du+\int_0^{\infty} u\int_0^{u} e^{-(\eta_0+\xi)v}\times$$
$$\sum_{k=1}^{N}\eta_0 q_k e^{-(\eta_k+\mu_k)(u-v)}(\eta_k+\mu_k)dvdu$$
$$=\frac{\xi}{(\eta_0+\xi)^2}+\eta_0\sum_{k=1}^{N}\frac{q_k(\eta_k+\mu_k+\eta_0+\xi)}{(\eta_0+\xi)^2(\eta_k+\mu_k)}. \tag{6.8}$$

6.6 Mean Sojourn time of a virtual customer arriving when the system is in the state $(j,0), j\geq 1$

Let $W_{j,0}$ denote the sojourn time of a virtual customer that arrives to the system when there are $j, j\geq 1$ customers in the system and server is under repair. Two mutually exclusive and exhaustive cases arise, namely (i) the repair is not completed and the virtual customer leaves the system due to impatience; or (ii) the virtual customer does not become impatient till the repair completion and the server enters into phase k, $k=1,2,\cdots,N$ with $j-r+1$ customers $(0\leq r\leq j)$ and in this phase the virtual customer either gets service completed or washed out by the occurrence of a disaster. Consequently, we get

$$E\left[W_{j,0}\right]=\int_0^{\infty} u e^{-(\eta_0+\xi)u}\xi du$$
$$+\int_0^{\infty} u\int_0^{u} e^{-\eta_0 v}\eta_0\sum_{k=1}^{N} q_k\sum_{r=1}^{j}\binom{j}{r}\left(1-e^{-\xi v}\right)^{r}\times$$
$$e^{-(j-r+1)\xi v}e^{-(\eta_k+\mu_k)(u-v)}\left(\frac{\{\mu_k(u-v)\}^{j-r}}{(j-r)!}\mu_k\right.$$
$$\left.+\sum_{n=0}^{j-r}\frac{\{\mu_k(u-v)\}^{n}}{n!}\eta_k\right)dvdu$$
$$=\frac{\xi}{(\eta_0+\xi)^2}+\sum_{k=1}^{N}\sum_{r=1}^{j}\sum_{l=0}^{r}\eta_0 q_k\binom{j}{r}\binom{r}{l}(-1)^{l}\frac{\mu_k^{j-r+1}}{(j-r)!}\times$$
$$\frac{(j-r)!}{(\beta-\alpha)^{j-r+1}}\left[\frac{1}{\alpha^2}-\sum_{i=0}^{j-r}\frac{(\beta-\alpha)^{i}(i+1)}{\beta^{i+2}}\right]$$
$$+\sum_{k=1}^{N}\sum_{r=1}^{j}\sum_{l=0}^{r}\sum_{n=0}^{j-r}\eta_0 q_k\binom{j}{r}\binom{r}{l}(-1)^{l}\frac{\mu_k^{n}\eta_k}{(j-r)!}\times$$
$$\frac{n!}{(\beta-\alpha)^{n+1}}\left[\frac{1}{\alpha^2}-\sum_{i=0}^{n}\frac{(\beta-\alpha)^{i}(i+1)}{\beta^{i+2}}\right], \tag{6.9}$$

where $\alpha(j,r,l)=\eta_0+(j-r+l+1)\xi, \beta(k)=\eta_k+\mu_k$.

6.7 Mean Sojourn time of a virtual customer arriving when the queueing system is in the state $(0,k), k=1,2,\cdots,N$

Let $W_{0,k}$ denote the sojourn time of a customer that arrives to the system when there are no customers in the system and server is in phase $k, k=1,2,\cdots,N$. Since there are no customers in the system before its arrival, the virtual customer either gets service completed or washed out by the occurrence of a disaster. Consequently, we get

$$E\left[W_{0,k}\right]=\int_0^{\infty} u e^{-(\eta_k+\mu_k)u}(\mu_k+\eta_k)du=\frac{1}{\eta_k+\mu_k}. \tag{6.10}$$

6.8 Mean Sojourn time of a virtual customer arriving when the system is in the state $(j,k), j=1,2,\cdots;k=1,2,\cdots,N$

Let $W_{j,k}$ denote the sojourn time of a customer that arrives to the system when there are already $j, j\geq 1$ customers in the system and the server is in phase $k, k=1,2,\cdots,N$. Then, $W_{j,k}$ is the time point at which either the $(j+1)-$th service is completed in the $k-$th phase without the occurrence of disaster, or a disaster occurs before the completion of the $(j+1)-$th service in phase k. Consequently, we obtain

$$E\left[W_{j,k}\right]=\int_0^{\infty} u e^{-(\eta_k+\mu_k)u}\frac{\mu_k^{j}u^{j}}{j!}\mu_k du$$
$$+\int_0^{\infty} u\left(1-e^{-\mu_k u}\frac{\mu_k^{j+1}u^{j+1}}{(j+1)!}\right)e^{-\eta_k u}\eta_k du$$
$$=\frac{(\eta_k+\mu_k)^{j+3}+(j+1)\mu_k^{j+2}\eta_k}{\eta_k(\eta_k+\mu_k)^{j+3}}. \tag{6.11}$$

7 A Numerical Illustration

Let there be 5 levels for the environment so that $N=5$. For the purpose of numerical illustration of the model, we have planned to choose impatience rate ξ and the repair rate both higher than the arrival rate λ_0. Accordingly, we fix $\lambda_0=5.0, \eta_0=6.0$, and $\xi=7.0$. As per the stability condition, we should have $\lambda_k<\mu_k, k=1,2,3,4,5$. So, we have taken λ_k, μ_k and η_k as in Table 1.

By (6.11), expressions for $E(W_{j,k}), j\neq 0, k=1,2,3,4,5$ do not depend on the distribution $\{q_k: k=1,2,3,4,5\}$. We have computed their values and plotted them against j in Figure 1. We observe in Figure 1 that the mean virtual time in each level of the environment increases as the number of customers in that level increases. This confirms the consistency of the model. In Figure 1, we also find that for a fixed j, $E(W_{j,k})$ decreases as the level k increases from 1 to 5. This is due to the fact that $\rho_k=\frac{\lambda_k}{\mu_k}$ increases as k increases from 1 to 5.

Table 1. Values of λ_k, μ_k and η_k

k	1	2	3	4	5
λ_k	10	11	12	13	14
μ_k	11	12	13	14	15
η_k	0.1	0.2	0.3	0.4	0.5

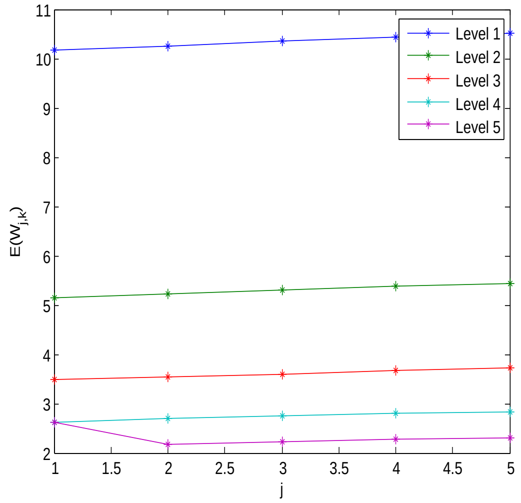

Figure 1. Variation of $E(W_{j,k}), j \neq 0$

The distribution for $\{q_k : k = 1, 2, 3, 4, 5\}$ is obtained by random number generation. We did 6 runs and listed distributions in Table 2.

For these runs, we have obtained the values of the other performance measures $E(L), E(C), E(D)$ and FT and plotted them against the 6 distributions in Figure 2. It is quite interesting and true to observe in Figure 2 that these values do not exhibit much dispersion. In fact, the coefficient of variation of $E(L_T), E(C), E(D)$ and FT are respectively 0.0524, 0.1417, 0.1864, and 0.0047. This confirms that the behaviour of the queueing system is governed by the values of λ_k, μ_k and η_k only.

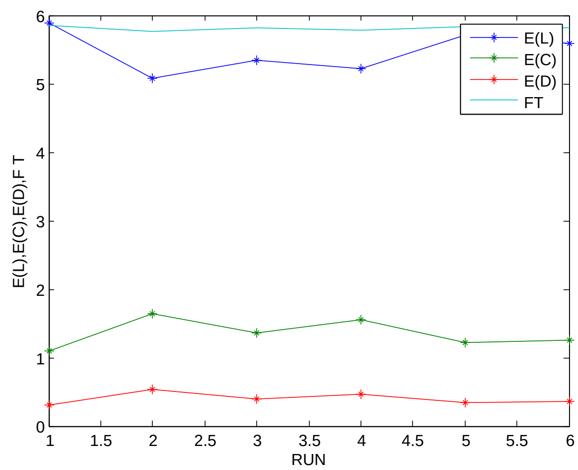

Figure 2. Dependence of $E(L), E(C), E(D)$ and FT on q_k

Next, we examine the behaviour of level dependent measures $E(L_r), r = 0, 1, 2, 3, 4, 5$. We have obtained the values of these measures for the 6 runs of $\{q_k\}$ and plotted them in Figure 3 against $\{q_k\}$. We find that the behaviour of each of $E(L_r), r = 0, 1, 2, 3, 4, 5$ depends on runs of $\{q_k\}$. This is quite expected since repair completion of the server and her subsequent movement to one of the five levels would affect the queue behaviour.

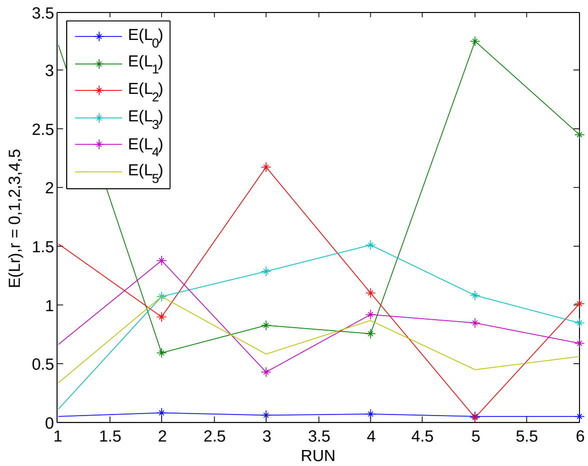

Figure 3. Dependence of $E(L_r), r = 0, 1, 2, 3, 4, 5$ on $\{q_k\}$

Finally, we examine the dependence of $E(W_{j,0}), j = 0, 1, 2, \cdots$ on $\{q_k\}$. We have found that, for any fixed j, $E(W_{j,0})$ does not exhibit much dispersion (see Figure 4). On the other hand, as a function of j, the expression $E(W_{j,0})$ behaves differently for different runs (see Figure 5). This is quite expected since the virtual waiting time of a customer joining the system when the server is in the down state would certainly depend not only on the customers in the system at that epoch but also on the time of repair completion and subsequent move of the server to one of the levels of the environment. This is also evident from the expression given in equation (6.9).

8 Conclusion

We considered a single server queueing system operating in a random environment subject to disaster, repair and customer impatience and obtained explicit expressions for some new performance measures of the system. We chose arbitrarily values of the parameters subject to the stability condition and examined the behaviour of the system. For the chosen values of the parameters, the performance measures indicated that the system did not exhibit much deviation by the presence of several phases of the environment.

Table 2. Distribution $\{q_k\}$

RUN	q_1	q_2	q_3	q_4	q_5
I	0.24	0.27	0.04	0.27	0.18
II	0.03	0.10	0.19	0.34	0.34
III	0.05	0.29	0.28	0.14	0.24
IV	0.04	0.13	0.28	0.25	0.30
V	0.21	0.01	0.27	0.3	0.21
VI	0.16	0.15	0.21	0.23	0.25

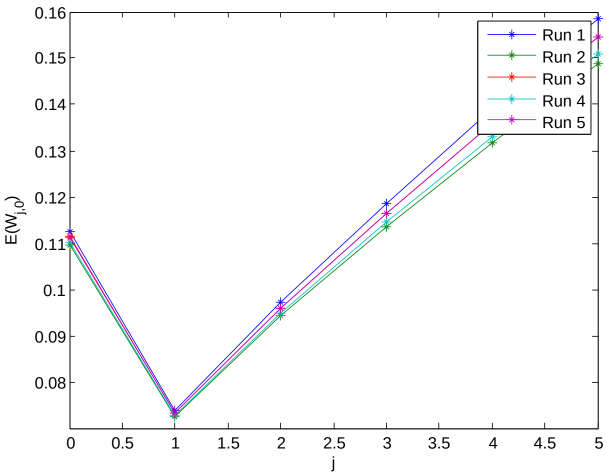

Figure 4. Dependence of $E(W_{j,0})$ on $\{q_k\}$ for fixed j

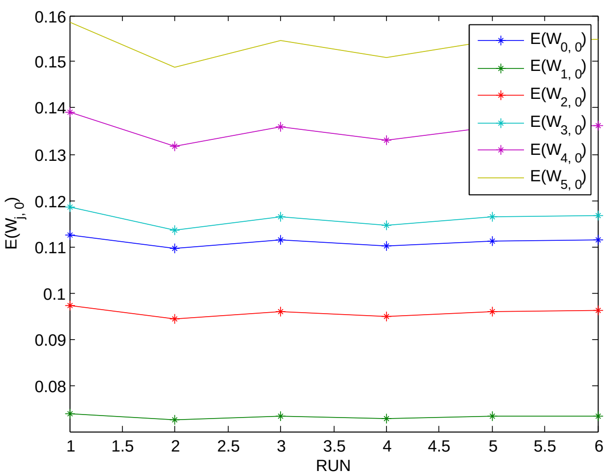

Figure 5. Dependence of $E(W_{j,0})$ on $\{q_k\}$

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