Mathematics and Statistics 11(1): 169-178, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110119

Half-sweep Modified SOR Approximation of A Two-dimensional Nonlinear Parabolic Partial Differential Equation

Jackel Vui Lung Chew[1,*], Jumat Sulaiman[2], Andang Sunarto[3], Zurina Patrick[4]

[1]Faculty of Computing and Informatics, Universiti Malaysia Sabah Labuan International Campus, Sungai Pagar, 87000, Labuan F.T., Malaysia
[2]Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Kota Kinabalu, 88400, Sabah, Malaysia
[3]Tadris Matematika, Universitas Islam Negeri (UIN) Fatmawati Sukarno, 38211, Bengkulu, Indonesia
[4]Labuan Faculty of International Finance, Universiti Malaysia Sabah Labuan International Campus, Sungai Pagar, 87000, Labuan F.T., Malaysia

Received September 22, 2022; Revised November16, 2022; Accepted December 9, 2022

Cite This Paper in the Following Citation Styles

(a): *[1] Jackel Vui Lung Chew, Jumat Sulaiman, Andang Sunarto, Zurina Patrick , "Half-sweep Modified SOR Approximation of A Two-dimensional Nonlinear Parabolic Partial Differential Equation," Mathematics and Statistics, Vol. 11, No. 1, pp. 169 - 178, 2023. DOI: 10.13189/ms.2023.110119.*

(b): *Jackel Vui Lung Chew, Jumat Sulaiman, Andang Sunarto, Zurina Patrick (2023). Half-sweep Modified SOR Approximation of A Two-dimensional Nonlinear Parabolic Partial Differential Equation. Mathematics and Statistics, 11(1), 169 - 178. DOI: 10.13189/ms.2023.110119.*

Abstract The sole subject of this numerical analysis was the half-sweep modified successive over-relaxation approach (HSMSOR), which takes the form of an iterative formula. This study computed a class of two-dimensional nonlinear parabolic partial differential equations subject to Dirichlet boundary conditions numerically using the implicit-type finite difference scheme. The computational cost optimization was considered by converting the traditional implicit finite difference approximation into the half-sweep finite difference approximation. The implementation required inner-outer iteration cycles, the second-order Newton method, and a linearization technique. The created HSMSOR is utilized to obtain an approximation of the linearized equations system through the inner iteration cycle. In contrast, the problem's numerical solutions are obtained using the outer iteration cycle. The study examined the local truncation error and the stability, convergence, and method analysis. Results from three initial-boundary value issues showed that the proposed method had competitive computational costs compared to the existing method.

Keywords Two-dimensional, Nonlinear, Partial Differential Equation, Finite Difference, Stability, Convergence, Iterative Method

1. Introduction

Nonlinear parabolic partial differential equations (NPPDE) have successfully modelled various realistic phenomena over the past years. NPPDE was used to develop a mathematical model of plant disease epidemiology [1], sediment transport [2], solar trough collector [3], reactive settling in wastewater treatment [4] and diffusion of the biological population [5]. These nonlinear mathematical models resemble phenomena more than linear partial differential equations. Researchers can describe the phenomena with greater accuracy using the developed models. NPPDE was also used to study the model of American options under the Heston model [6] and European and American options of risks in climate change [7]. By utilizing the nonlinear term from NPPDE, a better prediction can be made, leading to a wiser decision in options pricing.

Several numerical methods have been proposed to obtain valuable information from NPPDE models. For instance, [8] studied a hybrid spectral collocation method to precisely find the approximate solutions of NPPDE in studying gene propagation and transmission of nerve impulses. Then, [9] proposed a numerical method based on a weak L-stable time integration with a nonstandard finite difference method. Next, [10] introduced the combined quintic trigonometric B-spline collocation, finite difference, and Rubin-Graves linearization. Another article [11] investigated the solutions of NPPDE using a cubic B-spline collocation using a non-uniform mesh. Besides that, [12] proposed combining a multi-step Milne method with the central finite difference approach and a weighted linearization technique. Generally, different numerical methods have another degree of accuracy, efficiency, and difficulty in implementation. Moreover, some methods might not be adequate to solve a specific class of NPPDE.

A literature review of various proposed methods of solving NPPDE found a lack of efficient numerical methods that consider lowering computational complexity in simulation. This paper is motivated to present an efficient numerical method called the half-sweep modified successive over-relaxation method (HSMSOR) for solving NPPDE. The earliest study of the HSMSOR method was conducted by [13], in which they examined the method's effectiveness in solving linear systems by the discretized two-dimensional Helmholtz equations. They showed the efficiency level of the HSMSOR method is higher than the standard, modified successive over-relaxation (MSOR) method. Then, the results by [14] showed HSMSOR method has better efficiency than successive over-relaxation methods in solving linear systems after the cubic spline scheme was implemented. Due to the potential of the HSMSOR method, [15,16] applied the method to simulate indoor mobile robots' path planning and agent navigations, respectively. Since nonlinear problems are more realistic to be solved, [17] initiated the investigation of solving one-dimensional NPPDE using their HSMSOR method.

This paper extends the study of the HSMSOR method by solving a higher-dimension nonlinear problem. A class of two-dimensional NPPDE is chosen to illustrate the development of the HSMSOR method. An implicit-type finite difference method is applied to compute numerical solutions of the chosen NPPDE to minimize the computational cost. The minimization of the computational cost is further optimized with the introduction of the half-sweep finite difference approximation, which was proposed by [18]. The HSMSOR method is formulated for the efficient solution of the generated system of algebraic equations. This paper contributes to an efficient HSMSOR method to solve two-dimensional NPPDEs with rigorous proofs of local truncation errors, stability, and convergence. This paper covers the local truncation error of the approximation, the derivation of the half-sweep finite difference approximation equation, and the stability and convergence of the scheme are covered in Section 2. The HSMSOR iterative formula's development is shown in Section 3. The numerical experiment is described in Section 4, along with the results based on initial-boundary value problems for the selected class of NPPDE. The paper's final section finishes with discussion of possible future study topics.

2. Half-sweep Finite Difference Approximation

Half-sweep Finite Difference Scheme

Let's consider the following two-dimensional NPPDE that is famously used to model the flow through a porous medium [19]:

$$\frac{\partial u}{\partial t} = \alpha\left[\frac{\partial}{\partial x}\left(u^{\rho}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u^{\rho}\frac{\partial u}{\partial y}\right)\right], \alpha, \rho \in \mathbb{R} \tag{1}$$

subjects to an initial condition

$$u(x, y, 0) = G_0, 0 \le x \le L_x, 0 \le y \le L_y \tag{2}$$

and Dirichlet boundary conditions

$$u(0, y, t) = a_y, u(L_x, y, t) = b_y, 0 \le t \le L_t \tag{3}$$

And

$$u(x, 0, t) = a_x, u(x, L_y, t) = b_x, 0 < t \le L_t \tag{4}$$

To discretize Eq. (1) using a half-sweep finite difference scheme, let the approximate solution be denoted as $U_{p,q}^{n} = U(x_p, y_q, t_n)$ where $x_p = ph, y_q = qh, t_n = nk$ with uniform spatial and temporal steps $h = L_x/M = L_y/M$ and $k = L_t/N$, and $1 \le p, q \le M-1, 1 \le n \le N$. Then, the time and space derivative terms in Eq. (1) can be approximated using half-sweep finite difference operators defined as follows:

$$\frac{\partial u}{\partial t} \approx \frac{U_{p,q}^{n} - U_{p,q}^{n-1}}{k} \tag{5}$$

$$\frac{\partial u}{\partial x} \approx \frac{U_{p+1,q+1}^{n} - U_{p-1,q-1}^{n}}{2\sqrt{2}h} \tag{6}$$

$$\frac{\partial u}{\partial y} \approx \frac{U_{p-1,q+1}^{n} - U_{p+1,q-1}^{n}}{2\sqrt{2}h} \tag{7}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{U_{p+1,q+1}^{n} - 2U_{p,q}^{n} + U_{p-1,q-1}^{n}}{2h^2} \tag{8}$$

and

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{U_{p-1,q+1}^{n} - 2U_{p,q}^{n} + U_{p+1,q-1}^{n}}{2h^2} \tag{9}$$

Using Eq. (5)-(9) on Eq. (1) and some manipulations, an implicit-type finite difference approximation equation to Eq. (1) can be formulated into

$$
\begin{gathered}
U_{p,q}^{n} - A_1\left(U_{p,q}^{n}\right)^{\rho} U_{p+1,q+1}^{n} + 2A_1\left(U_{p,q}^{n}\right)^{\rho+1} - A_1\left(U_{p,q}^{n}\right)^{\rho} U_{p-1,q-1}^{n} \\
-A_2\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left(U_{p+1,q+1}^{n}\right)^2 + 2A_2\rho\left(U_{p,q}^{n}\right)^{\rho-1} U_{p+1,q+1}^{n} U_{p-1,q-1}^{n} - A_2\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left(U_{p-1,q-1}^{n}\right)^2 \\
-A_3\left(U_{p,q}^{n}\right)^{\rho} U_{p-1,q+1}^{n} + 2A_3\left(U_{p,q}^{n}\right)^{\rho+1} - A_3\left(U_{p,q}^{n}\right)^{\rho} U_{p+1,q-1}^{n} \\
-A_4\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left(U_{p-1,q+1}^{n}\right)^2 + 2A_4\rho\left(U_{p,q}^{n}\right)^{\rho-1} U_{p-1,q+1}^{n} U_{p+1,q-1}^{n} - A_4\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left(U_{p+1,q-1}^{n}\right)^2 = U_{p,q}^{n-1} \qquad (10)
\end{gathered}
$$

where $A_1 = A_3 = \alpha k/2h^2$ and $A_2 = A_4 = \alpha k/8h^2$.

Local Truncation Error

In this section, the local truncation error of Eq. (10) is investigated using Taylor's expansion of unknown points $U_{p,q}^{n-1}, U_{p+1,q+1}^{n}, U_{p-1,q-1}^{n}, U_{p+1,q-1}^{n}$ and $U_{p-1,q+1}^{n}$ as follows:

$$U_{p,q}^{n-1} = U_{p,q}^{n} - k\frac{\partial}{\partial t}U_{p,q}^{n} + \cdots \qquad (11)$$

$$U_{p+1,q+1}^{n} = U_{p,q}^{n} + h\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots \qquad (12)$$

$$U_{p-1,q-1}^{n} = U_{p,q}^{n} + h\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots \qquad (13)$$

$$U_{p+1,q-1}^{n} = U_{p,q}^{n} + h\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots \qquad (14)$$

And

$$U_{p-1,q+1}^{n} = U_{p,q}^{n} + h\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots \qquad (15)$$

Substituting Eq. (11)-(15) into Eq. (10) gives

$$
\begin{gathered}
U_{p,q}^{n} - A_1\left(U_{p,q}^{n}\right)^{\rho}\left(U_{p,q}^{n} + h\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right) \\
+2A_1\left(U_{p,q}^{n}\right)^{\rho+1} - A_1\left(U_{p,q}^{n}\right)^{\rho}\left(U_{p,q}^{n} + h\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right) \\
-A_2\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left(U_{p,q}^{n} + h\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right)^2 \\
+2A_2\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left(U_{p,q}^{n} + h\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right)\left(U_{p,q}^{n} + h\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right) \\
-A_2\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left(U_{p,q}^{n} + h\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right)^2 \\
-A_3\left(U_{p,q}^{n}\right)^{\rho}\left(U_{p,q}^{n} + h\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right) \\
+2A_3\left(U_{p,q}^{n}\right)^{\rho+1} - A_3\left(U_{p,q}^{n}\right)^{\rho}\left(U_{p,q}^{n} + h\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right) \\
-A_4\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left(U_{p,q}^{n} + h\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right)^2 \\
+2A_4\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left(U_{p,q}^{n} + h\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right)\left(U_{p,q}^{n} + h\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right) \\
-A_4\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left(U_{p,q}^{n} + h\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)U_{p,q}^{n} + \frac{h^2}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 U_{p,q}^{n} + \cdots\right) = U_{p,q}^{n} - k\frac{\partial}{\partial t}U_{p,q}^{n} + \cdots^2 \qquad (16)
\end{gathered}
$$

and is then simplified into

$$\frac{\partial}{\partial t}U_{p,q}^{n} + O(k) = \alpha\left(U_{p,q}^{n}\right)^{\rho}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)U_{p,q}^{n} + \alpha\rho\left(U_{p,q}^{n}\right)^{\rho-1}\left[\left(\frac{\partial}{\partial x}U_{p,q}^{n}\right)^2 + \left(\frac{\partial}{\partial y}U_{p,q}^{n}\right)^2\right] + O(h^2) \qquad (17)$$

Hence, the local truncation error of Eq. (10) is

$$\lim_{k,h\to 0}[O(k)+O(h^2)]=0 \tag{18}$$

which shows that the half-sweep finite difference scheme is consistent with Eq. (1), and the accuracy of Eq. (18) is first-order in time and second-order in space.

Stability Analysis

In this section, the stability of Eq. (10) is analyzed using a frozen nonlinear coefficient approach and Fourier analysis. Suppose that the solution $u(x,y,t)$ exists in the domain $0\le x\le L_x, 0\le y\le L_y$ and $0<t\le L_t$. Then, let u^ρ be frozen and denote

$$\mu=\max|u^\rho| \tag{19}$$

Substituting Eq. (19) into Eq. (1) yields

$$\frac{\partial u}{\partial t}=\alpha\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right] \tag{20}$$

Eq. (20) can be rewritten in the form of

$$\frac{\partial u}{\partial t}=\alpha\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right) \tag{21}$$

The corresponding half-sweep finite difference approximation to Eq. (21) is

$$U_{p,q}^n-\lambda\left(U_{p+1,q+1}^n-2U_{p,q}^n+U_{p-1,q-1}^n\right)-\lambda\left(U_{p-1,q+1}^n-2U_{p,q}^n+U_{p+1,q-1}^n\right)=U_{p,q}^{n-1} \tag{22}$$

where $\lambda=\mu\alpha k/2h^2$.

Then, let $U_{p,q}^n=\xi^n e^{i\theta_1 p}e^{i\theta_2 q}$ and substitute it into Eq. (22) to give

$$\xi\left(1-\lambda\left(e^{i(\theta_1+\theta_2)}-2+e^{-i(\theta_1+\theta_2)}\right)-\lambda\left(e^{i(\theta_1-\theta_2)}-2+e^{-i(\theta_1-\theta_2)}\right)\right)=1 \tag{23}$$

Since

$$e^{i(\theta_1+\theta_2)}-2+e^{-i(\theta_1+\theta_2)}=-4\sin^2\frac{(\theta_1+\theta_2)}{2} \tag{24}$$

and

$$e^{i(\theta_1-\theta_2)}-2+e^{-i(\theta_1-\theta_2)}=-4\sin^2\frac{(\theta_1-\theta_2)}{2} \tag{25}$$

we obtain

$$\xi=\frac{1}{1+4\lambda\left(\sin^2\frac{(\theta_1+\theta_2)}{2}+\sin^2\frac{(\theta_1-\theta_2)}{2}\right)} \tag{26}$$

which means for all positive values of λ and $\theta_1,\theta_2\in[-\pi,\pi]$, Eq. (10) is unconditionally stable.

Convergence Analysis

This section analyses the convergence of Eq. (10), and the proof begins with the following proposition.

Proposition 1

$$\|e^n\|_\infty\le c(k+h^2) \tag{27}$$

where $\|e^n\|_\infty=\max_{1\le p,q\le M-1}\left|e_{p,q}^n\right|$.

Theorem 1

Let $U_{p,q}^n$ be the approximation to $u(x_p,y_q,t_n)$. There exists a positive constant c such that $\left|U_{p,q}^n-u(x_p,y_q,t_n)\right|\le c(k+h^2)$.

Proof

Suppose $e_{p,q}^n=u(x_p,y_q,t_n)-U_{p,q}^n$ and $e^n=(e_{1,1}^n,\ e_{1,2}^n,\ \dots,\ e_{2,1}^n,\ e_{2,2}^n,\ \dots,\ e_{M-1,1}^n,\ e_{M-1,2}^n,\ \dots,\ e_{M-1,M-1}^n)$. Then, from Eq. (10), we get

$$e_{p,q}^{n} - A_1\left(e_{p,q}^{n}\right)^{\rho} e_{p+1,q+1}^{n} + 2A_1\left(e_{p,q}^{n}\right)^{\rho+1} - A_1\left(e_{p,q}^{n}\right)^{\rho} e_{p-1,q-1}^{n} - A_2\rho\left(e_{p,q}^{n}\right)^{\rho-1}\left(e_{p+1,q+1}^{n}\right)^2$$
$$+2A_2\rho\left(e_{p,q}^{n}\right)^{\rho-1} e_{p+1,q+1}^{n} e_{p-1,q-1}^{n} - A_2\rho\left(e_{p,q}^{n}\right)^{\rho-1}\left(e_{p-1,q-1}^{n}\right)^2 - A_3\left(e_{p,q}^{n}\right)^{\rho} e_{p-1,q+1}^{n} + 2A_3\left(e_{p,q}^{n}\right)^{\rho+1}$$
$$-A_3\left(e_{p,q}^{n}\right)^{\rho} e_{p+1,q-1}^{n} - A_4\rho\left(e_{p,q}^{n}\right)^{\rho-1}\left(e_{p-1,q+1}^{n}\right)^2 + 2A_4\rho\left(e_{p,q}^{n}\right)^{\rho-1} e_{p-1,q+1}^{n} e_{p+1,q-1}^{n}$$
$$-A_4\rho\left(e_{p,q}^{n}\right)^{\rho-1}\left(e_{p+1,q-1}^{n}\right)^2 = e_{p,q}^{n-1} + R_{p,q}^{n} \tag{28}$$

where

$$R_{p,q}^{n} = u(x_p, y_q, t_n) - u(x_p, y_q, t_{n-1}) - A_1 u^{\rho}(x_p, y_q, t_n) u(x_{p+1}, y_{q+1}, t_n) + 2A_1 u^{\rho+1}(x_p, y_q, t_n)$$
$$-A_1 u^{\rho}(x_p, y_q, t_n) u(x_{p-1}, y_{q-1}, t_n) - A_2\rho u^{\rho-1}(x_p, y_q, t_n) u^2(x_{p+1}, y_{q+1}, t_n)$$
$$+2A_2\rho u^{\rho-1}(x_p, y_q, t_n) u(x_{p+1}, y_{q+1}, t_n) u(x_{p-1}, y_{q-1}, t_n)$$
$$-A_2\rho u^{\rho-1}(x_p, y_q, t_n) u^2(x_{p-1}, y_{q-1}, t_n) - A_3 u^{\rho}(x_p, y_q, t_n) u(x_{p-1}, y_{q+1}, t_n) + 2A_3 u^{\rho+1}(x_p, y_q, t_n)$$
$$-A_3 u^{\rho}(x_p, y_q, t_n) u(x_{p+1}, y_{q-1}, t_n) - A_4\rho u^{\rho-1}(x_p, y_q, t_n) u^2(x_{p-1}, y_{q+1}, t_n)$$
$$+2A_4\rho u^{\rho-1}(x_p, y_q, t_n) u(x_{p-1}, y_{q+1}, t_n) u(x_{p+1}, y_{q-1}, t_n) - A_4\rho u^{\rho-1}(x_p, y_q, t_n) u^2(x_{p+1}, y_{q-1}, t_n) \tag{29}$$

From the two-dimensional half-sweep schemes are as follows,

$$\frac{\partial u(x_p,y_q,t_n)}{\partial t} + c_1 k = \frac{u(x_p,y_q,t_n) - u(x_p,y_q,t_{n-1})}{k} \tag{30}$$

$$\frac{\partial u(x_p,y_q,t_n)}{\partial x} + c_2 h = \frac{u(x_{p+1},y_{q+1},t_n) - u(x_{p-1},y_{q-1},t_n)}{2\sqrt{2}h} \tag{31}$$

$$\frac{\partial u(x_p,y_q,t_n)}{\partial y} + c_3 h = \frac{u(x_{p-1},y_{q+1},t_n) - u(x_{p+1},y_{q-1},t_n)}{2\sqrt{2}h} \tag{32}$$

$$\frac{\partial^2 u(x_p,y_q,t_n)}{\partial x^2} + c_4 h^2 = \frac{u(x_{p+1},y_{q+1},t_n) - 2u(x_p,y_q,t_n) + u(x_{p-1},y_{q-1},t_n)}{2h^2} \tag{33}$$

and

$$\frac{\partial^2 u(x_p,y_q,t_n)}{\partial y^2} + c_5 h^2 = \frac{u(x_{p-1},y_{q+1},t_n) - 2u(x_p,y_q,t_n) + u(x_{p+1},y_{q-1},t_n)}{2h^2} \tag{34}$$

Eq. (28) can be rewritten into

$$R_{p,q}^{n} = \frac{\partial u(x_p, y_q, t_n)}{\partial t} - \alpha u^{\rho}(x_p, y_q, t_n)\frac{\partial^2 u(x_p, y_q, t_n)}{\partial x^2} - \alpha\rho u^{\rho-1}(x_p, y_q, t_n)\left(\frac{\partial u(x_p, y_q, t_n)}{\partial x}\right)^2$$
$$-\alpha u^{\rho}(x_p, y_q, t_n)\frac{\partial^2 u(x_p,y_q,t_n)}{\partial y^2} - \alpha\rho u^{\rho-1}(x_p, y_q, t_n)\left(\frac{\partial u(x_p,y_q,t_n)}{\partial y}\right)^2 + c_1 k + (\widetilde{c_2} + \widetilde{c_3} + c_4 + c_5)h^2 \tag{35}$$

where $c_1, c_2, \widetilde{c_2}, c_3, \widetilde{c_3}, c_4$ and c_5 are arbitrary constants. Also, $\left|R_{p,q}^{n}\right| \le c(k + h^2)$.

By using mathematical induction and let $\|e^1\|_\infty = \left|e_{P,Q}^{1}\right| = \max_{1\le p,q\le M-1}\left|e_{p,q}^{1}\right|$, we have

$$\left|e_{P,Q}^{1}\right| \le \left|e_{P,Q}^{1}\right| - A_1\left|\left(e_{P,Q}^{1}\right)^{\rho}\right|\left|e_{P+1,Q+1}^{1}\right| + 2A_1\left|\left(e_{P,Q}^{1}\right)^{\rho+1}\right| - A_1\left|\left(e_{P,Q}^{1}\right)^{\rho}\right|\left|e_{P-1,Q-1}^{1}\right|$$
$$-A_2\rho\left|\left(e_{P,Q}^{1}\right)^{\rho-1}\right|\left|\left(e_{P+1,Q+1}^{1}\right)^2\right| + 2A_2\rho\left|\left(e_{P,Q}^{1}\right)^{\rho-1}\right|\left|e_{P+1,Q+1}^{1}\right|\left|e_{P-1,Q-1}^{1}\right| - A_2\rho\left|\left(e_{P,Q}^{1}\right)^{\rho-1}\right|\left|\left(e_{P-1,Q-1}^{1}\right)^2\right|$$
$$-A_3\left|\left(e_{P,Q}^{1}\right)^{\rho}\right|\left|e_{P-1,Q+1}^{1}\right| + 2A_3\left|\left(e_{P,Q}^{1}\right)^{\rho+1}\right| - A_3\left|\left(e_{P,Q}^{1}\right)^{\rho}\right|\left|e_{P+1,Q-1}^{1}\right| - A_4\rho\left|\left(e_{P,Q}^{1}\right)^{\rho-1}\right|\left|\left(e_{P-1,Q+1}^{1}\right)^2\right|$$
$$+2A_4\rho\left|\left(e_{P,Q}^{1}\right)^{\rho-1}\right|\left|e_{P-1,Q+1}^{1}\right|\left|e_{P+1,Q-1}^{1}\right| - A_4\rho\left|\left(e_{P,Q}^{1}\right)^{\rho-1}\right|\left|\left(e_{P+1,Q-1}^{1}\right)^2\right|$$
$$\le |\, e_{P,Q}^{1} - A_1\left(e_{P,Q}^{1}\right)^{\rho} e_{P+1,Q+1}^{1} + 2A_1\left(e_{P,Q}^{1}\right)^{\rho+1} - A_1\left(e_{P,Q}^{1}\right)^{\rho} e_{P-1,Q-1}^{1}$$
$$-A_2\rho\left(e_{P,Q}^{1}\right)^{\rho-1}\left(e_{P+1,Q+1}^{1}\right)^2 + 2A_2\rho\left(e_{P,Q}^{1}\right)^{\rho-1} e_{P+1,Q+1}^{1} e_{P-1,Q-1}^{1} - A_2\rho\left(e_{P,Q}^{1}\right)^{\rho-1}\left(e_{P-1,Q-1}^{1}\right)^2$$
$$-A_3\left(e_{P,Q}^{1}\right)^{\rho} e_{P-1,Q+1}^{1} + 2A_3\left(e_{P,Q}^{1}\right)^{\rho+1} - A_3\left(e_{P,Q}^{1}\right)^{\rho} e_{P+1,Q-1}^{1} - A_4\rho\left(e_{P,Q}^{1}\right)^{\rho-1}\left(e_{P-1,Q+1}^{1}\right)^2$$
$$+2A_4\rho\left(e_{P,Q}^{1}\right)^{\rho-1} e_{P-1,Q+1}^{1} e_{P+1,Q-1}^{1} - A_4\rho\left(e_{P,Q}^{1}\right)^{\rho-1}\left(e_{P+1,Q-1}^{1}\right)^2\, | = \left|R_{p,q}^{n}\right| \le c(k + h^2) \tag{36}$$

Suppose that $\left\|e^j\right\|_\infty \le c(k + h^2), j = 1, 2, \dots, n-1$ and $\left|e_{P,Q}^{n}\right| = \max_{1\le p,q\le M-1}\left|e_{p,q}^{n}\right|$. Then,

$$\left|e_{P,Q}^{n}\right| \le \left|e_{P,Q}^{n}\right| - A_1\left|\left(e_{P,Q}^{n}\right)^{\rho}\right|\left|e_{P+1,Q+1}^{n}\right| + 2A_1\left|\left(e_{P,Q}^{n}\right)^{\rho+1}\right| - A_1\left|\left(e_{P,Q}^{n}\right)^{\rho}\right|\left|e_{P-1,Q-1}^{n}\right| - A_2\rho\left|\left(e_{P,Q}^{n}\right)^{\rho-1}\right|\left|\left(e_{P+1,Q+1}^{n}\right)^{2}\right|$$
$$+2A_2\rho\left|\left(e_{P,Q}^{n}\right)^{\rho-1}\right|\left|e_{P+1,Q+1}^{n}\right|\left|e_{P-1,Q-1}^{n}\right| - A_2\rho\left|\left(e_{P,Q}^{n}\right)^{\rho-1}\right|\left|\left(e_{P-1,Q-1}^{n}\right)^{2}\right| - A_3\left|\left(e_{P,Q}^{n}\right)^{\rho}\right|\left|e_{P-1,Q+1}^{n}\right| + 2A_3\left|\left(e_{P,Q}^{n}\right)^{\rho+1}\right|$$
$$-A_3\left|\left(e_{P,Q}^{n}\right)^{\rho}\right|\left|e_{P+1,Q-1}^{n}\right| - A_4\rho\left|\left(e_{P,Q}^{n}\right)^{\rho-1}\right|\left|\left(e_{P-1,Q+1}^{n}\right)^{2}\right| + 2A_4\rho\left|\left(e_{P,Q}^{n}\right)^{\rho-1}\right|\left|e_{P-1,Q+1}^{n}\right|\left|e_{P+1,Q-1}^{n}\right|$$
$$-A_4\rho\left|\left(e_{P,Q}^{n}\right)^{\rho-1}\right|\left|\left(e_{P+1,Q-1}^{n}\right)^{2}\right| \le |\, e_{P,Q}^{n} - A_1\left(e_{P,Q}^{n}\right)^{\rho}e_{P+1,Q+1}^{n} + 2A_1\left(e_{P,Q}^{n}\right)^{\rho+1} - A_1\left(e_{P,Q}^{n}\right)^{\rho}e_{P-1,Q-1}^{n}$$
$$-A_2\rho\left(e_{P,Q}^{n}\right)^{\rho-1}\left(e_{P+1,Q+1}^{n}\right)^{2} + 2A_2\rho\left(e_{P,Q}^{n}\right)^{\rho-1}e_{P+1,Q+1}^{n}e_{P-1,Q-1}^{n} - A_2\rho\left(e_{P,Q}^{n}\right)^{\rho-1}\left(e_{P-1,Q-1}^{n}\right)^{2}$$
$$-A_3\left(e_{P,Q}^{n}\right)^{\rho}e_{P-1,Q+1}^{n} + 2A_3\left(e_{P,Q}^{n}\right)^{\rho+1} - A_3\left(e_{P,Q}^{n}\right)^{\rho}e_{P+1,Q-1}^{n} - A_4\rho\left(e_{P,Q}^{n}\right)^{\rho-1}\left(e_{P-1,Q+1}^{n}\right)^{2}$$
$$+2A_4\rho\left(e_{P,Q}^{n}\right)^{\rho-1}e_{P-1,Q+1}^{n}e_{P+1,Q-1}^{n} - A_4\rho\left(e_{P,Q}^{n}\right)^{\rho-1}\left(e_{P+1,Q-1}^{n}\right)^{2}\,|$$
$$= \left|\tilde{c}e_{P,Q}^{n-1} + R_{p,q}^{1}\right| \le \tilde{c}\left|e_{P,Q}^{n-1}\right| + c(k+h^2) \le \tilde{c}\|e^{n-1}\|_{\infty} + c(k+h^2) = c(k+h^2) \tag{37}$$

which completes the proof.

3. Modified Successive Over-relaxation Iterative Formula

This section discusses the iterative formula to obtain the solutions of the generated system of equations of Eq. (10). Let's consider a large-scale and complex system of nonlinear equations by using Eq. (10) as follows.

$$\underset{\sim}{F}(\underset{\sim}{U}^{n}) = 0 \tag{38}$$

where $\underset{\sim}{F} = \left(F_{1,1}^{n}, F_{1,2}^{n}, \dots, F_{2,1}^{n}, F_{2,2}^{n}, \dots, F_{M-1,1}^{n}, F_{M-1,2}^{n}, \dots, F_{M-1,M-1}^{n}\right)$ with

$$F_{1,1}^{n} = U_{1,1}^{n} - A_1\left(U_{1,1}^{n}\right)^{\rho}U_{2,2}^{n} + 2A_1\left(U_{1,1}^{n}\right)^{\rho+1} - A_1\left(U_{1,1}^{n}\right)^{\rho}U_{0,0}^{n} - A_2\rho\left(U_{1,1}^{n}\right)^{\rho-1}\left(U_{2,2}^{n}\right)^{2} +$$
$$2A_2\rho\left(U_{1,1}^{n}\right)^{\rho-1}U_{2,2}^{n}U_{0,0}^{n} - A_2\rho\left(U_{1,1}^{n}\right)^{\rho-1}\left(U_{0,0}^{n}\right)^{2} - A_3\left(U_{1,1}^{n}\right)^{\rho}U_{0,2}^{n} + 2A_3\left(U_{1,1}^{n}\right)^{\rho+1} - A_3\left(U_{1,1}^{n}\right)^{\rho}U_{2,0}^{n}$$
$$-A_4\rho\left(U_{1,1}^{n}\right)^{\rho-1}\left(U_{0,2}^{n}\right)^{2} + 2A_4\rho\left(U_{1,1}^{n}\right)^{\rho-1}U_{0,2}^{n}U_{2,0}^{n} - A_4\rho\left(U_{1,1}^{n}\right)^{\rho-1}\left(U_{2,0}^{n}\right)^{2} = U_{1,1}^{n-1} \tag{39}$$

$$F_{1,2}^{n} = U_{1,2}^{n} - A_1\left(U_{1,2}^{n}\right)^{\rho}U_{2,3}^{n} + 2A_1\left(U_{1,2}^{n}\right)^{\rho+1} - A_1\left(U_{1,2}^{n}\right)^{\rho}U_{0,1}^{n} - A_2\rho\left(U_{1,2}^{n}\right)^{\rho-1}\left(U_{2,3}^{n}\right)^{2}$$
$$+2A_2\rho\left(U_{1,2}^{n}\right)^{\rho-1}U_{2,3}^{n}U_{0,1}^{n} - A_2\rho\left(U_{1,2}^{n}\right)^{\rho-1}\left(U_{0,1}^{n}\right)^{2} - A_3\left(U_{1,2}^{n}\right)^{\rho}U_{0,3}^{n} + 2A_3\left(U_{1,2}^{n}\right)^{\rho+1} - A_3\left(U_{1,2}^{n}\right)^{\rho}U_{2,1}^{n}$$
$$A_4\rho\left(U_{1,2}^{n}\right)^{\rho-1}\left(U_{0,3}^{n}\right)^{2} + 2A_4\rho\left(U_{1,2}^{n}\right)^{\rho-1}U_{0,3}^{n}U_{2,1}^{n} - A_4\rho\left(U_{1,2}^{n}\right)^{\rho-1}\left(U_{2,1}^{n}\right)^{2} = U_{1,2}^{n-1} \tag{40}$$

$$F_{2,1}^{n} = U_{2,1}^{n} - A_1\left(U_{2,1}^{n}\right)^{\rho}U_{3,2}^{n} + 2A_1\left(U_{2,1}^{n}\right)^{\rho+1} - A_1\left(U_{2,1}^{n}\right)^{\rho}U_{1,0}^{n} - A_2\rho\left(U_{2,1}^{n}\right)^{\rho-1}\left(U_{3,2}^{n}\right)^{2}$$
$$+2A_2\rho\left(U_{2,1}^{n}\right)^{\rho-1}U_{3,2}^{n}U_{1,0}^{n} - A_2\rho\left(U_{2,1}^{n}\right)^{\rho-1}\left(U_{1,0}^{n}\right)^{2} - A_3\left(U_{2,1}^{n}\right)^{\rho}U_{1,2}^{n} + 2A_3\left(U_{2,1}^{n}\right)^{\rho+1} - A_3\left(U_{2,1}^{n}\right)^{\rho}U_{3,0}^{n}$$
$$-A_4\rho\left(U_{2,1}^{n}\right)^{\rho-1}\left(U_{1,2}^{n}\right)^{2} + 2A_4\rho\left(U_{2,1}^{n}\right)^{\rho-1}U_{1,2}^{n}U_{3,0}^{n} - A_4\rho\left(U_{2,1}^{n}\right)^{\rho-1}\left(U_{3,0}^{n}\right)^{2} = U_{2,1}^{n-1} \tag{41}$$

$$F_{2,2}^{n} = U_{2,2}^{n} - A_1\left(U_{2,2}^{n}\right)^{\rho}U_{3,3}^{n} + 2A_1\left(U_{2,2}^{n}\right)^{\rho+1} - A_1\left(U_{2,2}^{n}\right)^{\rho}U_{1,1}^{n} - A_2\rho\left(U_{2,2}^{n}\right)^{\rho-1}\left(U_{3,3}^{n}\right)^{2}$$
$$+2A_2\rho\left(U_{2,2}^{n}\right)^{\rho-1}U_{3,3}^{n}U_{1,1}^{n} - A_2\rho\left(U_{2,2}^{n}\right)^{\rho-1}\left(U_{1,1}^{n}\right)^{2} - A_3\left(U_{2,2}^{n}\right)^{\rho}U_{1,3}^{n} + 2A_3\left(U_{2,2}^{n}\right)^{\rho+1} - A_3\left(U_{2,2}^{n}\right)^{\rho}U_{3,1}^{n}$$
$$-A_4\rho\left(U_{2,2}^{n}\right)^{\rho-1}\left(U_{1,3}^{n}\right)^{2} + 2A_4\rho\left(U_{2,2}^{n}\right)^{\rho-1}U_{1,3}^{n}U_{3,1}^{n} - A_4\rho\left(U_{2,2}^{n}\right)^{\rho-1}\left(U_{3,1}^{n}\right)^{2} = U_{2,2}^{n-1} \tag{42}$$

$$F_{M-1,1}^{n} = U_{M-1,1}^{n} - A_1\left(U_{M-1,1}^{n}\right)^{\rho}U_{M,2}^{n} + 2A_1\left(U_{M-1,1}^{n}\right)^{\rho+1} - A_1\left(U_{M-1,1}^{n}\right)^{\rho}U_{M-2,0}^{n}$$
$$-A_2\rho\left(U_{M-1,1}^{n}\right)^{\rho-1}\left(U_{M,2}^{n}\right)^{2} + 2A_2\rho\left(U_{M-1,1}^{n}\right)^{\rho-1}U_{M,2}^{n}U_{M-2,0}^{n} - A_2\rho\left(U_{M-1,1}^{n}\right)^{\rho-1}\left(U_{M-2,0}^{n}\right)^{2}$$
$$-A_3\left(U_{M-1,1}^{n}\right)^{\rho}U_{M-2,2}^{n} + 2A_3\left(U_{M-1,1}^{n}\right)^{\rho+1} - A_3\left(U_{M-1,1}^{n}\right)^{\rho}U_{M,0}^{n}$$
$$-A_4\rho\left(U_{M-1,1}^{n}\right)^{\rho-1}\left(U_{M-2,2}^{n}\right)^{2} + 2A_4\rho\left(U_{M-1,1}^{n}\right)^{\rho-1}U_{M-2,2}^{n}U_{M,0}^{n} - A_4\rho\left(U_{M-1,1}^{n}\right)^{\rho-1}\left(U_{M,0}^{n}\right)^{2} = U_{M-1,1}^{n-1} \tag{43}$$

$$F_{M-1,2}^{n}=U_{M-1,2}^{n}-A_1\left(U_{M-1,2}^{n}\right)^{\rho}U_{M,3}^{n}+2A_1\left(U_{M-1,2}^{n}\right)^{\rho+1}-A_1\left(U_{M-1,2}^{n}\right)^{\rho}U_{M-2,1}^{n}$$
$$-A_2\rho\left(U_{M-1,2}^{n}\right)^{\rho-1}\left(U_{M,3}^{n}\right)^2+2A_2\rho\left(U_{M-1,2}^{n}\right)^{\rho-1}U_{M,3}^{n}U_{M-2,1}^{n}-A_2\rho\left(U_{M-1,2}^{n}\right)^{\rho-1}\left(U_{M-2,1}^{n}\right)^2$$
$$-A_3\left(U_{M-1,2}^{n}\right)^{\rho}U_{M-2,3}^{n}+2A_3\left(U_{M-1,2}^{n}\right)^{\rho+1}-A_3\left(U_{M-1,2}^{n}\right)^{\rho}U_{M,1}^{n}$$
$$-A_4\rho\left(U_{M-1,2}^{n}\right)^{\rho-1}\left(U_{M-2,3}^{n}\right)^2+2A_4\rho\left(U_{M-1,2}^{n}\right)^{\rho-1}U_{M-2,3}^{n}U_{M,1}^{n}-A_4\rho\left(U_{M-1,2}^{n}\right)^{\rho-1}\left(U_{M,1}^{n}\right)^2=U_{M-1,2}^{n-1} \quad (44)$$
$$F_{M-1,M-1}^{n}=U_{M-1,M-1}^{n}-A_1\left(U_{M-1,M-1}^{n}\right)^{\rho}U_{M,M}^{n}+2A_1\left(U_{M-1,M-1}^{n}\right)^{\rho+1}-A_1\left(U_{M-1,M-1}^{n}\right)^{\rho}U_{M-2,M-2}^{n}$$
$$-A_2\rho\left(U_{M-1,M-1}^{n}\right)^{\rho-1}\left(U_{M,M}^{n}\right)^2+2A_2\rho\left(U_{M-1,M-1}^{n}\right)^{\rho-1}U_{M,M}^{n}U_{M-2,M-2}^{n}-A_2\rho\left(U_{M-1,M-1}^{n}\right)^{\rho-1}\left(U_{M-2,M-2}^{n}\right)^2$$
$$-A_3\left(U_{M-1,M-1}^{n}\right)^{\rho}U_{M-2,M}^{n}+2A_3\left(U_{M-1,M-1}^{n}\right)^{\rho+1}-A_3\left(U_{M-1,M-1}^{n}\right)^{\rho}U_{M,M}^{n}-A_4\rho\left(U_{M-1,M-1}^{n}\right)^{\rho-1}\left(U_{M-2,M}^{n}\right)^2+$$
$$2A_4\rho\left(U_{M-1,M-1}^{n}\right)^{\rho-1}U_{M-2,M}^{n}U_{M,M-2}^{n}-A_4\rho\left(U_{M-1,M-1}^{n}\right)^{\rho-1}\left(U_{M,M-2}^{n}\right)^2=U_{M-1,M-1}^{n-1} \quad (45)$$

and $\underset{\sim}{U}^n=\left(U_{1,1}^n,U_{1,2}^n,\dots,U_{2,1}^n,U_{2,2}^n,\dots,U_{M-1,1}^n,U_{M-1,2}^n,\dots,U_{M-1,M-1}^n\right)$.

From the implementation of the second-order Newton method, a system of linearized equations can be constructed in the form of

$$F'^n\underset{\sim}{W}^n=-\underset{\sim}{F}(\underset{\sim}{U}^n) \quad (46)$$

where F'^n represents the penta-diagonal Jacobian matrix at time level n and $\underset{\sim}{W}^n$ is the approximation to the system of linearized equations that can be computed iteratively using

$$\underset{\sim}{W}^n=\underset{\sim}{U}^{n(l+1)}-\underset{\sim}{U}^{n(l)}, l=0,1,2,\dots \quad (47)$$

Next, to derive the standard MSOR iterative formula, let the $(M-1)^2$ coefficient matrix of F'^n be decomposed into three component matrices as follows.

$$F'^n=D^n+L^n+V^n \quad (48)$$

where $D^n, L^n,$ and V^n are the diagonal, lower triangular and upper triangular parts of the leading coefficient matrix. A single parameter successive over-relaxation iterative formula can be derived using Eq. (46) and (48) to give

$$\underset{\sim}{W}^{n(l+1)}=(1-\omega)\underset{\sim}{W}^{n(l)}+\omega(D^n+L^n)^{-1}\left(-V^n\underset{\sim}{W}^{n(l)}-\underset{\sim}{F}\left(\underset{\sim}{U}^{n(l)}\right)\right), l=0,1,2,\dots \quad (49)$$

where $1<\omega<2$.

In this paper, we proposed a modification of Eq. (49) that uses two independent parameters to run the odd- and even-indexed successive over-relaxation iterative formula. The modified iterative formula, which is also known as the HSMSOR, can be expressed in the form of

$$\underset{\sim}{W}_j^{n(l+1)}=(1-\omega_o)\underset{\sim}{W}_j^{n(l)}+\omega_o\left(D_j^n+L_j^n\right)^{-1}\left(-V_j^n\underset{\sim}{W}_j^{n(l)}-\underset{\sim}{F}\left(\underset{\sim}{U}_j^{n(l)}\right)\right) \quad (50)$$

where $j=1,3,\dots,M-1$ and $1<\omega_o<2$, and

$$\underset{\sim}{W}_j^{n(l+1)}=(1-\omega_e)\underset{\sim}{W}_j^{n(l)}+\omega_e\left(D_j^n+L_j^n\right)^{-1}\left(-V_j^n\underset{\sim}{W}_j^{n(l)}-\underset{\sim}{F}\left(\underset{\sim}{U}_j^{n(l)}\right)\right) \quad (51)$$

where $j=2,4,\dots,M-2$ and $1<\omega_e<2$.

The implementation of the HSMSOR to solve Eq. (1) subjects to Eq. (2)-(4) can be illustrated in the form of a numerical algorithm as follows.

ALGORITHM 1 HSMSOR method for solving a two-dimensional NPPDE

Output: Total number of iterations (l_N), total elapsed time (t_N), and maximum absolute errors (ε_{max}).

Define ω_o and ω_e;
Initialize $l_N = 0, \underset{\sim}{U}_j^{n(l)} = 1.000$, and $\underset{\sim}{W}_j^{n(l)} = 0$;

while $n \le N$ **do**
- Initialize $l = l_{outer} = 0$;
- **while** $\left|\underset{\sim}{U}_j^{n(l+1)} - \underset{\sim}{U}_j^{n(l)}\right| > 10^{-10}$ **do**
  - Initialize $l_{inner} = 0$;
  - **while** $\left|\underset{\sim}{W}_j^{n(l+1)} - \underset{\sim}{W}_j^{n(l)}\right| > 10^{-10}$ **do**
    - For $j = 1, 3, \ldots, M-1$, iterate Eq. (50);
    - For $j = 2, 4, \ldots, M-2$, iterate Eq. (51);
    - $l_{inner} + +$;
  - **end**
  - $\underset{\sim}{U}_j^{n(l+1)} = \underset{\sim}{U}_j^{n(l)} + \underset{\sim}{W}_j^{n(l+1)}$;
  - $l_{outer} = l_{outer} + l_{inner}$;
- **end**
- $l_N = l_N + l_{outer}$;

end

Based on the developed algorithm, the implementation of HSMSOR involves inner and outer iteration cycles. The inner iteration cycle is used to obtain the converged approximations to the system of linearized equations. Meanwhile, the outer iteration cycle obtains the converged numerical solutions to the problem.

4. Numerical Experiment

In the numerical experiment, three initial-boundary value problems of nonlinear porous medium equations are used to demonstrate the performance of the HSMSOR method. The competitive computational cost attained by the HSMSOR method is compared against the standard implicit finite difference method and the developed MSOR method [20]. The total number of iterations (l_N), total elapsed time by the developed simulation program (t_N) and maximum absolute errors (ε_{max}) are the criteria to be measured and discussed in this section. Five different sizes of matrices are used for consistency checking, which are $M \times M = 16 \times 16, 32 \times 32, 64 \times 64, 128 \times 128,$ and 256×256. Below are the following problems used in the paper.

Problem 1 [21]

Given that

$$\frac{\partial u}{\partial t} = \frac{1}{5}\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right)\right] \tag{52}$$

subjects to the conditions $G_0 = x + y, a_y = y + 0.4t, b_y = 1 + y + 0.4t, a_x = x + 0.4t,$ and $b_x = x + 1 + 0.4t$ with the domain $0 \le x \le 1, 0 \le y \le 1,$ and $0 < t \le 1$. The exact solution used for accuracy checking is given by $u(x, y, t) = x + y + 0.4t$.

Problem 2 [21]

Given that

$$\frac{\partial u}{\partial t} = \frac{1}{5}\left[\frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u^2\frac{\partial u}{\partial y}\right)\right] \tag{53}$$

with the conditions $G_0 = \sqrt{5x + 5y}, a_y = \sqrt{5y + 5t}, b_y = \sqrt{5 + 5y + 5t}, a_x = \sqrt{5x + 5t}$ and $b_x = \sqrt{5x + 5 + 5t}$, where $0 \le x \le 1, 0 \le y \le 1,$ and $0 < t \le 1$. The suggested exact solution is $u(x, y, t) = \sqrt{5x + 5y + 5t}$.

Problem 3 [22]

Given that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^5\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u^5\frac{\partial u}{\partial y}\right) \tag{54}$$

subjects to the initial condition $G_0 = \sqrt[4]{0.8x + 0.8y}, 0 \le x \le 1, 0 \le y \le 1,$ and boundary conditions are $a_y = \sqrt[4]{0.8y + 1.6t}, b_y = \sqrt[4]{0.8 + 0.8y + 1.6t}, a_x = \sqrt[4]{0.8x + 1.6t}$ and $b_x = \sqrt[4]{0.8x + 0.8 + 1.6t}$ for $0 < t \le 1$. The exact solution is $u(x, y, t) = \sqrt[4]{0.8x + 0.8y + 1.6t}$.

Based on the collected data from the experiment, the study found that the total number of iterations required by the HSMSOR method to obtain the final solutions to selected problems is lesser than the standard implicit finite difference and MSOR methods, see Tables 1, 2, and 3. Following the reduced number of iterations, the total elapsed time by the HSMSOR program is shorter than the two compared methods. It can be said that HSMSOR is more efficient than the implicit finite difference and MSOR methods in solving the considered two-dimensional NPPDE. When the accuracy of the solutions obtained by the methods is compared against each other, again, the HSMSOR method is superior to both implicit finite difference and MSOR methods. The maximum absolute errors produced by the HSMSOR method are significantly smaller than these compared methods at all different sizes of matrix used. It can be said that HSMSOR is more accurate than the implicit finite difference and MSOR methods in computing the solutions of two-dimensional NPPDE.

Table 1. Comparison between methods after solving Problem 1

$M \times M$	Method	l_N	t_N	ε_{max}
16×16	Implicit	136	0.47	8.86E-11
	MSOR	61	0.36	1.28E-12
	HSMSOR	46	0.24	5.09E-13
32×32	Implicit	436	2.43	3.25E-10
	MSOR	127	2.23	9.46E-12
	HSMSOR	95	1.17	3.84E-12
64×64	Implicit	1525	19.59	1.90E-09
	MSOR	258	8.65	2.68E-11
	HSMSOR	191	5.51	1.37E-11
128×128	Implicit	5462	260.89	9.02E-09
	MSOR	512	57.47	3.86E-11
	HSMSOR	378	36.92	2.70E-11
256×256	Implicit	19404	4586.69	3.86E-08
	MSOR	1010	437.43	6.16E-11
	HSMSOR	750	223.63	4.27E-11

Table 2. Comparison between methods after solving Problem 2

$M \times M$	Method	l_N	t_N	ε_{max}
16×16	Implicit	130	0.47	7.57E-11
	MSOR	73	0.38	6.24E-13
	HSMSOR	53	0.36	7.95E-14
32×32	Implicit	400	2.28	2.31E-09
	MSOR	135	1.96	3.28E-11
	HSMSOR	103	1.26	1.03E-11
64×64	Implicit	1380	19.26	1.31E-08
	MSOR	270	6.55	5.83E-11
	HSMSOR	207	4.95	2.81E-11
128×128	Implicit	4901	248.82	4.95E-08
	MSOR	537	42.40	1.27E-10
	HSMSOR	389	36.26	1.93E-11
256×256	Implicit	17458	4243.79	1.75E-07
	MSOR	1059	405.57	1.58E-10
	HSMSOR	780	226.13	2.67E-11

Table 3. Comparison between methods after solving Problem 3

$M \times M$	Method	l_N	t_N	ε_{max}
16×16	Implicit	739	0.98	1.10E-09
	MSOR	242	0.69	1.72E-11
	HSMSOR	181	0.57	8.07E-12
32×32	Implicit	2630	8.51	6.69E-09
	MSOR	474	2.78	4.42E-11
	HSMSOR	355	1.98	2.18E-11
64×64	Implicit	9478	113.63	3.56E-08
	MSOR	926	14.98	1.32E-10
	HSMSOR	691	10.89	3.06E-11
128×128	Implicit	34098	1653.85	1.72E-07
	MSOR	1806	120.19	1.73E-10
	HSMSOR	1336	96.34	5.69E-11
256×256	Implicit	121649	29234.80	7.78E-07
	MSOR	3516	1131.24	2.28E-10
	HSMSOR	2640	775.09	6.23E-11

5. Conclusions

This paper presented the development of an efficient numerical method called the HSMSOR method for solving two-dimensional NPPDE problems. The half-sweep finite difference approximation equation is formulated using the defined operators. The proofs of local truncation errors, stability, and convergence are well-established. In addition, the HSMSOR iterative formula is derived, and the numerical algorithm is designed. The numerical experiment showed the superiority of the proposed HSMSOR method in terms of efficiency and accuracy in obtaining the desired solutions, which have been compared against the standard implicit finite difference and MSOR methods. Future research will investigate the capability of the method to solve multi-dimensional NPPDE problems and NPPDE with fractional derivative orders.

Acknowledgements

The corresponding author wants to acknowledge the financial support received from University Malaysia Sabah, Malaysia, as a research grant (SBK0462-2021).

REFERENCES

[1] N. Bazarra, M. Colturato, J. R. Fernández, M. G. Naso, A. Simonetto, and G. Gilioli, Appl. Math. Optim. 85, article ID 19, 2022.

[2] M. Kumbhakar, S. Mohan, K. Ghoshal, J. Kumar, and V. P. Singh, J. Hydrol. Eng. 27(2), 2022.

[3] A. El Kouche and F. Ortegón Gallego, Math. Comput. Simul. 192, 430-451, 2022.

[4] R. Bürger, J. Careaga, and S. Diehl, IMA J. Appl. Math. 86(3), 514-546, 2021.

[5] O. Nikan, Z. Avazzadeh, and J. A. Tenreiro Machado, Chaos Solitons Fractals 151, article ID 111220, 2021.

[6] Q. Zhang, H. Song, and Y. Hao, Comput. Appl. Math. 41, article ID 73, 2022.

[7] F. Huang, Z. Lu, L. Li, X. Wu, S. Liu, and Y. Yang, J. Numer. Math. 30(1), 23-42, 2022.

[8] M. Izadi and P. Roul, Appl. Math. Comput. 429, article ID 127226, 2022.

[9] M. K. Rawani, L. Verma, A. K. Verma, and R. P. Agarwal, Math. Methods Appl. Sci. 45(3), 1276-1298, 2022.

[10] M. J. Huntul, M. Tamsir, A. A. H. Ahmadini, and S. R. Thottoli, Ain Shams Eng. J. 13(1), article ID 101497, 2022.

[11] S. Singh, S. Bhatt, and S. Singh, Comput. Methods Differ. Equ. 10(1), 28-43, 2022.

[12] S. Chonladed and K. Wuttanachamsri, IAENG Int. J. Appl. Math. 51(2), article ID IJAM_51_2_20, 2021.

[13] M. K. M. Akhir, M. Othman, J. Sulaiman, Z. A. Majid, and M. Suleiman, Aust. J. Basic Appl. Sci. 5(12), 3033-3039, 2011.

[14] N. I. M. Fauzi, and J. Sulaiman, Int. J. Contemp. Math. Sciences 7(32), 1579-1589, 2012.

[15] S. Suparmin, and A. Saudi, IEEE 2nd International Conference on Automatic Control and Intelligent Systems (I2CACIS), 46-51, 2017.

[16] F. A. Musli, J. Sulaiman, and A. Saudi, J. Phys.: Conf. Ser. 1988, article ID 012035, 2021.

[17] J. V. L. Chew, and J. Sulaiman, Adv. Sci. Lett. 24(3), 1906-1911, 2018.

[18] A. R. Abdullah, Int. J. Comput. Math. 38(1-2), 61-70, 1991.

[19] J. L. Vazquez, *The Porous Medium Equation: Mathematical Theory* (Oxford University Press, New York, 2007

[20] J. C. V. Lung and J. Sulaiman, Far East J. Math. Sci. 108(1), 1-12, 2018.

[21] A. D. Polyanin and V. F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations* (Chapman and Hall/CRC Press, Boca Raton, 2004

[22] B. P. Sommeijer and P. J. van der Houwen, ACM Trans. Math. Softw. 10(4), 378-396, 1984.