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A facet defining of the dicycle polytope

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Abstract In this paper, we consider the polytope $\mathcal{P}(G)$ of all elementary dicycles of a digraph G. Dicycles problem, in graph theory and combinatorial optimization, solved by polyhedral approaches has been extensively studied in literature. Therefore cutting plane and branch and cut algorithms are unavoidable to exactly solve such a combinatorial optimization problem. For this purpose, we introduce a new family of valid inequalities called $s-t$ alternating 3-arc path inequalities for the polytope of elementary dicycles $\mathcal{P}(G)$. Indeed, these inequalities can be used in cutting plane and branch and cut algorithms to construct strengthened relaxations of a linear formulation of the dicycle problem. To prove the facetness of $s-t$ alternating 3-arc path inequalities, in opposite to what is usually done that consists basically to determine the affine subspace of a linear description of the considered polytope, we resort to constructive algorithms. Given the set of arcs of the digraph G, algorithms devised and introduced are based on the fact that from a first elementary dicycle, all other dicycles are iteratively generated by replacing some arcs of previously generated dicycles by others such that the current elementary dicycle contains an arc that does not belong to any other previously generated dicycles. These algorithms generate dicyles with affinely independent incidence vectors that satisfy $s-t$ alternating 3-arc path inequalities with equality. It can easily be verified that all these devised algorithms are polynomial from time complexity point of view.

Keywords digraph, dicycle, valid inequality, facet, polytope.

1 Introduction

Let $G=(V,A)$ be a connected digraph with V as vertex set and A as arc set. We mean by dicycle a sequence $(v_0,a_1,\dots,a_k,v_k)$, where k is an integer, $v_0,v_1,\dots v_k$ are vertices such that $v_0=v_k$. For every index i, a_i is an arc connecting vertices v_{i-1} and v_i (where $i\in\{1,\dots k\}$) and, finally, all arcs a_i have the same direction. An *elementary dicycle* is a directed cycle $(v_0,a_1,\dots,a_k,v_k)$ in which each vertex v_i, for every index belonging to $\{0\dots k\}$, appears once.
We denote by $\mathcal{P}(G)$ the polytope of all *elementary dicycles* of G. That is, the convex hull of the set of incidence vectors of elementary dicycles of the digraph G. Thus,

$$\mathcal{P}(G) = conv\{x \in \{0,1\}^A : x \text{ is an incidence vector of an elementary dicycle}\}.$$

The polytope $\mathcal{P}(G)$ has been already studied by Balas and Oosten [1]. The authors present a linear description of the cycle polytope in digraphs. They study the facial structure of valid inequalities defining the polytope $\mathcal{P}(G)$. Balas and Stephan [2] consider the dominant of the polytope $\mathcal{P}(G)$ and derive other facet-defining inequalities for $\mathcal{P}(G)$. Hartmann et al. [8] provide a polyhedral analysis of the p-cycle polytope, which is the convex hull of incidence vectors of all the p-elementary dicycles with p arcs of the complete directed graph G.

In the case of undirected graphs, the cycle polytope has been studied by Coullard and Pulleyblank [6], and after by Bauer [3]. Kovalev et al. [15] and Bauer et al. [4] study the cardinality constrained cycle polytope which is the convex hull of all cycles with at most p nodes on a complete undirected graph. The p-cycle polytope has been also studied by Nguyen and Maurras [17,18] for $p=3$ and for $2<p<n$.

Note that cycles in graphs or digraphs play an important role in many applications. One of the most interesting and important applications has to do with testing circuits. A circuit can be modeled by a directed graph where the vertices represent gates (which compute boolean functions) and the arcs which represent the wires which connect gates (see, [16]. In literature one can find other applications of cycle problem in other areas as analysis of electrical networks, analysis of chemical and biological pathways, ... For some examples of cycle problem applications, we refer the reader to Serafini and Ukovich [19], Bollob`as and Kavitha et al. [5, 14].

In this paper, consider the complete digraph G and the polytope $\mathcal{P}(G)$ of all elementary directed cycles, we study the facial structure of a family of valid inequalities of the polytope $\mathcal{P}(G)$ called $s-t$ alternating 3-arc path inequalities. For this, we address several algorithms constructing elementary dicycles with affinely independent incidence vectors that satisfy a given $s-t$ alternating

3-arc path inequality with equality.

In section 2, we introduce what we call $s-t$ alternating 3-arc path inequalities for the polytope $\mathcal{P}(G)$. After, using constructive algorithms based on the approach that consists to generate elementary dicycles with affinely independent incidence vectors, we study the facetness of such valid inequalities.

In the rest of this section, we give further definitions and notations. Consider a loopless complete digraph $G = (V, A)$, with $V = \{v_1, v_2, \ldots, v_n\}$ and $A = \{a_1, a_2, \ldots, a_m\}$. n and m are vertex and arc numbers of G, respectively. As G is complete, we have $m = n(n-1)$. Let $\{v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(n)}\}$ be a permutation of elements of V, where τ is a bijection function from $[n]$ to $[n]$, with $[n] = \{1, 2, \ldots, n\}$. Given a sub-digraph, say, H, we denote by $A(H)$, its arc set. Particularly, an elementary directed cycle C has $A(C)$ as arc set. We mean by a minimal elementary dicycle C an elementary dicycle which has only two arcs. Given a vertex $v \in V$, let $\delta^-(v)$ and $\delta^+(v)$ be the set of arcs incoming to v and the set of arcs outgoing from v, respectively. We mean by incoming half degree and outgoing half degree, the numbers of arcs of $\delta^-(v)$ and $\delta^+(v)$, respectively. We recall from the definition of affine independence that vectors $\gamma_i,\ i = 1, \ldots, q$ are said affinely independent, if there exists some coefficients $\lambda_i,\ i = 1, \ldots, q$ such that the unique solution of systems $\sum_{i=1}^{q} \lambda_i \gamma_i = 0$ and $\sum_{i=1}^{q} \lambda_i = 0$ is $\lambda_i = 0,\ i = 1, \ldots, q$. In the sequel, we denote by $P_{s,t}^{u,v}$ a $s-t$ elementary dipath with $A(P_{s,t}^{u,v}) = \{(s,u), (u,v), (v,t)\}$ and by $P_{s,t}^{v}$ a $s-t$ elementary dipath with $A(P_{s,t}^{v}) = \{(s,v), (v,t)\}$.

2 A family of valid inequalities for the polytope $\mathcal{P}(G)$

2.1 $s-t$ *alternating 3-arc path inequalities*

Given a digraph $G = (V, A)$, we say that a directed subgraph denoted $P_{s,v_i,v_j,t}$ is a $s-t$ *alternating 3-arc path* if it consists to a directed path that passes by 3 arcs $(s, v_i), (v_i, v_j), (v_j, t)$ such that $x(P_{s,v_i,v_j,t}) = x(s, v_i) - x(v_i, v_j) + x(v_j, t)$.

Figure 1 shows a $s-t$ alternating 3-arc path. The dashed line represents the unique term of the $s-t$ *alternating 3-arc path* inequality that has -1 as coefficient. The other terms have $+1$ as coefficients. We say *alternating* as the coefficient of the terms that represent the arcs $(s, v_i), (v_i, v_j), (v_j, t)$ in the inequality are respectively $+1$, -1 and $+1$, respectively.

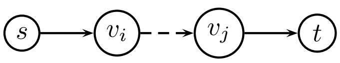

Figure 1: A $s-t$ *alternating 3-arc path*

Consider a permutation $\{v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(n)}\}$ of V. Define subgraphs P^{n+} and P^{n-} of G such that

- P^{n+} is a subgraph of G that spans n vertices. It contains the $v_{\tau(3)} - v_{\tau(n)}$ *alternating 3-arc path* $(v_{\tau(3)}, v_{\tau(2)}), (v_{\tau(2)}, v_{\tau(1)}), (v_{\tau(1)}, v_{\tau(n)})$, the arc $(v_{\tau(1)}, v_{\tau(2)})$ and all other arcs $(v_{\tau(2)}, v_{\tau(k)}) \in A$ outgoing from the vertex $v_{\tau(2)}$, with $k \in \{4, 5, \ldots, n-1\}$, (see Figure 2).
- Similarly, P^{n-} denotes a subgraph of G that spans n vertices. It contains the $v_{\tau(3)} - v_{\tau(n)}$ *alternating 3-arc path* $(v_{\tau(3)}, v_{\tau(2)}), (v_{\tau(2)}, v_{\tau(1)}), (v_{\tau(1)}, v_{\tau(n)})$, the arc $(v_{\tau(1)}, v_{\tau(2)})$ and all other arcs $(v_{\tau(k)}, v_{\tau(1)}) \in A$ incoming to the vertex $v_{\tau(1)}$, with $k \in \{4, 5, \ldots, n-1\}$, (see, Figure 3).

We have:

$$x(P^{n+}) = x(v_{\tau(3)}, v_{\tau(2)}) - x(v_{\tau(2)}, v_{\tau(1)}) - x(v_{\tau(2)}, v_{\tau(4)}) - x(v_{\tau(2)}, v_{\tau(5)}) - \ldots - x(v_{\tau(2)}, v_{\tau(n-2)}) - x(v_{\tau(2)}, v_{\tau(n-1)}) + x(v_{\tau(1)}, v_{\tau(2)}) + x(v_{\tau(1)}, v_{\tau(n)})$$

and

$$x(P^{n-}) = x(v_{\tau(3)}, v_{\tau(2)}) - x(v_{\tau(2)}, v_{\tau(1)}) - x(v_{\tau(4)}, v_{\tau(1)}) - x(v_{\tau(5)}, v_{\tau(1)}) \ \ldots - x(v_{\tau(n-2)}, v_{\tau(1)}) \ - x(v_{\tau(n-1)}, v_{\tau(1)}) + x(v_{\tau(1)}, v_{\tau(2)}) + x(v_{\tau(1)}, v_{\tau(n)}).$$

Terms with negative coefficients correspond to dashed lines in the subgraph supporting the inequality.

Subgraphs P^{n+} and P^{n-} are depicted in Figures 2 and 3, respectively.

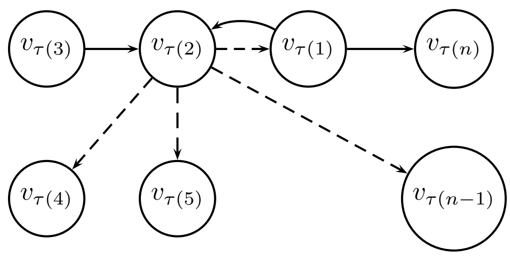

Figure 2: The subgraph P^{n+}

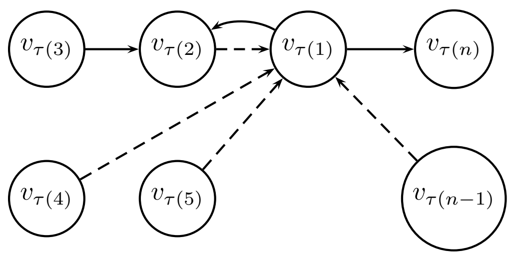

Figure 3: The subgraph P^{n-}

Theorem 1 *Following inequalities*

$$x(P^{n+}) \leq 1 \quad (1)$$

$$x(P^{n-}) \leq 1 \quad (2)$$

are valid for $\mathcal{P}(G)$.

Proof. Indeed, refering to subgraphs (Figures 2 and 3) that support inequalities (1) and (2), one can verify that there is no solution C (elementary dicycle) that intersects subgraphs P^{n+} and P^{n-} more than three arcs. Formally, this implies that $|A(C) \cap A(P^{n+})| \leq 3$ and $|A(C) \cap A(P^{n-})| \leq 3$. Without loss the generality, consider the subgraph P^{n+} and let show the validity of an inequality of type (1). We distinguish four cases.

- Case 1. If $|A(C) \cap A(P^{n+})| = 0$ then $x(P^{n+}) = 0$. This implies that C do not intersect any arc of P^{n+}. Such a dicyle exists. As an example, incidence vectors of all dicycles C with $V(C) \subset \{v_{\tau(4)}, \ldots, v_{\tau(n-1)}\}$ are such that $x(P^{n+}) = 0$. $V(C)$ is the vertex set of the elementary dicycle C.
- Case 2. If $|A(C) \cap A(P^{n+})| = 1$ then $x(P^{n+}) = -1$ or $+1$ dependly on the fact that every dicycle C intersects subgraph P^{n+} to an arc represented by a dashed line or a full line.
- Case 3. If $|A(C) \cap A(P^{n+})| = 2$, it follows that $x(P^{n+}) = 0$. This because, every dicycle that intersects the subgraph P^{n+} to exactly two arcs, may pass by one arc represented with a full line and another arc represented with a dashed line.
- Case 4. $|A(C) \cap A(P^{n+})| = 3$. That is, if a solution C intersects P^{n+} to exactly three arcs. In such a case, $x(P^{n+}) = 1$. Indeed, every dicycle that intersects the subgraph P^{n+} to three arcs may pass by arcs $(v_{\tau(3)}, v_{\tau(2)}), (v_{\tau(2)}, v_{\tau(1)}), (v_{\tau(1)}, v_{\tau(n)})$.

We then conclude that an inequality of type (1) is valid for the polytope $\mathcal{P}(G)$.
We proceed similarly to show the validity of an inequality of type (2).

2.2 facetness of $s - t$ *alternating 3-arc path inequalities*

Let introduce the lemma and theorem below that will be useful in what follows.

Lemma 1 *Consider a set $\{C_i,\ i = 1, \ldots, q\}$ of some elementary dicycles that incidence vectors γ_i, with $i \in \{1, \ldots, q\}$ are affinely independent. Let $C_{q+1} \notin \{C_i,\ i = 1, \ldots, q\}$, with incidence vector γ_{q+1}, be an elementary dicycle that contains an arc $a_j \in A(C_{q+1})$ such that $a_j \notin A(C_i),\ \forall\ C_i \in \{C_i,\ i = 1, \ldots, q\}$. Then, vectors $\gamma_1, \gamma_2, \ldots, \gamma_q, \gamma_{q+1}$ are affinely independent.*

Proof. The proof follows directly from the definition of affine independence.

Remark *It is well known that the dimension of the polytope $\mathcal{P}(G)$ is $(n-1)^2$* [1].

Consider the valid inequality (1) and its support depicted in Figure 2. One can easily distinguish five cases for which the valid inequality (1) is satisfied with equality. There are,

- Case 1. When an elementary dicycle passes through only the arc $(v_{\tau(1)}, v_{\tau(2)})$,
- Case 2. When an elementary dicycle passes through only the arc $(v_{\tau(3)}, v_{\tau(2)})$,
- Case 3. When an elementary dicycle passes through only the arc $(v_{\tau(1)}, v_{\tau(n)})$,
- Case 4. When an elementary dicycle passes by the *alternating 3-arc path* $(v_{\tau(3)},\ v_{\tau(2)})$, $(v_{\tau(2)}, v_{\tau(1)})$, and $(v_{\tau(1)}, v_{\tau(n)})$,
- Case 5. When an elementary dicycle passes by extreme arcs $(v_{\tau(3)}, v_{\tau(2)})$ and $(v_{\tau(1)}, v_{\tau(n)})$ of an *alternating 3-arc path* and one of the $(n-4)$ arcs outgoing from the vertex $v_{\tau(2)}$.

According to Lemma 1, we construct $(n-1)^2$ elementary dicycles that incidence vectors are affinely independent and satisfy the inequality (1) with equality. These dicycles may satisfy one of five cases defined above.

Let first generate $(n^2 - 5n + 8)$ elementary dicyles that incidence vectors are affinely independent and pass by the unique arc $(v_{\tau(1)}, v_{\tau(2)})$ of P^{n+}, (Case 1).

Consider the hamiltonian dicycle C_1, (with γ_1 as incidence vector), that passes through the arc $(v_{\tau(1)}, v_{\tau(2)})$ such that
$A(C_1) = \{a_1 = (v_{\tau(1)}, v_{\tau(2)}), a_2 = (v_{\tau(2)}, v_{\tau(3)}), a_3 = (v_{\tau(3)}, v_{\tau(4)}), \ldots, a_{n-2} = (v_{\tau(n-2)}, v_{\tau(n-1)}),$
$a_{n-1} = (v_{\tau(n-1)}, v_{\tau(n)}), a_n = (v_{\tau(n)}, v_{\tau(1)})\}$. Let partition the arc set A as follow

$$A = A(C_1) \cup A(P^{n+}) \cup A_1 \cup A_2,$$

where A_2 is the set of arcs that do not belong to any elementary dicycle passing by the arc $a_1 = (v_{\tau(1)}, v_{\tau(2)})$, except arcs $(v_{\tau(3),\tau(2)})$ and $(v_{\tau(1),\tau(n)})$. Thus,

$$A_2 = \{(v_{\tau(1)}, v_{\tau(k)}), (v_{\tau(k)}, v_{\tau(2)}) : k \in \{3, 4, \ldots, n\}\} \setminus \{(v_{\tau(3)}, v_{\tau(2)}),\ (v_{\tau(1)}, v_{\tau(n)})\}$$

with $| A_2 |= 2n - 6$.

Therefore, as

$|A(C_1)| = n$ and $|(A(P^{n+}) \setminus \{(v_{\tau(1)}, v_{\tau(2)})\}| = n - 1$,

we have

$| A_1 |= n(n-1) - (2n-6) - n - (n-1) = n^2 - 5n + 7.$

Formally, we describe the arc set

$A_1 = \{a_{n+1}, a_{n+2}, \ldots, a_{n^2-4n+7}\}$

such that

$$A_1 = A_{11} \cup A_{12} \cup A_{13} \cup A_{14} \cup A_{15},$$

where

$$A_{11} = \{a_{n+1} = (v_{\tau(2)}, v_{\tau(n)})\},$$

$$A_{12} = \{a_i = (v_{\tau(k)}, v_{\tau(1)}), k = 3, \ldots, n-1, \\ i = n+2, \ldots, 2n-2\},$$

$$A_{13} = \{a_i = (v_{\tau(3)}, v_{\tau(k)}), k = 5, \ldots, n, \\ i = 2n-1, \ldots, 3n-6\},$$

$$A_{14} = \{a_i = (v_{\tau(n)}, v_{\tau(k)}),\ k = 3, \ldots, n-1,\ i = 3n-5, \ldots, 4n-9\}$$

and

$$A_{15} = \{a_i = (v_{\tau(j)}, v_{\tau(k)}), j = 4, \ldots, n-1,\ k = 3, \ldots, n,\ k \neq j+1,\ i = 4n-8, \ldots, n^2-4n+7\}.$$

Let $P_{v_{\tau(1)},v_{\tau(k)}},\ k \in \{3, \ldots, n\}$ be a $v_{\tau(1)} - v_{\tau(k)}$ elementary dipath such that

$$A(P_{v_{\tau(1)},v_{\tau(k)}}) \subset A(C_1).$$

Consider the elementary dicycle C_2

with

$$A(C_2) = \{a_1 = (v_{\tau(1)}, v_{\tau(2)}), a_{n+1} = (v_{\tau(2)}, v_{\tau(n)}),\ a_n = (v_{\tau(n)}, v_{\tau(1)})\}.$$

The following constructive algorithm generates $(n^2 - 5n + 8)$ elementary directed cycles.

Algorithm 1: Affinely independent cycle vectors (Case 1)

Data: A loopless digraph $G = (V, A)$ and a valid inequality $x(P^{n+}) \leq 1$.

Result: Set $\mathcal{C}_1$ of dicycles with affinely independent incidence vectors

begin
- $\mathcal{C}_1 \leftarrow \{C_1, C_2\},\ a_1 \leftarrow (v_{\tau(1)}, v_{\tau(2)})$
- $l \leftarrow 3$
- $i \leftarrow n+2$
- **for** $k \leftarrow 3$ *To* $(n-1)$ **do**
  - $C_l \leftarrow P_{(v_{\tau(1)},v_{\tau(k)})} \cup \{(v_{\tau(k)}, v_{\tau(1)})\}$
  - $\mathcal{C}_1 \leftarrow \mathcal{C}_1 \cup \{C_l\}$;
  - $l \leftarrow l+1$; $i \leftarrow i+1$
- **end**
- $i \leftarrow 2n-1,\ l \leftarrow n$
- **for** $k \leftarrow 5$ *To* n **do**
  - $C_l \leftarrow P_{(v_{\tau(1)},v_{\tau(3)})} \cup \{(v_{\tau(3)}, v_{\tau(k)})\} \cup \{(v_{\tau(k)}, v_{\tau(1)})\}$
  - $\mathcal{C}_1 \leftarrow \mathcal{C}_1 \cup \{C_l\}$;
  - $l \leftarrow l+1$; $i \leftarrow i+1$
- **end**
- $i \leftarrow 3n-5,\ l \leftarrow 2n-4$
- **for** $k \leftarrow 3$ *To* $n-1$ **do**
  - $C_l \leftarrow P^{v_{\tau(2)}}_{v_{\tau(1)},v_{\tau(n)}} \cup \{(v_{\tau(n)}, v_{\tau(k)})\} \cup \{(v_{\tau(k)}, v_{\tau(1)})\}$
  - $\mathcal{C}_1 \leftarrow \mathcal{C}_1 \cup \{C_l\}$;
  - $l \leftarrow l+1$; $i \leftarrow i+1$
- **end**
- $i \leftarrow 4n-8,\ l \leftarrow 3n-7$
- **for** $j \leftarrow 4$ *To* $n-1$ **do**
  - **for** $k \leftarrow 3$ *To* n **do**
    - **if** $(k \neq j+1)$ *and* $(k = n)$ **then**
      - $C_l \leftarrow P_{(v_{\tau(1)},v_{\tau(j)})} \cup \{(v_{\tau(j)}, v_{\tau(k)})\} \cup \{(v_{\tau(k)}, v_{\tau(1)})\}$
    - **end**
    - **if** $(k \neq j+1)$ *and* $(k \neq n)$ **then**
      - $C_l \leftarrow P^{v_{\tau(2)},v_{\tau(n)}}_{v_{\tau(1)},v_{\tau(j)}} \cup \{(v_{\tau(j)}, v_{\tau(k)})\} \cup \{(v_{\tau(k)}, v_{\tau(1)})\}$
    - **end**
    - $\mathcal{C}_1 \leftarrow \mathcal{C}_1 \cup \{C_l\}$;
    - $l \leftarrow l+1$; $i \leftarrow i+1$
  - **end**
- **end**
- **return** $\mathcal{C}_1$

end

Secondly, w.r.t. Case 2, in addition to the minimal dicycle C_{n^2-4n+6} containing the arc $(v_{\tau(1)}, v_{\tau(n)})$, we generate $(n-3)$ elementary dicycles that pass simultaneously by unique arc $(v_{\tau(3)}, v_{\tau(2)})$ of P^{n+} and an arc of type $(v_{\tau(1)}, v_{\tau(k)}),\ k \in \{3, \ldots, n-1\}$ of the arc set A_2 defined above implying that its corresponding incidence vectors verify the valid inequality (1) with equality.
Similarly, w.r.t. Case 3, in addition to the minimal dicycle C_{n^2-4n+7} that contains the arc $(v_{\tau(3)}, v_{\tau(2)})$, we generate $(n-3)$ elementary dicycles that pass simultaneously by unique arc $(v_{\tau(1)}, v_{\tau(n)})$ of P^{n+} and an arc of type $(v_{\tau(k)}, v_{\tau(2)}),\ k \in \{4, \ldots, n\}$ of the arc set A_2. Note that incidence vectors of such dicyles satisfy valid inequality (1) with equality.

Let

$$a_{n^2-3n+5} = (v_{\tau(1)}, v_{\tau(n)})$$

and

$$a_{n^2-3n+6} = (v_{\tau(3)}, v_{\tau(2)}).$$

Consider the arc set $A_2 = A_{21} \cup A_{22}$ that elements do not belong to any elementary dicycle that passes by $a_1 = (v_{\tau(1)}, v_{\tau(2)})$,

with

$$A_{21} = \{a_i = (v_{\tau(1)}, v_{\tau(k)}),\ k = 3, \ldots, n-1,\ i = n^2-4n+8, \ldots, n^2-3n+4\}$$

and

$$A_{22} = \{a_i = (v_{\tau(k)}, v_{\tau(2)}),\ k = 4, \ldots, n,\ i = n^2-3n+7, \ldots, n^2-2n+3\},$$

We have C_{n^2-4n+6} and C_{n^2-4n+7} are such that

$$A(C_{n^2-4n+6}) = \{(v_{\tau(1)}, v_{\tau(n)}), (v_{\tau(n)}, v_{\tau(1)})\}$$

and

$$A(C_{n^2-4n+7}) = \{(v_{\tau(3)}, v_{\tau(2)}), (v_{\tau(2)}, v_{\tau(3)})\}.$$

Taking into account dicycles C_{n^2-4n+6} and C_{n^2-4n+7}, the following algorithm (Algorithm 2) generates $(n-2)$ elementary dicycles that pass by the arc $(v_{\tau(3)}, v_{\tau(2)})$ and $(n-2)$ elementary dicycles passing by $(v_{\tau(1)}, v_{\tau(n)})$, (satisfying Cases 2 and 3).

Algorithm 2: Affinely independent cycle vectors verifying Case 2 and 3

Data: A loopless digraph $G = (V, A)$ and a valid inequality $x(P^{n+}) \leq 1$.

Result: Set $\mathcal{C}_2$ of dicycles with affinely independent incidence vectors

begin
- $\mathcal{C}_2 \leftarrow \{C_{n^2-4n+6}, C_{n^2-4n+7}\}$
- $l \leftarrow n^2 - 5n + 9$
- $i \leftarrow n^2 - 4n + 8$
- **for** $k \leftarrow 3$ *To* $(n-1)$ **do**
  - **if** $k \leftarrow 3$ **then**
    - $C_l \leftarrow P^{v_{\tau(2)}, v_{\tau(n)}}_{v_{\tau(3)}, v_{\tau(1)}} \cup \{(v_{\tau(1)}, v_{\tau(k)})\}$
  - **end**
  - **else**
    - $C_l \leftarrow P^{v_{\tau(2)}, v_{\tau(n)}}_{v_{\tau(3)}, v_{\tau(1)}} \cup \{(v_{\tau(1)}, v_{\tau(k)})\} \cup \{(v_{\tau(k)}, v_{\tau(3)})\}$
  - **end**
  - $\mathcal{C}_2 \leftarrow \mathcal{C}_2 \cup \{C_l\}$;
  - $l \leftarrow l + 1$; $i \leftarrow i + 1$
- **end**
- $l \leftarrow n^2 - 4n + 8$
- $i \leftarrow n^2 - 3n + 7$
- **for** $k \leftarrow 4$ *To* n **do**
  - **if** $k \neq n$ **then**
    - $C_l \leftarrow P^{v_{\tau(n)}}_{(v_{\tau(1)}, v_{\tau(k)})} \cup \{(v_{\tau(k)}, v_{\tau(2)})\} \cup P^{v_{\tau(3)}}_{(v_{\tau(2)}, v_{\tau(1)})}$
  - **end**
  - **if** $k \leftarrow n$ **then**
    - $C_l \leftarrow P^{v_{\tau(n)}}_{(v_{\tau(1)}, v_{\tau(2)})} \cup P^{v_{\tau(3)}}_{(v_{\tau(2)}, v_{\tau(1)})}$
  - **end**
  - $\mathcal{C}_2 \leftarrow \mathcal{C}_2 \cup \{C_l\}$;
  - $l \leftarrow l + 1$; $i \leftarrow i + 1$
- **end**
- **return** $\mathcal{C}_2$

end

At the end, w.r.t. Case 4, consider the dicycle C_{n^2-3n+5} such that

$$A(C_{n^2-3n+5}) = \{(v_{\tau(3)}, v_{\tau(2)}), (v_{\tau(2)}, v_{\tau(1)}), (v_{\tau(1)}, v_{\tau(n)}), (v_{\tau(n)}, v_{\tau(3)})\}.$$

Note that the dicycle C_{n^2-3n+5} pass by the arc $a_{n^2-2n+4} = (v_{\tau(2)}, v_{\tau(1)})$ and let generate $(n-4)$ elementary dicycles that verify the condition of Case 5. Such dicycles pass simultaneously through arcs $(v_{\tau(3)}, v_{\tau(2)})$, $(v_{\tau(1)}, v_{\tau(n)})$ and an unique arc of P^{n+} that outgoing from the vertex $v_{\tau(2)}$. Let A_3 be the set of these arcs. Thus,

$$A_3 = \{(v_{\tau(2)}, v_{\tau(k)}),\ k = 4, \ldots, n-1\}$$

.

Algorithm 3: Affinely independent cycle vectors verifying Case 4 and 5

Data: A loopless digraph $G = (V, A)$ and a valid inequality $x(P^{n+}) \leq 1$.

Result: Set $\mathcal{C}_3$ of dicycles with affinely independent incidence vectors

begin
- $\mathcal{C}_3 \leftarrow \{C_{n^2-3n+5}\}$
- $l \leftarrow n^2 - 3n + 6$
- $i \leftarrow n^2 - 2n + 5$
- **for** $k \leftarrow 4$ *To* $(n-1)$ **do**
  - $C_l \leftarrow \{(v_{\tau(3)}, v_{\tau(2)})\} \cup \{(v_{\tau(2)}, v_{\tau(k)})\} \cup P^{v_{\tau(1)}, v_{\tau(n)}}_{v_{\tau(k)}, v_{\tau(3)}}$
  - $\mathcal{C}_3 \leftarrow \mathcal{C}_3 \cup \{C_l\}$;
  - $l \leftarrow l + 1$; $i \leftarrow i + 1$
- **end**
- **return** $\mathcal{C}_3$

end

Theorem 2 *Algorithms 1, 2 and 3 construct $(n-1)^2$ elementary dicycles that incidence vectors are affinely independent.*

Proof. One can easily verify that Algorithm 1, 2 and 3 generate $(n^2 - 5n + 8)$, $(2n - 4)$ and $(n - 3)$ elementary dicyles, respectively. All these dicycles are such that its corresponding incidence vectors satisfy the valid inequality (1) with equality. Moreover, according to Lemma 1, incidence vectors of elementary dicycles of sets $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_2$ returned by Algorithms 1, 2, 3, respectively, are affinely independent. Indeed, applying the algorithm, one can verify that the current constructed dicycle, $C_l, l \in \{1, \ldots, (n-1)^2\}$, passes by an arc $a_i \in A_1 \cup A_2 \cup A_3$ that does not belong to any already constructed elementary dicycles $C_1, C_2, \ldots, C_{l\,1}$. –

Remark *One can easily verify that, in polynomial time, Algorithms 1, 2 and 3 generate $(n-1)^2$ points (represented by incidence vectors of dicycles) that are affinely independent. Indeed, the time complexity of Algorithms 1, 2 and 3 are respectively $O(n^2)$, $O(n)$ and $O(n)$.*

Theorem 3 *$s - t$ alternating 3-arc path inequalities define facets of $\mathcal{P}(G)$, with $n \geq 4$.*

Proof. By virtue of Theorem 2, the affine sub-space $\{x : x(P^{n+}) = 1\}$ contains $(n-1)^2$ points (represented by incidence vectors of dicycles) that are affinely independent. On the other hand, there exists points that strictly verify the valid inequality (1). As an example, incidence vectors of all minimal dicycles, that contain an arc of the arc set A_3 defined above, strictly verify the valid inequality (1). Such a minimal dicycle, say C, is such that $A(C) = \{(v_{\tau(2)}, v_{\tau(k)}), (v_{\tau(k)}, v_{\tau(2)}), \ k = 4, \ldots, n-1\}$. This shows that inequality (1) is a facet and is not an equation. This completes the proof of Theorem 3.

To show the facetness of the valid inequality (2), we proceed similarly.

3 Conclusions

In general, in combinatorial optimization and particularly in polyhedral theory to show the facetness of a valid inequality, one may prove that the valid inequality is essential in the description of the polyhedron or show the existence of affinely independent points of the polyhedron that number corresponds to the dimension of the polyhedron.
The main contribution of this paper is, consider the elementary dicycle polytope, to show that what we call $s - t$ alternating 3-arc path inequalities are facet of the dicycle polytope. For this purpose, after proving that $s - t$ alternating 3-arc path inequalities are valid for the dicycle polytope, taking into account different cases that permit the incidence vector of a given dicycle to satisfy a $s - t$ alternating 3-arc path inequality with equality, we devise some algorithms generating elementary dicycles with affinely independent incidence vectors. These latter vectors satisfy the $s - t$ alternating 3-arc path inequality with equality. At the end, we resort to such algorithms to prove the facetness of *alternating three arc path* inequalities.

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