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Construction of the A_4 Graph of Mathieu Group M_{11}

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Abstract Suppose that G is a group and X is a subset of G. Then, the A_4 graph of a group G, denoted by $A_4(G,X)$, is the simple undirected graph in which two distinct vertices $x,y \in X$ are connected to each other by an edge if and only if both vertices satisfy $xy^{-1} = yx^{-1}$. The main contribution of this paper is to construct the A_4 graph using the elements of Mathieu group, M_{11}. Additionally, the connectivity of $A_4(G,X)$ has been proven as a connected graph. Finally, an open problem is highlighted in addressing future research.

Keywords A_4 Graph, Finite Simple Group, Simple Graph, Mathieu Group, Conjugacy Class, Connectivity, Diameter

1. Introduction

Let G be a finite group and X be a subset of G. The actions of a group play the most pivotal role in a construction of a graph of group so-called A_4 graph. The graph can be denoted by $A_4(G,X)$ where it consists of vertex set X and two distinct vertices $x,y \in X$ are adjacent if and only if $xy^{-1} = yx^{-1}$. This A_4 graph is dependent on a specific set of generators. When examining the algebraic characteristics of finite simple groups, the involution components are crucial. Nevertheless, the elements of order 3 are equally essential in finite simple groups. Let $t \in G$ and $x \in X$. The alternating groups A_4 can be formed by a class of elements of order 3 such that the subgroup $\langle t,x\rangle \cong A_4$. For example, suppose that there is a symmetric group of degree 6, $Sym(6)$. A fixed element $t = (1,2,3)(4,5,6) \in G$ of order 3 is selected, thus 40 elements of the conjugacy class X have been produced. If $x = (1,5,2)(3,4,6)$ be one of the elements in X, then it purposely generates a subgroup generated by elements t,x, so-called $\langle t,x\rangle \cong A_4$.

Recently, there are deep investigations of employing various group properties which are used for the characterization of graphs depending on their structures. This approach becomes an exciting research topic in highlighting behaviors of the A_4 graph within a finite group. Aside from this, various graphs have also been associated with finite groups such as commuting graphs, non-commuting graphs, symmetric graphs, degree graphs, prime graphs, and power graphs, [1-6].

Interest in A_4 graph is stemming from Aubad's work [7] who initiated a new type of graph, denoted by $A_4(G,X)$. During this work, certain features have been obtained, for instances, the descriptions of disc structures and the diameter of $A_4(G,X)$ is equal to 5, when G is an exceptional Lie type group of characteristics 2, and X is a G-conjugacy class of order 3.

In a work of [8], an extended investigation was done by selecting a Higman-Sims group, $G \cong HS$ as well as a G-conjugacy class of elements of order 3. This research scrutinized the disc structures as well as the diameter of $A_4(G,X)$. One of the results discussed the diameter of $A_4(G,X)$ is equal to 3 with $G \cong HS$ and X is the G-conjugacy class of type $3A$. Meanwhile, the outcome observed in [9] demonstrates the disc structures, diameter, girth and clique number of $A_4(G,X)$ when G is either Tits group T, or the Mathieu group M_{20}, and X is the conjugacy class of elements of order 3. As the results, the diameter of $A_4(T,3A)$ and $A_4(M_{20},3A)$ are equal to 4 and 3, respectively.

This current work intends to determine the connectivity of the graph $A_4(G,X)$ by choosing the group $G \cong M_{11}$

and G-conjugacy class of elements of order 3, which is $X = 3A$. This paper is organized as follows. Detailed background and notations of this research will be given in Section 2. In Section 3, methods used in this study are discussed, followed by the results achieved for the graph $A_4(G, X)$ in Section 4. Finally, the study will be summarized in Section 5.

2. Related Works

Along this work, the concepts of graph and group have been applied throughout this study. Generally, a graph Γ that has a path starting from one vertex. It will connect with all the other vertices, then the graph is connected. Otherwise, it is a disconnected graph. A graph is said to be simple and undirected if the set of nodes $V(\Gamma)$ are linked by lines called edges $E(\Gamma)$. The number of edges connecting two vertices u and v in a shortest path is known as the distance, $d(u, v)$. The largest number of two vertices is called the graph diameter [10].

Now, some useful notations in group theory are introduced [11]. The Mathieu group M_{11} is one of the five sporadic simple groups which multiplies transitive permutation groups on 11 points. Notice that there is a group action of G on a finite set X. Let $t \in G$ is of order 3 and be an initial point of the graph $A_4(G, X)$. Evidently, G is acting by conjugation, thus $X = t^G$. The group G also induces graph automorphisms of $A_4(G, X)$ and is transitive on its vertices. As $x \in X$ and $i \in \mathbb{N}$, $\Delta_i(x)$ demonstrates the vertex set of $A_4(G, X)$ or it can illustrate $d(t, x) = i$. Now, let C be a G-conjugacy class. It is implemented in the set $X_C = \{x \in X : tx \in C\}$ under the action X on the centralizer $C_G(t)$. The size of the set X_C is given by

$$|X_C| = \frac{|G|}{|C_G(t)||C_G(z)|}\sum_{\chi \in Irr(G)} \frac{\chi(t)^2\chi(z)}{\chi(1)},$$

for any $z \in C$ [12]. Besides that, the Orbit-Stabilizer theorem can be used to determine the index of orbit for an element $x \in X$. Let $Orb(x)$ denotes the orbit of x and $Stab(x)$ is its stabilizer in G. There is the index of $Stab(x)$ in G, namely, $[G: Stab(x)]$. Then,

$$|Orb(x)| = [G: Stab(x)] = \frac{|G|}{|Stab(x)|}.$$

It may be simple to determine the conjugacy class of a specific element, hence, it is useful to distinguish the set X_C for all classes in a particular order.

3. Methodology

In the analysis of A_4 graph, the fixed vertex $t \in G$ was selected from the Mathieu group M_{11}, whereas X was a G-conjugacy class of order 3, known as $X = 3A$. The conjugacy class X_{3A} has 440 elements and the centralizer $C_G(t)$ is of order 18. By adapting computational methods in [12], Algorithm 1 has been performed to calculate whether $(t, x) \leq k$.

In essence, the algorithm strives to build the A_4 graph when $xy^{-1} = yx^{-1}$, for $x, y \in X$. It is observed that Algorithm 1 only uses the group G to choose a random element t, compute products when creating the condition of $A_4(G, X)$ and the centralizing elements, to decide the order of $\Delta_i(t)$. Calculating $\Delta_i(t)$ repeatedly about $1 \leq i \leq k$, then a connected component with a set of vertices centralizing at t is of order $|X|$. Else, multiple connected components of identical size will be produced. For the useful information regarding group generators and notations, those can be gleaned from the ATLAS [13]. The website includes generators for numerous finite simple groups in matrix and permutation representations. In addition, the small-degree permutation representations are only considered here. Utilization of the mathematical package, MAGMA [14] has operating this work successfully.

Algorithm 1	**Centralizing Element Set**
Input:	$G \leftarrow$ the Mathieu group $t \leftarrow$ a random element of order three in G
	1. $nX \leftarrow$ order of conjugacy class of G
	2. $nCt \leftarrow$ order of centralizer of G
	3. $d_1 \leftarrow$ set of elements in which $tx^{-1} = xt^{-1}$ excluding t
	4. $nd_1 \leftarrow$ order of set d_1
	5. **for** y in d_i and i in $[2..k]$ **do** $xy^{-1} = yx^{-1}$ **end for**
	6. $d_i \leftarrow$ set of elements for $2 \leq i \leq k$ excluding t and d_1
Output:	7. $nd_i \leftarrow$ order of set d_i $sd_i \leftarrow$ sum of set of elements in d_j where $1 \leq j \leq k$

Table 1. Connectivity of $A_4(G,X)$ for $G = M_{11}$ and $X = 3A$.

	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$
$\lvert\Delta_i(t)\rvert$	18	144	266	11
$tx \in C$	$3A$	$2A, 5A, 11A, 11B$	$1A, 3A, 4A, 5A, 6A, 8A, 8B$	$3A, 4A$
$\lvert\langle t,x\rangle\rvert$	12	12, 60, 660	3, 9, 24, 360, 660, 7920	9, 24

4. Results and Discussion

The main results pertaining to the A_4 graph for $G = M_{11}$ and $X = 3A$ are summarized in this section. The diameter of $A_4(G,X)$ is presented in Theorem 1. While the disc structure of this graph can be found in Lemma 1.

Theorem 1

Let $G = M_{11}$, $X = 3A$ and let $A_4(G,X)$ be the A_4 graph for the elements of order 3 on X. The graph $A_4(G,X)$ has diameter exactly 4.

Proof of Theorem 1

The support given by Algorithm 1 makes the proof of Theorem 1 straightforward. Consulting Table 1 yields a connected graph of $A_4(G,X)$ such that $X = \{t\} \cup \Delta_i(t)$ belongs to 440 vertices for $1 \leq i \leq 4$, where each disc contains 18, 144, 266 and 11 vertices, respectively. Thus, the diameter of $A_4(G,X)$ is equal to 4.

Lemma 1

Consider $G = M_{11}$, $X = 3A$ and $t, x \in X$. Let m and H be the order of tx and subgroup generated by $t, x \in X$, respectively. Then,

(1) $x \in \Delta_1(t)$ if and only if $m = 3$ and $H \cong A_4$.

(2) Let $\xi = |tx^{-1}| = |xt^{-1}| = |t^{-1}x|$. For the cases $tx \in 4A \cup 5A$, one of the following statements is satisfied:

(a) If $tx \in 4A$,

$$x \in \begin{cases} \Delta_3(t), & if\ \ \xi \in 5A, \\ \Delta_4(t), & if\ \ \xi \in 6A. \end{cases}$$

(b) If $tx \in 5A$,

$$x \in \begin{cases} \Delta_2(t), & if\ \ \xi \in 6A, \\ \Delta_3(t), & if\ \ \xi \in 11A \cup 11B. \end{cases}$$

Proof

(1) Since H is isomorphic to A_4 and tx has order 3 which is equivalent to $tx^{-1} = xt^{-1}$, then it is clear that $x \in \Delta_1(t)$.

(2) By looking at Table 1, observe that $2 \leq d(t,x) \leq 4$ whenever $tx \in 4A \cup 5A$. In such instances, the use of an inverted element of a product of two X's, such that $\xi = |tx^{-1}| = |xt^{-1}| = |t^{-1}x|$ helps to differentiate them from the discs in $A_4(G,X)$. Therefore, the statements (a) and (b) clearly hold.

Example 1

Assume that $t = (1,6,5)(2,3,9)(4,8,7) \in M_{11}$ be an element and considering $X = t^G = 3A$ for the A_4 graph. By taking an element $x = (1,11,2)(3,10,5)(4,7,8)$ in X, the computation of $tx \in C$ yields a product of the set $X_C = 4A$. To simplify this work, Lemma 1(2a) is applied thus it produces ξ of length 6. Therefore, $x \in \Delta_4(t),\ if\ \ \xi \in 6A$. Now, let's taking a vertex $y \in X$ such that $y = (1,8,6)(2,9,7)(3,11,5)$. Then, it is proven that ξ is of length 5. It can be summarized that $y \in \Delta_3(t),\ if\ \ \xi \in 5A$.

5. Conclusions

The graph $A_4(G,X)$ was constructed by taking the elements of order 3 such that $G = M_{11}$ and $t \in X = 3A$. A connected graph has been obtained with the diameter of 4. Moreover, the properties of conjugacy classes tx and the subgroup H were considered in this work to determine $d(t,x)$ for particular $x \in X$. However, there is an unresolved problem to determine the distance between t and x. It is due to some similar characterizations of $x \in X$, for examples, $tx \in 3A$, $|\langle t,x\rangle| = 9$ and $\langle t,x\rangle$ is isomorphic to an elementary group of order 9 with the following presentation.

$$\langle t, x | t^9 = x^9 = (tx)^9 = (tx^{-1})^9 = (t^{-1}x^{-1})^9 = x^{-9} = (xt^{-1})^9 = t^{-9} = 1\rangle$$

Thus, the distance for this case can only be estimated by $d(t,x) \geq 3$. Therefore, more exploration should be carried out in addressing various group properties to determine the exact disc, $\Delta_i(t)$.

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REFERENCES

[1] Everette and P. Rowley, "Commuting Involution Graphs for 4-Dimensional Projective Symplectic Groups," *Graphs and Combinatorics*, 36(4), 2020, pp. 959 – 1000

[2] N. I. Alimon, N. H. Sarmin and A. Erfanian, "Constructing Non-Commuting Graph of Symmetric Groups by Using Maple's Software," *International Conference on Information Technology, Engineering, Science and Its Applications.* Available at http://dx.doi.org/10.2139/ssrn.3248133/, 2018

[3] J. J. Li, H. C. Liao, Z. P. Lu and W. Y. Zhu, "Symmetric Graph of Prime Valency Associated with Some Almost Simple Groups," *Discrete Mathematics,* 344(112547), 2021, pp. 1 – 12

[4] Z. Akhlaghi, S. Dolfi, E. Pacifici and L. Sanus, "Bounding the number of vertices in the degree graph of a finite group," in *Journal of Pure and Applied Algebra*, 224(2020), 2019, pp. 725 – 731

[5] S. Dolfi, E. Pacifici, L. Sanus and V. Sotomayor, "The Prime Graph on Class Sizes on a Finite Group Has a Bipartite Complement," *Journal of Algebra*, 542(2020), 2019, pp. 35 – 42

[6] P. J. Cameron, P. Manna and R. Mehatari, "On Finite Groups Whose Power Graph is a Cograph," *Journal of Algebra*, 591(2022), 2021, pp. 59 – 74

[7] A. Aubad, "A_4-graph of finite simple groups," in *Iraqi Journal of Science*, 62(1), 2021, pp. 289 – 294

[8] J. Oudah and A. Aubad, "Computational Investigation of A_4-Graphs for Certain Leech Lattice Groups," *Minar Journal*, 3(3), 2021, pp. 51 – 57

[9] Z. H. Msheree, M. M. Abed and W. F. Abid, "Studying the A_4-Graph for Elements of Order 3 in Tits Group T and Mathieu Group M_{20}," *International Journal of Nonlinear Analysis and Applications*, 12(2), 2021, pp. 1 – 6

[10] P. H. Guzzi and S. Roy, "*Preliminaries of Graph Theory. Biological Network Analysis*," Academic Press, 2020, pp. 7 – 24

[11] U. Jetzek, "*Finite Groups and Fields. Galois Fields, Linear Feedback Shift Registers and their Applications*," Hanser, 2018, pp. 17 – 44

[12] Bates, D. Bundy, S. Hart and P. Rowley, "Commuting Involution Graphs for Sporadic Simple Groups," *Journal of Algebra*, 316, 2007, pp. 849 – 868

[13] ATLAS of finite group representations - Version 3. Available at *http://brauer.maths.qmul.ac.uk/Atlas/v3/*, 2021

[14] MAGMA Computer Algebra - Version V2.26-10 (2021). Available at *http://magma.maths.usyd.edu.au/magma/*