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A Note on External Direct Products of BP-algebras

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Abstract The notion of BP-algebras was introduced by Ahn and Han [2] in 2013, which is related to several classes of algebra. It has been examined by several researchers. In the group, the concept of the direct product (DP) [21] was initially developed and given some features. Then, other algebraic structures are subjected to DP groups. Lingcong and Endam [16] examined the idea of the DP of (0-commutative) B-algebras and B-homomorphisms in 2016 and discovered several related features, one of which is a DP of two B-algebras that is a B-algebra. Later on, the concept of the DP of B-algebra was expanded to include finite family B-algebra, and some of the connected issues were researched. In this work, the external direct product (EDP), a general concept of the DP, is established, and the results of the EDP for certain subsets of BP-algebras are determined. In addition, we define the weak direct product (WDP) of BP-algebras. In light of the EDP BP-algebras, we conclude by presenting numerous essential theorems of (anti-)BP-homomorphisms.

Keywords BP-algebra, External Direct Product, Weak Direct Product, BP-homomorphism, Anti-BP-homomorphism

1 Introduction and Preliminaries

The notion of BP-algebras was introduced by Ahn and Han [2] in 2013, which is related to several classes of algebra. It has been examined by several researchers, for example, in 2015, Jefferson and Chandramouleeswaran [9] introduced the notion of fuzzy BP-algebras and discussed the properties of fuzzy BP-subalgebras. In 2016, Kandaraj and Devi [14] introduced f-derivations on BP-algebras. In the same year, Jefferson and Chandramouleeswaran [11, 10, 12, 13] introduced the concept of fuzzy T-ideals and L-fuzzy T-ideals in BP-algebras. In 2019, Prasanna et al. [18] introduced the concept of doubt fuzzy BP-ideals and lower level cuts of a fuzzy set. In 2020, Prasanna et al. [20] introduced ω-fuzzy BP-subalgebras and fuzzy BP-ideals. Elgendy [6] introduced the concept of bipolar fuzzy α-ideals of BP-algebras. Prasanna et al. [19] introduced the concept of doubt Q-fuzzy BP-ideals.

First defined in the group, the DP idea was given some attributes [21]. A group's DP, for instance, is also a group, and an abelian group's DP is likewise an abelian group. The next step is to apply DP groups to additional algebraic structures. In their study of the DP of B-algebras in 2016, Lingcong and Endam [16] identified a number of associated properties, one of which is a DP of two B-algebras that is itself a B-algebra. Endam and Teves [7] defined the DP of BF-algebras in the same year and obtained associated features. The idea of the DP of BRK-algebras was first suggested in 2018 by Abebe [1], who also demonstrated that the finite DP of BRK-algebras is a BRK-algebra. Widianto et al. [22] defined the DP of BG-algebras, as well as associated BG-algebraic characteristics, in 2019. The DP of BP-algebras, which are identical to B-algebras, was defined by Setiani et al. [21] in 2020. In order to define the DP of BP-algebras as applicable to finite sets of BP-algebras, they first obtained the appropriate property of BP-algebras' DP. The DP of GK-algebra was defined in 2021 by Kavitha and Gowri [15]. The idea of the EDP, also known as the DP of infinite family of B-algebras, was first suggested by Chanmanee et al. [4] in 2022. They also established the idea of the B-algebraic WDP. Chanmanee et al. [5] proposed the idea of the DP of infinite family of UP-algebras and showed that it is a DUP-algebra in the same year, referring to the EDP as a DUP-algebra induced by UP-algebras. Chanmanee et al. [3] established the idea of the DP of an infinite family of UP (BCC)-algebras, which they dubbed the EDP, and discovered the outcome of the EDP of certain UP (BCC)-algebras.

The EDP, which we refer to as a broad notion of the DP in the meaning of Lingcong and Endam [16], is an idea of the DP of the infinite family of BP-algebras that we present in this study. We also present the idea of the WDP of BP-algebras. Several (anti-)BP-homomorphism theorems are then discussed in light of the EDP BP-algebras. It is common to define the external direct product using a universal property in the appropriate category. We hope to investigate universal property in future work.

The definitions and examples of BP-algebras and other terms that are pertinent to the study in this article are given below.

Definition 1.1. [2] An algebra $\mathfrak{U} = (\mathfrak{U}; \diamond, 0)$ of type $(2, 0)$ is referred to be a *BP-algebra* if it meets the criteria listed below:

$$(\forall \delta \in \mathfrak{U})(\delta \diamond \delta = 0), \tag{BP-1}$$
$$(\forall \delta, \gamma \in \mathfrak{U})(\delta \diamond (\delta \diamond \gamma) = \gamma), \tag{BP-2}$$
$$(\forall \delta, \gamma, \zeta \in \mathfrak{U})((\delta \diamond \zeta) \diamond (\gamma \diamond \zeta) = \delta \diamond \gamma). \tag{BP-3}$$

From [2], it is known that every BP-algebra is a BF-algebra.

Example 1.2. Let $\mathfrak{U} = \{0, 1, 2, 3, 4, 5, 6\}$. Then $\mathfrak{U} = (\mathfrak{U}; \diamond, 0)$ is a BP-algebra in which a binary operation $\diamond$ is defined as follows:

$\diamond$	0	1	2	3	4	5	6
0	0	6	5	4	3	2	1
1	1	0	6	5	4	3	2
2	2	1	0	6	5	4	3
3	3	2	1	0	6	5	4
4	4	3	2	1	0	6	5
5	5	4	3	2	1	0	6
6	6	5	4	3	2	1	0

In a BP-algebra $\mathfrak{U} = (\mathfrak{U}; \diamond, 0)$, the following assertions are valid (see [2]).

$$(\forall \delta \in \mathfrak{U})(0 \diamond (0 \diamond \delta) = \delta), \tag{1.1}$$
$$(\forall \delta, \gamma \in \mathfrak{U})(0 \diamond (\gamma \diamond \delta) = \delta \diamond \gamma), \tag{1.2}$$
$$(\forall \delta \in \mathfrak{U})(\delta \diamond 0 = \delta), \tag{1.3}$$
$$(\forall \delta, \gamma \in \mathfrak{U})(\delta \diamond \gamma = 0 \Rightarrow \gamma \diamond \delta = 0), \tag{1.4}$$
$$(\forall \delta, \gamma \in \mathfrak{U})(0 \diamond \delta = 0 \diamond \gamma \Rightarrow \delta = \gamma), \tag{1.5}$$
$$(\forall \delta, \gamma \in \mathfrak{U})(0 \diamond \delta = \gamma \Rightarrow 0 \diamond \gamma = \delta), \tag{1.6}$$
$$(\forall \delta, \gamma \in \mathfrak{U})(0 \diamond \delta = \delta \Rightarrow \delta \diamond \gamma = \gamma \diamond \delta). \tag{1.7}$$

Definition 1.3. [2, 11] A subset $\mathfrak{S} \neq \emptyset$ of a BP-algebra $\mathfrak{U} = (\mathfrak{U}; \diamond, 0)$ is called

(i) a *subalgebra* of $\mathfrak{U}$ if it meets the following:

$$(\forall \delta, \gamma \in \mathfrak{S})(\delta \diamond \gamma \in \mathfrak{S}), \tag{1.8}$$

(ii) an *ideal* of $\mathfrak{U}$ if it meets the following:

$$0 \in \mathfrak{S}, \tag{1.9}$$
$$(\forall \delta, \gamma \in \mathfrak{S})(\delta \diamond \gamma \in \mathfrak{S}, \gamma \in \mathfrak{S} \Rightarrow \delta \in \mathfrak{S}), \tag{1.10}$$

(iii) a *T-ideal* of $\mathfrak{U}$ if it meets (1.9) and the following:

$$(\forall \delta, \gamma, \zeta \in \mathfrak{S})((\delta \diamond \gamma) \diamond \zeta \in \mathfrak{S}, \gamma \in \mathfrak{S} \Rightarrow \delta \diamond \zeta \in \mathfrak{S}). \tag{1.11}$$

Definition 1.4. [11] An ideal $\mathfrak{S}$ of a BP-algebra $\mathfrak{U} = (\mathfrak{U}; \diamond, 0)$ is said to be *closed* if

$$(\forall \delta \in \mathfrak{S})(0 \diamond x \in \mathfrak{S}). \tag{1.12}$$

Definition 1.5. [2] A BP-algebra $\mathfrak{U} = (\mathfrak{U}; \diamond, 0)$ is said to be 0-*commutative* if

$$(\forall \delta, \gamma \in \mathfrak{U})(\delta \diamond (0 \diamond \gamma) = \gamma \diamond (0 \diamond \delta)). \tag{1.13}$$

Let $\mathfrak{U}_1 = (\mathfrak{U}_1; \diamond_1, 0_1)$ and $\mathfrak{U}_2 = (\mathfrak{U}_2; \diamond_2, 0_2)$ be two BP-algebras. A map $\varphi : \mathfrak{U}_1 \to \mathfrak{U}_2$ is called a *BP-homomorphism* if $\varphi(\delta \diamond_1 \gamma) = \varphi(\delta) \diamond_2 \varphi(\gamma)$ for any $\delta, \gamma \in \mathfrak{U}$, and an *anti-BP-homomorphism* if $\varphi(\delta \diamond_1 \gamma) = \varphi(\gamma) \diamond_2 \varphi(\delta)$ for any $\delta, \gamma \in \mathfrak{U}$. The kernel of φ denoted by $\ker \varphi$ is defined to be the set $\ker \varphi = \{\delta \in \mathfrak{U}_1 : \varphi(\delta) = 0_2\}$. A BP-homomorphism φ is called a BP-monomorphism, BP-epimorphism, or BP-isomorphism if it is one-one, onto, or bijective, respectively.

2 External Direct Product of BP-algebras

Definition 2.1. [16] Let $\forall \xi \in \{1, 2, ..., k\}, \mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi)$ be an algebra. Define the *direct product* (DP) of algebras $\mathfrak{U}_1, \mathfrak{U}_2, ..., \mathfrak{U}_k$ to be the structure $\mathfrak{U}_1 \times \mathfrak{U}_2 \times ... \times \mathfrak{U}_k = (\mathfrak{U}_1 \times \mathfrak{U}_2 \times ... \times \mathfrak{U}_k; \otimes)$, where

$$\mathfrak{U}_1 \times \mathfrak{U}_2 \times ... \times \mathfrak{U}_k = \{(\delta_1, \delta_2, ..., \delta_k) \mid \delta_\xi \in \mathfrak{U}_\xi, \forall \xi = 1, 2, ..., k\}$$

and whose operation $\otimes$ is defined by $\forall (\delta_1, \delta_2, ..., \delta_k), (\gamma_1, \gamma_2, ..., \gamma_k) \in \mathfrak{U}_1 \times \mathfrak{U}_2 \times ... \times \mathfrak{U}_k$,

$$(\delta_1, \delta_2, ..., \delta_k) \otimes (\gamma_1, \gamma_2, ..., \gamma_k) = (\delta_1 \diamond_1 \gamma_1, \delta_2 \diamond_2 \gamma_2, ..., \delta_k \diamond_k \gamma_k).$$

We now provide some of the DP's features and extend the idea to the infinite family of BP-algebras.

Definition 2.2. [3] Let $\mathfrak{U}_\xi \neq \emptyset, \forall \xi \in \triangle$. Define the *external direct product* (EDP) of sets $\mathfrak{U}_\xi, \forall \xi \in \triangle$ to be the set $\prod_{\xi \in \triangle} \mathfrak{U}_\xi$, where

$$\prod_{\xi \in \triangle} \mathfrak{U}_\xi = \{f : \triangle \to \bigcup_{\xi \in \triangle} \mathfrak{U}_\xi \mid f(\xi) \in \mathfrak{U}_\xi, \forall \xi \in \triangle\}.$$

To make things easier, we define an element of $\prod_{\xi \in \triangle} \mathfrak{U}_\xi$ as a function $(\delta_\xi)_{\xi \in \triangle} : \triangle \to \bigcup_{\xi \in \triangle} \mathfrak{U}_\xi$, where $\xi \mapsto \delta_\xi \in \mathfrak{U}_\xi, \forall \xi \in \triangle$.

Remark 2.3. [3] Let $\mathfrak{U}_\xi \neq \emptyset$ and $\mathfrak{S}_\xi \subseteq \mathfrak{U}_\xi, \forall \xi \in \triangle$. Then $\emptyset \neq \prod_{\xi \in \triangle} \mathfrak{S}_\xi \subseteq \prod_{\xi \in \triangle} \mathfrak{U}_\xi$ if and only if $\emptyset \neq \mathfrak{S}_\xi \subseteq \mathfrak{U}_\xi, \forall \xi \in \triangle$.

Definition 2.4. [3] Let $\forall \xi \in \triangle, \mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi)$ be an algebra. Define the binary operation $\otimes$ on the EDP $\prod_{\xi \in \triangle} \mathfrak{U}_\xi = (\prod_{\xi \in \triangle} \mathfrak{U}_\xi; \otimes)$ as follows: $\forall (\delta_\xi)_{\xi \in \triangle}, (\gamma_\xi)_{\xi \in \triangle} \in \prod_{\xi \in \triangle} \mathfrak{U}_\xi$,

$$(\delta_\xi)_{\xi \in \triangle} \otimes (\gamma_\xi)_{\xi \in \triangle} = (\delta_\xi \diamond_\xi \gamma_\xi)_{\xi \in \triangle}. \tag{2.1}$$

Let $\forall \xi \in \triangle, \mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ be a BP-algebra. For $\xi \in \triangle$, let $\delta_\xi \in \mathfrak{U}_\xi$. We define the function $f_{\delta_\xi} : \triangle \to \bigcup_{\xi \in \triangle} \mathfrak{U}_\xi$ as follows:

$$(\forall \lambda \in \triangle) \left(f_{\delta_\xi}(\lambda) = \begin{cases} \delta_\xi & \text{if } \lambda = \xi \\ 0_\lambda & \text{otherwise} \end{cases} \right). \tag{2.2}$$

Then $f_{\delta_\xi} \in \prod_{\xi \in \triangle} \mathfrak{U}_\xi$.

Lemma 2.5. *Let $\forall \xi \in \triangle, \mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ be a BP-algebra. For $\xi \in \triangle$, let $\delta_\xi, \gamma_\xi \in \mathfrak{U}_\xi$. Then $f_{\delta_\xi} \otimes f_{\gamma_\xi} = f_{\delta_\xi \diamond_\xi \gamma_\xi}$.*

Proof. Now, $\forall \lambda \in \triangle$,

$$(f_{\delta_\xi} \otimes f_{\gamma_\xi})(\lambda) = \begin{cases} \delta_\xi \diamond_\xi \gamma_\xi & \text{if } \lambda = \xi \\ 0_\lambda \diamond_\lambda 0_\lambda & \text{otherwise} \end{cases}.$$

By (BP-1), we have $\forall \lambda \in \triangle$,

$$(f_{\delta_\xi} \otimes f_{\gamma_\xi})(\lambda) = \begin{cases} \delta_\xi \diamond_\xi \gamma_\xi & \text{if } \lambda = \xi \\ 0_\lambda & \text{otherwise} \end{cases}.$$

By (2.2), we have $f_{\delta_\xi} \otimes f_{\gamma_\xi} = f_{\delta_\xi \diamond_\xi \gamma_\xi}$. □

The DP of BP-algebras in terms of the infinite family of BP-algebras is shown to be a BP-algebra by the following theorem.

Theorem 2.6. *$\forall \xi \in \triangle, \mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ is a BP-algebra if and only if $\prod_{\xi \in \triangle} \mathfrak{U}_\xi = (\prod_{\xi \in \triangle} \mathfrak{U}_\xi; \otimes, (0_\xi)_{\xi \in \triangle})$ is a BP-algebra.*

Proof. Suppose $\forall \xi \in \triangle, \mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ is a BP-algebra.

(BP-1) Let $(\delta_\xi)_{\xi \in \triangle} \in \prod_{\xi \in \triangle} \mathfrak{U}_\xi$. Since $\mathfrak{U}_\xi$ satisfies (BP-1), we have $\delta_\xi \diamond_\xi \delta_\xi = 0_\xi, \forall \xi \in \triangle$. Thus $(\delta_\xi)_{\xi \in \triangle} \otimes (\delta_\xi)_{\xi \in \triangle} = (\delta_\xi \diamond_\xi \delta_\xi)_{\xi \in \triangle} = (0_\xi)_{\xi \in \triangle}$.

(BP-2) Let $(\delta_\xi)_{\xi \in \triangle}, (\gamma_\xi)_{\xi \in \triangle} \in \prod_{\xi \in \triangle} \mathfrak{U}_\xi$. Since $\mathfrak{U}_\xi$ satisfies (BP-2), we have $\delta_\xi \diamond_\xi (\delta_\xi \diamond_\xi \gamma_\xi) = \gamma_\xi, \forall \xi \in \triangle$. Thus $(\delta_\xi)_{\xi \in \triangle} \otimes ((\delta_\xi)_{\xi \in \triangle} \otimes (\gamma_\xi)_{\xi \in \triangle}) = (\delta_\xi \diamond_\xi (\delta_\xi \diamond_\xi \gamma_\xi))_{\xi \in \triangle} = (\gamma_\xi)_{\xi \in \triangle}$.

(BP-3) Let $(\delta_\xi)_{\xi \in \triangle}, (\gamma_\xi)_{\xi \in \triangle}, (\zeta_\xi)_{\xi \in \triangle} \in \prod_{\xi \in \triangle} \mathfrak{U}_\xi$. Since $\mathfrak{U}_\xi$ satisfies (BP-3), we have $(\delta_\xi \diamond_\xi \zeta_\xi) \diamond_\xi (\gamma_\xi \diamond_\xi \zeta_\xi) = \delta_\xi \diamond_\xi \gamma_\xi, \forall \xi \in \triangle$. Thus $((\delta_\xi)_{\xi \in \triangle} \otimes (\zeta_\xi)_{\xi \in \triangle}) \otimes ((\gamma_\xi)_{\xi \in \triangle} \otimes (\zeta_\xi)_{\xi \in \triangle}) = ((\delta_\xi \diamond_\xi \zeta_\xi) \diamond_\xi (\gamma_\xi \diamond_\xi \zeta_\xi))_{\xi \in \triangle} = (\delta_\xi \diamond_\xi \gamma_\xi)_{\xi \in \triangle} = (\delta_\xi)_{\xi \in \triangle} \otimes (\gamma_\xi)_{\xi \in \triangle}$.

As a result, $\prod_{\xi \in \triangle} \mathfrak{U}_\xi = (\prod_{\xi \in \triangle} \mathfrak{U}_\xi; \otimes, (0_\xi)_{\xi \in \triangle})$ is a BP-algebra.

On the other hand, suppose that $\prod_{\xi \in \triangle} \mathfrak{U}_\xi = (\prod_{\xi \in \triangle} \mathfrak{U}_\xi; \otimes, (0_\xi)_{\xi \in \triangle})$ is a BP-algebra. Let $\xi \in \triangle$.

(BP-1) Let $\delta_\xi \in \mathfrak{U}_\xi$. Then $f_{\delta_\xi} \in \prod_{\xi \in \triangle} \mathfrak{U}_\xi$. Since $\prod_{\xi \in \triangle} \mathfrak{U}_\xi$ satisfies (BP-1), we have $f_{\delta_\xi} \otimes f_{\delta_\xi} = (0_\xi)_{\xi \in \triangle}$. By Lemma 2.5, we have $f_{\delta_\xi \diamond_\xi \delta_\xi} = (0_\xi)_{\xi \in \triangle}$ and so $\delta_\xi \diamond_\xi \delta_\xi = 0_\xi$.

(BP-2) Let $\delta_\xi, \gamma_\xi \in \mathfrak{U}_\xi$. Then $f_{\delta_\xi}, f_{\gamma_\xi} \in \prod_{\xi \in \triangle} \mathfrak{U}_\xi$. Since $\prod_{\xi \in \triangle} \mathfrak{U}_\xi$ satisfies (BP-2), we have $f_{\delta_\xi} \otimes (f_{\delta_\xi} \otimes f_{\gamma_\xi}) = f_{\gamma_\xi}$. By Lemma 2.5, we have $f_{\delta_\xi \diamond_\xi (\delta_\xi \diamond_\xi \gamma_\xi)} = f_{\gamma_\xi}$ and so $\delta_\xi \diamond_\xi (\delta_\xi \diamond_\xi \gamma_\xi) = \gamma_\xi$.

(BP-3) Let $\delta_\xi, \gamma_\xi, \zeta_\xi \in \mathfrak{U}_\xi$. Then $f_{\delta_\xi}, f_{\gamma_\xi}, f_{\zeta_\xi} \in \prod_{\xi \in \triangle} \mathfrak{U}_\xi$. Since $\prod_{\xi \in \triangle} \mathfrak{U}_\xi$ satisfies (BP-3), we have $(f_{\delta_\xi} \otimes f_{\zeta_\xi}) \otimes (f_{\gamma_\xi} \otimes f_{\zeta_\xi}) = f_{\delta_\xi} \otimes f_{\gamma_\xi}$. By Lemma 2.5, we have $f_{(\delta_\xi \diamond_\xi \zeta_\xi) \diamond_\xi (\gamma_\xi \diamond_\xi \zeta_\xi)} = f_{\delta_\xi \diamond_\xi \gamma_\xi}$ and so $(\delta_\xi \diamond_\xi \zeta_\xi) \diamond_\xi (\gamma_\xi \diamond_\xi \zeta_\xi) = \delta_\xi \diamond_\xi \gamma_\xi$.

As a result, $\forall \xi \in \triangle, \mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ is a BP-algebra. □

We call the BP-algebra $\prod_{\xi \in \triangle} \mathfrak{U}_\xi = (\prod_{\xi \in \triangle} \mathfrak{U}_\xi; \otimes, (0_\xi)_{\xi \in \triangle})$ in Theorem 2.6 the EDP BP-algebra induced by a BP-algebra $\mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi), \forall \xi \in \triangle$.

The WDP of infinite family of BP-algebras is then discussed, and the following are some of its properties:

Definition 2.7. Let $\forall \xi \in \triangle, \mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ be a BP-algebra. Define the *weak direct product* (WDP) of $\mathfrak{U}_\xi, \forall \xi \in \triangle$ to be the structure $\prod^{\mathrm{w}}_{\xi \in \triangle} \mathfrak{U}_\xi = (\prod^{\mathrm{w}}_{\xi \in \triangle} \mathfrak{U}_\xi; \otimes)$, where

$$\prod^{\mathrm{w}}_{\xi \in \triangle} \mathfrak{U}_\xi = \{(\delta_\xi)_{\xi \in \triangle} \in \prod_{\xi \in \triangle} \mathfrak{U}_\xi \mid \delta_\xi \neq 0_\xi, \text{ where the number of such } \xi \text{ is finite}\}.$$

Then $(0_\xi)_{\xi \in \triangle} \in \prod^{\mathrm{w}}_{\xi \in \triangle} \mathfrak{U}_\xi \subseteq \prod_{\xi \in \triangle} \mathfrak{U}_\xi$.

Theorem 2.8. *Let $\forall \xi \in \triangle, \mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ be a BP-algebra. Then $\prod^{\mathrm{w}}_{\xi \in \triangle} \mathfrak{U}_\xi$ is a subalgebra of the EDP BP-algebra $\prod_{\xi \in \triangle} \mathfrak{U}_\xi = (\prod_{\xi \in \triangle} \mathfrak{U}_\xi; \otimes, (0_\xi)_{\xi \in \triangle})$.*

Proof. We see that $(0_\xi)_{\xi \in \triangle} \in \prod^{\mathrm{w}}_{\xi \in \triangle} \mathfrak{U}_\xi \neq \emptyset$. Let $(\delta_\xi)_{\xi \in \triangle}, (\gamma_\xi)_{\xi \in \triangle} \in \prod^{\mathrm{w}}_{\xi \in \triangle} \mathfrak{U}_\xi$, where $\triangle_1 = \{\xi \in \triangle \mid \delta_\xi \neq 0_\xi\}$ and $\triangle_2 = \{\xi \in \triangle \mid \gamma_\xi \neq 0_\xi\}$ are finite. Then $|\triangle_1 \cup \triangle_2|$ is finite. Thus $\forall \lambda \in \triangle$,

$$((\delta_\xi)_{\xi \in \triangle} \otimes (\gamma_\xi)_{\xi \in \triangle})(\lambda) = \begin{cases} \delta_\lambda \diamond_\lambda 0_\lambda & \text{if } \lambda \in \triangle_1 - \triangle_2 \\ \delta_\lambda \diamond_\lambda \gamma_\lambda & \text{if } \lambda \in \triangle_1 \cap \triangle_2 \\ 0_\lambda \diamond_\lambda \gamma_\lambda & \text{if } \lambda \in \triangle_2 - \triangle_1 \\ 0_\lambda \diamond_\lambda 0_\lambda & \text{otherwise} \end{cases}.$$

By (BP-1) and (1.3), we have $\forall\lambda \in \triangle$,

$$((\delta_\xi)_{\xi\in\triangle} \otimes (\gamma_\xi)_{\xi\in\triangle})(\lambda) = \begin{cases} \delta_\lambda & \text{if } \lambda \in \triangle_1 - \triangle_2 \\ \delta_\lambda \diamond_\lambda \gamma_\lambda & \text{if } \lambda \in \triangle_1 \cap \triangle_2 \\ 0_\lambda \diamond_\lambda \gamma_\lambda & \text{if } \lambda \in \triangle_2 - \triangle_1 \\ 0_\lambda & \text{otherwise} \end{cases}.$$

This implies that the number of such $((\delta_\xi)_{\xi\in\triangle} \otimes (\gamma_\xi)_{\xi\in\triangle})(\lambda) \neq 0_\lambda$ is not more than $|\triangle_1 \cup \triangle_2|$, that is, it is finite. Thus $(\delta_\xi)_{\xi\in\triangle} \otimes (\gamma_\xi)_{\xi\in\triangle} \in \prod^{\mathrm{w}}_{\xi\in\triangle} \mathfrak{U}_\xi$. As a result, $\prod^{\mathrm{w}}_{\xi\in\triangle} \mathfrak{U}_\xi$ is a subalgebra of $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$. □

Theorem 2.9. *Let $\mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ be a BP-algebra and $\mathfrak{S}_\xi \subseteq \mathfrak{U}_\xi, \forall\xi \in \triangle$. Then $\mathfrak{S}_\xi$ is a subalgebra of $\mathfrak{U}_\xi, \forall\xi \in \triangle$ if and only if $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is a subalgebra of the EDP BP-algebra $\prod_{\xi\in\triangle} \mathfrak{U}_\xi = (\prod_{\xi\in\triangle} \mathfrak{U}_\xi; \otimes, (0_\xi)_{\xi\in\triangle})$.*

Proof. Suppose $\mathfrak{S}_\xi$ is a subalgebra of $\mathfrak{U}_\xi, \forall\xi \in \triangle$. Since $\emptyset \neq \mathfrak{S}_\xi \subseteq \mathfrak{U}_\xi, \forall\xi \in \triangle$ and by Remark 2.3, we have $\emptyset \neq \prod_{\xi\in\triangle} \mathfrak{S}_\xi \subseteq \prod_{\xi\in\triangle} \mathfrak{U}_\xi$. Let $(\delta_\xi)_{\xi\in\triangle}, (\gamma_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. Then $\delta_\xi, \gamma_\xi \in \mathfrak{S}_\xi, \forall\xi \in \triangle$. By (1.8), we have $\delta_\xi \diamond_\xi \gamma_\xi \in \mathfrak{S}_\xi, \forall\xi \in \triangle$ and so $(\delta_\xi)_{\xi\in\triangle} \otimes (\gamma_\xi)_{\xi\in\triangle} = (\delta_\xi \diamond_\xi \gamma_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. As a result, $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is a subalgebra of $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$.

On the other hand, suppose that $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is a subalgebra of $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$. Since $\emptyset \neq \prod_{\xi\in\triangle} \mathfrak{S}_\xi \subseteq \prod_{\xi\in\triangle} \mathfrak{U}_\xi$ and by Remark 2.3, we have $\emptyset \neq \mathfrak{S}_\xi \subseteq \mathfrak{U}_\xi, \forall\xi \in \triangle$. Let $\xi \in \triangle$ and let $\delta_\xi, \gamma_\xi \in \mathfrak{S}_\xi$. Then $f_{\delta_\xi}, f_{\gamma_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. By (1.8) and Lemma 2.5, we have $f_{\delta_\xi \diamond_\xi \gamma_\xi} = f_{\delta_\xi} \otimes f_{\gamma_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. By (2.2), we have $\delta_\xi \diamond_\xi \gamma_\xi \in \mathfrak{S}_\xi$. As a result, $\mathfrak{S}_\xi$ is a subalgebra of $\mathfrak{U}_\xi, \forall\xi \in \triangle$. □

Theorem 2.10. *Let $\mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ be a BP-algebra and $\mathfrak{S}_\xi \subseteq \mathfrak{U}_\xi, \forall\xi \in \triangle$. Then $\mathfrak{S}_\xi$ is an ideal of $\mathfrak{U}_\xi, \forall\xi \in \triangle$ if and only if $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is an ideal of the EDP BP-algebra $\prod_{\xi\in\triangle} \mathfrak{U}_\xi = (\prod_{\xi\in\triangle} \mathfrak{U}_\xi; \otimes, (0_\xi)_{\xi\in\triangle})$.*

Proof. Suppose $\mathfrak{S}_\xi$ is an ideal of $\mathfrak{U}_\xi, \forall\xi \in \triangle$. Then $0_\xi \in \mathfrak{S}_\xi, \forall\xi \in \triangle$, so $(0_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi \neq \emptyset$. Let $(\delta_\xi)_{\xi\in\triangle}, (\gamma_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{U}_\xi$ be such that $(\delta_\xi)_{\xi\in\triangle} \otimes (\gamma_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$ and $(\gamma_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. Then $(\delta_\xi \diamond_\xi \gamma_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. Thus $\delta_\xi \diamond_\xi \gamma_\xi \in \mathfrak{S}_\xi$ and $\gamma_\xi \in \mathfrak{S}_\xi$, it follows from (1.10) that $\delta_\xi \in \mathfrak{S}_\xi, \forall\xi \in \triangle$. Thus $(\delta_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. As a result, $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is an ideal of $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$.

On the other hand, suppose that $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is an ideal of $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$. Then $(0_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$, so $0_\xi \in \mathfrak{S}_\xi \neq \emptyset, \forall\xi \in \triangle$. Let $\xi \in \triangle$ and let $\delta_\xi, \gamma_\xi \in \mathfrak{U}_\xi$ be such that $\delta_\xi \diamond_\xi \gamma_\xi \in \mathfrak{S}_\xi$ and $\gamma_\xi \in \mathfrak{S}_\xi$. Then $f_{\delta_\xi}, f_{\gamma_\xi} \in \prod_{\xi\in\triangle} \mathfrak{U}_\xi$ and $f_{\delta_\xi \diamond_\xi \gamma_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$ and $f_{\gamma_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. By Lemma 2.5, we have $f_{\delta_\xi} \otimes f_{\gamma_\xi} = f_{\delta_\xi \diamond_\xi \gamma_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. By (1.10), we have $f_{\delta_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. By (2.2), we have $\delta_\xi \in \mathfrak{S}_\xi$. As a result, $\mathfrak{S}_\xi$ is an ideal of $\mathfrak{U}_\xi, \forall\xi \in \triangle$. □

Theorem 2.11. *Let $\mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ be a BP-algebra and $\mathfrak{S}_\xi \subseteq \mathfrak{U}_\xi, \forall\xi \in \triangle$. Then $\mathfrak{S}_\xi$ is a T-ideal of $\mathfrak{U}_\xi, \forall\xi \in \triangle$ if and only if $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is a T-ideal of the EDP BP-algebra $\prod_{\xi\in\triangle} \mathfrak{U}_\xi = (\prod_{\xi\in\triangle} \mathfrak{U}_\xi; \otimes, (0_\xi)_{\xi\in\triangle})$.*

Proof. Suppose $\mathfrak{S}_\xi$ is a T-ideal of $\mathfrak{U}_\xi, \forall\xi \in \triangle$. Then $0_\xi \in \mathfrak{S}_\xi, \forall\xi \in \triangle$, so $(0_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi \neq \emptyset$. Let $(\delta_\xi)_{\xi\in\triangle}, (\gamma_\xi)_{\xi\in\triangle}, (\zeta_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{U}_\xi$ be such that $((\delta_\xi)_{\xi\in\triangle} \otimes (\gamma_\xi)_{\xi\in\triangle}) \otimes (\zeta_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$ and $(\gamma_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. Then $((\delta_\xi \diamond_\xi \gamma_\xi) \diamond_\xi \zeta_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. Thus $(\delta_\xi \diamond_\xi \gamma_\xi) \diamond_\xi \zeta_\xi \in \mathfrak{S}_\xi$ and $\gamma_\xi \in \mathfrak{S}_\xi$, it follows from (1.11) that $\delta_\xi \diamond_\xi \zeta_\xi \in \mathfrak{S}_\xi, \forall\xi \in \triangle$. Thus $(\delta_\xi)_{\xi\in\triangle} \otimes (\zeta_\xi)_{\xi\in\triangle} = (\delta_\xi \diamond_\xi \zeta_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. As a result, $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is a T-ideal of $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$.

On the other hand, suppose that $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is a T-ideal of $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$. Then $(0_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$, so $0_\xi \in \mathfrak{S}_\xi \neq \emptyset, \forall\xi \in \triangle$. Let $\xi \in \triangle$ and let $\delta_\xi, \gamma_\xi, \zeta_\xi \in \mathfrak{U}_\xi$ be such that $(\delta_\xi \diamond_\xi \gamma_\xi) \diamond_\xi \zeta_\xi \in \mathfrak{S}_\xi$ and $\gamma_\xi \in \mathfrak{S}_\xi$. Then $f_{\delta_\xi}, f_{\gamma_\xi}, f_{\zeta_\xi} \in \prod_{\xi\in\triangle} \mathfrak{U}_\xi$ and $f_{(\delta_\xi \diamond_\xi \gamma_\xi) \diamond_\xi \zeta_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$ and $f_{\gamma_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. By Lemma 2.5, we have $(f_{\delta_\xi} \otimes f_{\gamma_\xi}) \otimes f_{\zeta_\xi} = f_{(\delta_\xi \diamond_\xi \gamma_\xi) \diamond_\xi \zeta_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. Since $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is a T-ideal of $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$ and by Lemma 2.5, we have $f_{\delta_\xi \diamond_\xi \zeta_\xi} = f_{\delta_\xi} \otimes f_{\zeta_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. By (2.2), we have $\delta_\xi \diamond_\xi \zeta_\xi \in \mathfrak{S}_\xi$. As a result, $\mathfrak{S}_\xi$ is a T-ideal of $\mathfrak{U}_\xi, \forall\xi \in \triangle$. □

Theorem 2.12. *Let $\mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ be a BP-algebra and $\mathfrak{S}_\xi \subseteq \mathfrak{U}_\xi, \forall\xi \in \triangle$. Then $\mathfrak{S}_\xi$ is a closed ideal of $\mathfrak{U}_\xi, \forall\xi \in \triangle$ if and only if $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is a closed ideal of the EDP BP-algebra $\prod_{\xi\in\triangle} \mathfrak{U}_\xi = (\prod_{\xi\in\triangle} \mathfrak{U}_\xi; \otimes, (0_\xi)_{\xi\in\triangle})$.*

Proof. By Theorem 2.10, we are left to prove that $\mathfrak{U}_\xi$ satisfies (1.12), $\forall\xi \in \triangle$ if and only if $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$ satisfies (1.12).

Suppose $\mathfrak{S}_\xi$ satisfies (1.12), $\forall\xi \in \triangle$. Let $(\delta_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. Then $\delta_\xi \in \mathfrak{S}_\xi, \forall\xi \in \triangle$. By (1.12), we have $0_\xi \diamond_\xi \delta_\xi \in \mathfrak{S}_\xi, \forall\xi \in \triangle$ and so $(0_\xi)_{\xi\in\triangle} \otimes (\delta_\xi)_{\xi\in\triangle} = (0_\xi \diamond_\xi \delta_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. As a result, $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ is a closed ideal of $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$.

On the other hand, suppose that $\prod_{\xi\in\triangle} \mathfrak{S}_\xi$ satisfies (1.12). Let $\xi \in \triangle$ and let $\delta_\xi \in \mathfrak{S}_\xi$. Then $f_{\delta_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. By (1.12), we have $(0_\xi)_{\xi\in\triangle} \otimes f_{\delta_\xi} \in \prod_{\xi\in\triangle} \mathfrak{S}_\xi$. Now, $\forall\lambda \in \triangle$,

$$((0_\xi)_{\xi\in\triangle} \otimes f_{\delta_\xi})(\lambda) = \begin{cases} 0_\xi \diamond_\xi \delta_\xi & \text{if } \lambda = \xi \\ 0_\lambda \diamond_\lambda 0_\lambda & \text{otherwise} \end{cases},$$

we have $0_\xi \diamond_\xi \delta_\xi \in \mathfrak{S}_\xi$. As a result, $\mathfrak{S}_\xi$ is a closed ideal of $\mathfrak{U}_\xi, \forall\xi \in \triangle$. □

Theorem 2.13. *Let $\forall\xi \in \triangle, \mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ be a BP-algebra. Then $\mathfrak{U}_\xi$ is 0_ξ-commutative, $\forall\xi \in \triangle$ if and only if $\prod_{\xi\in\triangle} \mathfrak{U}_\xi = (\prod_{\xi\in\triangle} \mathfrak{U}_\xi; \otimes, (0_\xi)_{\xi\in\triangle})$ is $(0_\xi)_{\xi\in\triangle}$-commutative.*

Proof. By Theorem 2.6, we are left to prove that $\mathfrak{U}_\xi$ satisfies (1.13), $\forall\xi \in \triangle$ if and only if $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$ satisfies (1.13).

Suppose $\mathfrak{U}_\xi$ satisfies (1.13), $\forall\xi \in \triangle$. Let $(\delta_\xi)_{\xi\in\triangle}, (\gamma_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{U}_\xi$. Then $\delta_\xi, \gamma_\xi \in \mathfrak{U}_\xi, \forall\xi \in \triangle$. By (1.13), we have $\delta_\xi \diamond_\xi (0_\xi \diamond_\xi \gamma_\xi) = \gamma_\xi \diamond_\xi (0_\xi \diamond_\xi \delta_\xi), \forall\xi \in \triangle$. Thus

$$\begin{aligned} &(\delta_\xi)_{\xi\in\triangle} \otimes ((0_\xi)_{\xi\in\triangle} \otimes (\gamma_\xi)_{\xi\in\triangle}) \\ &= (\delta_\xi \diamond_\xi (0_\xi \diamond_\xi \gamma_\xi))_{\xi\in\triangle} \\ &= (\gamma_\xi \diamond_\xi (0_\xi \diamond_\xi \delta_\xi))_{\xi\in\triangle} \\ &= (\gamma_\xi)_{\xi\in\triangle} \otimes ((0_\xi)_{\xi\in\triangle} \otimes (\delta_\xi)_{\xi\in\triangle}). \end{aligned}$$

As a result, $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$ is $(0_\xi)_{\xi\in\triangle}$-commutative.

On the other hand, suppose that $\prod_{\xi\in\triangle} \mathfrak{U}_\xi$ satisfies (1.13). Let $\xi \in \triangle$. Let $\delta_\xi, \gamma_\xi \in \mathfrak{U}_\xi$. Then $f_{\delta_\xi}, f_{\gamma_\xi} \in \prod_{\xi\in\triangle} \mathfrak{U}_\xi$. By

(1.13), we have $f_{\delta_\xi}\otimes((0_\xi)_{\xi\in\triangle}\otimes f_{\gamma_\xi}) = f_{\gamma_\xi}\otimes((0_\xi)_{\xi\in\triangle}\otimes f_{\delta_\xi})$. Now, $\forall\lambda\in\triangle$,

$$(f_{\delta_\xi}\otimes((0_\xi)_{\xi\in\triangle}\otimes f_{\gamma_\xi}))(\lambda) = \begin{cases}\delta_\xi \diamond_\xi (0_\xi \diamond_\xi \gamma_\xi) & \text{if } \lambda=\xi\\ 0_\lambda \diamond_\lambda (0_\lambda \diamond_\lambda 0_\lambda) & \text{otherwise}\end{cases}$$

and $\forall\lambda\in\triangle$,

$$(f_{\gamma_\xi}\otimes((0_\xi)_{\xi\in\triangle}\otimes f_{\delta_\xi}))(\lambda) = \begin{cases}\gamma_\xi \diamond_\xi (0_\xi \diamond_\xi \delta_\xi) & \text{if } \lambda=\xi\\ 0_\lambda \diamond_\lambda (0_\lambda \diamond_\lambda 0_\lambda) & \text{otherwise}\end{cases},$$

this implies that

$$\delta_\xi \diamond_\xi (0_\xi \diamond_\xi \gamma_\xi) = \gamma_\xi \diamond_\xi (0_\xi \diamond_\xi \delta_\xi).$$

As a result, $\mathfrak{U}_\xi$ is 0_ξ-commutative, $\forall\xi\in\triangle$. □

As well as that, we talk about a number of BP-homomorphism theorems in terms of the EDP of BP-algebras.

Definition 2.14. [4] Let $\mathfrak{U}_\xi = (\mathfrak{U}_\xi;\diamond_\xi)$ and $\mathfrak{S}_\xi = (\mathfrak{S}_\xi;\circ_\xi)$ be algebras and $\varpi_\xi : \mathfrak{U}_\xi \to \mathfrak{S}_\xi$ be a function, $\forall\xi\in\triangle$. Define the function $\varpi : \prod_{\xi\in\triangle}\mathfrak{U}_\xi \to \prod_{\xi\in\triangle}\mathfrak{S}_\xi$ given by

$$(\forall(\delta_\xi)_{\xi\in\triangle}\in\prod_{\xi\in\triangle}\mathfrak{U}_\xi)(\varpi(\delta_\xi)_{\xi\in\triangle} = (\varpi_\xi(\delta_\xi))_{\xi\in\triangle}). \quad (2.3)$$

Then, $\varpi : \prod_{\xi\in\triangle}\mathfrak{U}_\xi \to \prod_{\xi\in\triangle}\mathfrak{S}_\xi$ is a function. (see [4])

Theorem 2.15. *[4] Let $\mathfrak{U}_\xi = (\mathfrak{U}_\xi;\diamond_\xi)$ and $\mathfrak{S}_\xi = (\mathfrak{S}_\xi;\circ_\xi)$ be algebras and $\varpi_\xi : \mathfrak{U}_\xi \to \mathfrak{S}_\xi$ be a function, $\forall\xi\in\triangle$.*

(i) *$\forall\xi\in\triangle, \varpi_\xi$ is one-one if and only if ϖ is one-one,*

(ii) *$\forall\xi\in\triangle, \varpi_\xi$ is onto if and only if ϖ is onto,*

(iii) *$\forall\xi\in\triangle, \varpi_\xi$ is bijective if and only if ϖ is bijective.*

Theorem 2.16. *Let $\mathfrak{U}_\xi = (\mathfrak{U}_\xi;\diamond_\xi,0_\xi)$ and $\mathfrak{S}_\xi = (\mathfrak{S}_\xi;\circ_\xi,1_\xi)$ be BP-algebras and $\varpi_\xi : \mathfrak{U}_\xi \to \mathfrak{S}_\xi$ be a function, $\forall\xi\in\triangle$. Then*

(i) *$\forall\xi\in\triangle, \varpi_\xi$ is a BP-homomorphism if and only if ϖ is a BP-homomorphism,*

(ii) *$\forall\xi\in\triangle, \varpi_\xi$ is a BP-monomorphism if and only if ϖ is a BP-monomorphism,*

(iii) *$\forall\xi\in\triangle, \varpi_\xi$ is a BP-epimorphism if and only if ϖ is a BP-epimorphism,*

(iv) *$\forall\xi\in\triangle, \varpi_\xi$ is a BP-isomorphism if and only if ϖ is a BP-isomorphism,*

(v) *$\ker\varpi = \prod_{\xi\in\triangle}\ker\varpi_\xi$ and $\varpi(\prod_{\xi\in\triangle}\mathfrak{U}_\xi) = \prod_{\xi\in\triangle}\varpi_\xi(\mathfrak{U}_\xi)$.*

Proof. (i) Suppose $\forall\xi\in\triangle, \varpi_\xi$ is a BP-homomorphism. Let $(\delta_\xi)_{\xi\in\triangle}, (\delta'_\xi)_{\xi\in\triangle}\in\prod_{\xi\in\triangle}\mathfrak{U}_\xi$. Then

$$\begin{aligned}\varpi((\delta_\xi)_{\xi\in\triangle}\otimes(\delta'_\xi)_{\xi\in\triangle}) &= \varpi(\delta_\xi\diamond_\xi\delta'_\xi)_{\xi\in\triangle}\\ &= (\varpi_\xi(\delta_\xi\diamond_\xi\delta'_\xi))_{\xi\in\triangle}\\ &= (\varpi_\xi(\delta_\xi)\diamond_\xi\varpi_\xi(\delta'_\xi))_{\xi\in\triangle}\\ &= (\varpi_\xi(\delta_\xi))_{\xi\in\triangle}\otimes(\varpi_\xi(\delta'_\xi))_{\xi\in\triangle}\\ &= \varpi(\delta_\xi)_{\xi\in\triangle}\otimes\varpi(\delta'_\xi)_{\xi\in\triangle}.\end{aligned}$$

As a result, ϖ is a BP-homomorphism.

On the other hand, suppose that ϖ is a BP-homomorphism. Let $\xi\in\triangle$. Let $\delta_\xi,\gamma_\xi\in\mathfrak{U}_\xi$. Then $f_{\delta_\xi}, f_{\gamma_\xi}\in\prod_{\xi\in\triangle}\mathfrak{U}_\xi$. Since ϖ is a BP-homomorphism, we have $\varpi(f_{\delta_\xi}\otimes f_{\gamma_\xi}) = \varpi(f_{\delta_\xi})\otimes\varpi(f_{\gamma_\xi})$. Since $\forall\lambda\in\triangle$,

$$(f_{\delta_\xi}\otimes f_{\gamma_\xi})(\lambda) = \begin{cases}\delta_\xi\diamond_\xi\gamma_\xi & \text{if } \lambda=\xi\\ 0_\lambda\diamond_\lambda 0_\lambda & \text{otherwise}\end{cases},$$

we have $\forall\lambda\in\triangle$,

$$\varpi(f_{\delta_\xi}\otimes f_{\gamma_\xi})(\lambda) = \begin{cases}\varpi_\xi(\delta_\xi\diamond_\xi\gamma_\xi) & \text{if } \lambda=\xi\\ \varpi_\lambda(0_\lambda\diamond_\lambda 0_\lambda) & \text{otherwise}\end{cases}. \quad (2.4)$$

Since $\forall\lambda\in\triangle$,

$$\varpi(f_{\delta_\xi})(\lambda) = \begin{cases}\varpi_\xi(\delta_\xi) & \text{if } \lambda=\xi\\ \varpi_\lambda(0_\lambda) & \text{otherwise}\end{cases}$$

and $\forall\lambda\in\triangle$,

$$\varpi(f_{\gamma_\xi})(\lambda) = \begin{cases}\varpi_\xi(\gamma_\xi) & \text{if } \lambda=\xi\\ \varpi_\lambda(0_\lambda) & \text{otherwise}\end{cases},$$

we have $\forall\lambda\in\triangle$,

$$(\varpi(f_{\delta_\xi})\otimes\varpi(f_{\gamma_\xi}))(\lambda) = \begin{cases}\varpi_\xi(\delta_\xi)\circ_\xi\varpi_\xi(\gamma_\xi) & \text{if } \lambda=\xi\\ \varpi_\lambda(0_\lambda)\circ_\lambda\varpi_\lambda(0_\lambda) & \text{otherwise}\end{cases}. \quad (2.5)$$

By (2.4) and (2.5), we have $\varpi_\xi(\delta_\xi\diamond_\xi\gamma_\xi) = \varpi_\xi(\delta_\xi)\circ_\xi\varpi_\xi(\gamma_\xi)$. As a result, ϖ_ξ is a BP-homomorphism, $\forall\xi\in\triangle$.

(ii)-(iv) It follows naturally from (i) and Theorem 2.15 (i)-(iii).

(v) Let $(\delta_\xi)_{\xi\in\triangle}\in\prod_{\xi\in\triangle}\mathfrak{U}_\xi$. Then

$$\begin{aligned}(\delta_\xi)_{\xi\in\triangle}\in\ker\varpi &\Leftrightarrow \varpi(\delta_\xi)_{\xi\in\triangle} = (1_\xi)_{\xi\in\triangle}\\ &\Leftrightarrow (\varpi_\xi(\delta_\xi))_{\xi\in\triangle} = (1_\xi)_{\xi\in\triangle}\\ &\Leftrightarrow \varpi_\xi(\delta_\xi) = 1_\xi, \forall\xi\in\triangle\\ &\Leftrightarrow \delta_\xi\in\ker\varpi_\xi, \forall\xi\in\triangle\\ &\Leftrightarrow (\delta_\xi)_{\xi\in\triangle}\in\prod_{\xi\in\triangle}\ker\varpi_\xi.\end{aligned}$$

As a result, $\ker\varpi = \prod_{\xi\in\triangle}\ker\varpi_\xi$. Now,

$$\begin{aligned}&(\gamma_\xi)_{\xi\in\triangle}\in\varpi(\prod_{\xi\in\triangle}\mathfrak{U}_\xi)\\ &\Leftrightarrow \exists(\delta_\xi)_{\xi\in\triangle}\in\prod_{\xi\in\triangle}\mathfrak{U}_\xi \text{ s.t. } (\gamma_\xi)_{\xi\in\triangle} = \varpi(\delta_\xi)_{\xi\in\triangle}\\ &\Leftrightarrow \exists(\delta_\xi)_{\xi\in\triangle}\in\prod_{\xi\in\triangle}\mathfrak{U}_\xi \text{ s.t. } (\gamma_\xi)_{\xi\in\triangle} = (\varpi_\xi(\delta_\xi))_{\xi\in\triangle}\\ &\Leftrightarrow \exists\delta_\xi\in\mathfrak{U}_\xi \text{ s.t. } \gamma_\xi = \varpi_\xi(\delta_\xi)\in\varpi(\mathfrak{U}_\xi), \forall\xi\in\triangle\\ &\Leftrightarrow (\gamma_\xi)_{\xi\in\triangle}\in\prod_{\xi\in\triangle}\varpi_\xi(\mathfrak{U}_\xi).\end{aligned}$$

As a result, $\varpi(\prod_{\xi\in\triangle}\mathfrak{U}_\xi) = \prod_{\xi\in\triangle}\varpi_\xi(\mathfrak{U}_\xi)$. □

Theorem 2.17. *Let* $\mathfrak{U}_\xi = (\mathfrak{U}_\xi; \diamond_\xi, 0_\xi)$ *and* $\mathfrak{S}_\xi = (\mathfrak{S}_\xi; \circ_\xi, 1_\xi)$ *be BP-algebras and* $\varpi_\xi : \mathfrak{U}_\xi \to \mathfrak{S}_\xi$ *be a function,* $\forall \xi \in \triangle$. *Then*

(i) $\forall \xi \in \triangle, \varpi_\xi$ *is an anti-BP-homomorphism if and only if* ϖ *is an anti-BP-homomorphism,*

(ii) $\forall \xi \in \triangle, \varpi_\xi$ *is an anti-BP-monomorphism if and only if* ϖ *is an anti-BP-monomorphism,*

(iii) $\forall \xi \in \triangle, \varpi_\xi$ *is an anti-BP-epimorphism if and only if* ϖ *is an anti-BP-epimorphism,*

(iv) $\forall \xi \in \triangle, \varpi_\xi$ *is an anti-BP-isomorphism if and only if* ϖ *is an anti-BP-isomorphism.*

Proof. (i) Suppose $\forall \xi \in \triangle, \varpi_\xi$ is an anti-BP-homomorphism. Let $(\delta_\xi)_{\xi\in\triangle}, (\delta'_\xi)_{\xi\in\triangle} \in \prod_{\xi\in\triangle} \mathfrak{U}_\xi$. Then

$$\begin{aligned}\varpi((\delta_\xi)_{\xi\in\triangle} \otimes (\delta'_\xi)_{\xi\in\triangle}) &= \varpi(\delta_\xi \diamond_\xi \delta'_\xi)_{\xi\in\triangle} \\ &= (\varpi_\xi(\delta_\xi \diamond_\xi \delta'_\xi))_{\xi\in\triangle} \\ &= (\varpi_\xi(\delta'_\xi) \diamond_\xi \varpi_\xi(\delta_\xi))_{\xi\in\triangle} \\ &= (\varpi_\xi(\delta'_\xi))_{\xi\in\triangle} \otimes (\varpi_\xi(\delta_\xi))_{\xi\in\triangle} \\ &= \varpi(\delta'_\xi)_{\xi\in\triangle} \otimes \varpi(\delta_\xi)_{\xi\in\triangle}.\end{aligned}$$

As a result, ϖ is an anti-BP-homomorphism.

On the other hand, suppose that ϖ is an anti-BP-homomorphism. Let $\xi \in \triangle$. Let $\delta_\xi, \gamma_\xi \in \mathfrak{U}_\xi$. Then $f_{\delta_\xi}, f_{\gamma_\xi} \in \prod_{\xi\in\triangle} \mathfrak{U}_\xi$. Since ϖ is an anti-BP-homomorphism, we have $\varpi(f_{\delta_\xi} \otimes f_{\gamma_\xi}) = \varpi(f_{\gamma_\xi}) \otimes \varpi(f_{\delta_\xi})$. Since $\forall \lambda \in \triangle$,

$$(f_{\delta_\xi} \otimes f_{\gamma_\xi})(\lambda) = \begin{cases} \delta_\xi \diamond_\xi \gamma_\xi & \text{if } \lambda = \xi \\ 0_\lambda \diamond_\lambda 0_\lambda & \text{otherwise} \end{cases},$$

we have $\forall \lambda \in \triangle$,

$$\varpi(f_{\delta_\xi} \otimes f_{\gamma_\xi})(\lambda) = \begin{cases} \varpi_\xi(\delta_\xi \diamond_\xi \gamma_\xi) & \text{if } \lambda = \xi \\ \varpi_\lambda(0_\lambda \diamond_\lambda 0_\lambda) & \text{otherwise} \end{cases}. \quad (2.6)$$

Since $\forall \lambda \in \triangle$,

$$\varpi(f_{\gamma_\xi})(\lambda) = \begin{cases} \varpi_\xi(\gamma_\xi) & \text{if } \lambda = \xi \\ \varpi_\lambda(0_\lambda) & \text{otherwise} \end{cases}$$

and $\forall \lambda \in \triangle$,

$$\varpi(f_{\delta_\xi})(\lambda) = \begin{cases} \varpi_\xi(\delta_\xi) & \text{if } \lambda = \xi \\ \varpi_\lambda(0_\lambda) & \text{otherwise} \end{cases},$$

we have $\forall \lambda \in \triangle$,

$$(\varpi(f_{\gamma_\xi}) \otimes \varpi(f_{\delta_\xi}))(\lambda) = \begin{cases} \varpi_\xi(\gamma_\xi) \circ_\xi \varpi_\xi(\delta_\xi) & \text{if } \lambda = \xi \\ \varpi_\lambda(0_\lambda) \circ_\lambda \varpi_\lambda(0_\lambda) & \text{otherwise} \end{cases}. \quad (2.7)$$

By (2.6) and (2.7), we have $\varpi_\xi(\delta_\xi \diamond_\xi \gamma_\xi) = \varpi_\xi(\gamma_\xi) \circ_\xi \varpi_\xi(\delta_\xi)$. As a result, ϖ_ξ is an anti-BP-homomorphism, $\forall \xi \in \triangle$.

(ii)-(iv) It follows naturally from (i) and Theorem 2.15 (i)-(iii). □

3 Conclusions and Future Work

The idea of the DP of an infinite family of BP-algebras, or the EDP, which is a broad concept of the DP, is introduced in this work. We established that BP-algebras also include their EDPs. Additionally, we proposed the idea of the WDP of BP-algebras. We established that the WDP of BP-algebras is a subalgebra and that the EDP of subalgebras (resp., ideals, T-ideals, closed ideals, and 0_ξ-commutative) is likewise a subalgebra of the EDP BP-algebras. In light of the EDP BP-algebras, we have established a number of essential theorems of (anti-)BP-homomorphisms.

We may use the idea of the EDP of BP-algebras presented in this article to examine the EDP in other algebraic systems. IUP-algebras, as defined by Iampan et al. [8], will be used to research the EDPs and WDPs that we will examine in the future.

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