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An Effective Spectral Approach to Solving Fractal Differential Equations of Variable Order Based on the Non-singular Kernel Derivative

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Abstract A new differential operators class has been discovered utilising fractional and variable-order fractal Atangana-Baleanu derivatives that have inspired the development of differential equations' new class. Physical phenomena with variable memory and fractal variable dimension can be described using these operators. In addition, the primary goal of this study is to use the operation matrix based on shifted Legendre polynomials to obtain numerical solutions with respect to this new differential equations' class, which will aid us in solving the issue and transforming it into an algebraic equation system. This method is employed in solving two forms of fractal fractional differential equations: non-linear and linear. The suggested strategy is contrasted with the mixture of two-step Lagrange polynomials, the predictor-corrector algorithm, as well as the fractional calculus methods' fundamental theorem, using numerical examples to demonstrate its accuracy and simplicity. The estimation error was proposed to contrast the results of the suggested methods and the exact solution to the problems. The proposed approach could apply to a wider class of biological systems, such as mathematical modelling of infectious disease dynamics and other important areas of study, such as economics, finance, and engineering. We are confident that this paper will open many new avenues of investigation for modelling real-world system problems.

Keywords Nonsingular Kernel Derivatives, Fractal Differential Equations, Variable-order Spectral Method

1 Introduction

Classical differential operators, integral operators, integral as well as differential operators with respect to a fading memory, and integral and differential operators with regards to the power-law process, which includes integral differential and operators concerning generalised Mittag-Leffler functions, have all been suggested in multiple research. These diverse classes of integral and differential operators paved the way for newer theories, ideas, innovations, and implementations in the real world [1,2].

Professionals working in the fractional calculus field discovered that these three differential operator classes are unable to describe a variety of real-world ideas. As a result, numerous novel differential operators [3-19] have been proposed to address the problem, which is part of fractal derivatives convolution with the generalised Mittag-Leffler function, exponential decay law, or power-law and continuous function. They have been regarded as the upper classes of integral and differential operators since they may be turned into fractal, classical, and integral operators, as well as fractional differential operators in the limit instances [20-28].

Since they have memory effects and self-similar features, these novel operators are greater than present operators in that they can effectively describe the varied complexities of real-world notions. The variable-order fractal-fractional Atangana-Baleanu Caputo (VOFFABC) derivatives type includes novel operators such as the Riemann-Liouville derivative as well as Atangana-Baleanu Caputo (ABC) fractional derivative.

Even though multiple articles have emphasised the capability of this method in describing the heterogeneity occurring in real-world occasions, the current fractal-fractional differential operators class is still not widely known among the academic community due to its novelty. One of the most significant advantages of these new differential operators is their ability to restore current differential operators.

The operational matrix method (OMM) may be used to answer a variety of fractional calculus problems with respect to the conventional Caputo sense, which is based on orthogonal functions and can minimise the intricacy of issues to an algebraic equations system. The OMM technique may also be employed to solve variable-order fractional differential equations (VOFFDEs). Furthermore, the OMM was created with respect to various types of polynomials in [29-32], in which real world implementations of Legendre polynomials (LPs) have been studied [33-37].

For OMM based on shifted Legendre polynomials (SLPs) will be generalised with the aim of using it in solving VOFFABC differential equations. It is also compared at first method of Predictor-Corrector [38], the second method mixture of two-step Lagrange polynomial, as well as fundamental theorem of fractional calculus [39]. Upon transforming issues to an algebraic equations system, the OMM has strong accuracy, demonstrating convergent outcomes, applicability, and simplicity. Compared to other methodologies, the OMM requires more subroutines to produce accurate values, resulting in extra complex computation and lengthy duration. The suggested project has several intriguing benefits. The approach for displaying the operational matrix of integration, which is achieved with great accuracy, is a major benefit. This property is reflected in the computational technique, resulting in an approximate solution close enough to the definite solution.

The following are the five sections of this publication: Section 2 focuses on the definitions of fractal fractional derivatives. The Legendre operational matrix method (LOMM) will be utilised for the fractal fractional Atangana–Baleanu Caputo (FFABC) derivative in Section 3. Meanwhile, the OMM for the VOFFABC derivative is specified in Section 4. Lastly, Section 5 comprises a few numerical examples and solutions.

2 Mathematical preliminaries

The fractal fractional derivatives with regard to the Mittag-Leffler kernel will be presented in this section.

Definition 2.1. Let $0 < \nu < 1$. Then, $\mathfrak{u}(\tau) \in H^1(\mathfrak{a}, \mathfrak{b})$ with $\mathfrak{b} > \mathfrak{a}$. The Atangana-Baleanu Caputo (ABC)-derivative with respect to the fractional-order is expressed in [4]

$$^{ABC}\mathfrak{D}^{\nu}\mathfrak{u}(\tau) = \frac{M(\nu)}{1-\nu}\int_0^{\tau} \mathfrak{u}^{'}(s) E_{\nu}\left[\frac{-\nu(\tau-s)^{\nu}}{1-\nu}\right] ds, \quad (1)$$

in which $0 < \nu < 1$, $M(\nu)$ resembles a normalization function, meanwhile E_{ν} denotes Mittag Leffler function.

Expand the ABC definition for the $n < \nu < n+1$ case with regard to $\mathfrak{u}^{(s)}(\mathfrak{a}) = 0 for \quad s = 1, 2, ..., n$ yields

$$\begin{aligned} ^{ABC}\mathfrak{D}^{\nu}\mathfrak{u}(\tau) = &\ ^{ABC}\mathfrak{D}^{\nu}(\mathfrak{D}^{n}\mathfrak{u}(\tau)) \\ &= \frac{M(\nu)}{1-\nu}\int_0^{\tau} \mathfrak{u}^{(n+1)}(s) E_{\nu}\left[\frac{-\nu(\tau-s)^{\nu}}{1-\nu}\right] ds \\ \mathfrak{u}^{(n+1)}(\tau) = \mathfrak{D}^{(n+1)}\mathfrak{u}(\tau) &= \mathfrak{D}^{\lceil\nu\rceil}\mathfrak{u}(\tau) \end{aligned} \quad (2)$$

in which $\lceil\nu\rceil$ denotes ceil ν.

Definition 2.2. The fractional integral with respect to the ABC-derivative can be written as given below:

$$^{AB}_{0}I^{\nu}_{\tau}\{\mathfrak{u}(\tau)\} = \frac{1-\nu}{M(\nu)}\mathfrak{u}(\tau) + \frac{\nu}{M(\nu)\Gamma(\nu)}\int_0^{\tau}\mathfrak{u}(s)(\tau-s)^{\nu-1}ds. \quad (3)$$

Definition 2.3. The ABC-derivative having variable-order $\nu(\tau)$, $0 < \nu(\tau) < 1$ of function $\mathfrak{u}(\tau)$ is expressed as [40]:

$$^{ABC}\mathfrak{D}^{\nu(\tau)}\mathfrak{u}(\tau) = \frac{M(\nu(\tau))}{1-\nu(\tau)}\int_0^{\tau}\mathfrak{u}^{'}(s)E_{\nu(\tau)}\left[\frac{-\nu(\tau)}{1-\nu(\tau)}(\tau-s)^{\nu(\tau)}\right] ds, \quad (4)$$

where $E_{\nu(\tau)}(\tau)$ denotes the Mittag Leffer function.

Lemma 1. Assume $\rho > 0$. It follows that the ABC-derivative having variable order may be expressed as follows [40]:

$$^{ABC}\mathfrak{D}^{\nu(\tau)}\tau^{\rho} = \frac{M(\nu(\tau))}{1-\nu(\tau)}\Gamma(\rho+1)\tau^{\rho}E_{\nu(\tau),\rho+1}\left[\frac{-\nu(\tau)}{1-\nu(\tau)}\tau^{\nu(\tau)}\right] \quad (5)$$

Definition 2.4. The fractal fractional Atangana–Baleanu Caputo (FFABC)-a derivative of the function $\mathfrak{u}(\tau)$ with order ν is expressed in [3-41] provided that the function $\mathfrak{u}(\tau)$ is fractal differentiable and continuous over the interval $(\mathfrak{a}, \mathfrak{b})$ having to order q given by

$$^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) = \frac{AB(\nu)}{(1-\nu)}\int_0^{t}\frac{d\mathfrak{u}(s)}{ds^q}E_{\nu}\left[-\frac{\nu}{1-\nu}(\tau-s)^{\nu}\right] ds, \quad (6)$$

$$0 < \nu, \quad q \leq 1, \quad \frac{d\mathfrak{u}(s)}{ds^q} = \lim_{\tau\to s}\frac{\mathfrak{u}(\tau)-\mathfrak{u}(s)}{\tau^q - s^q}.$$

The integral presented possesses a Mittag-Leffler kernel (non-local as well as non-singular), in which E_{ν} refers to Mittag-Leffler function that may be expressed as follows

$$E_{\nu}(-\tau^{\nu}) = \sum_{\kappa=0}^{\infty}\frac{(-\tau)^{\nu\kappa}}{\Gamma(\nu\kappa+1)},$$

as well as

$$AB(\nu) = 1 - \nu + \frac{\nu}{\Gamma(\nu)}.$$

Moreover, order $\nu(\tau)$ and $q(\tau)$ yields

$$^{VOFFABC}\mathfrak{D}^{\nu(\tau),q(\tau)}\mathfrak{u}(\tau) = \frac{AB(\nu(\tau))}{(1-\nu(\tau))}\int_0^{t}\frac{du(s)}{ds^{q(\tau)}}E_{\nu(\tau)}\left[-\frac{\nu(\tau)}{1-\nu(\tau)}(\tau-s)^{\nu(\tau)}\right] ds. \quad (7)$$

2.1 Several SLPs properties

The Legendre polynomials (LPs) expressed on the interval $[-1,1]$ refer to the orthogonal polynomial as well as specific recurrence formula relation [37].

$$\bar{\mathcal{L}}_{\iota+1}(\mathcal{Z})=\frac{2\iota+1}{\iota+1}\mathcal{Z}\bar{\mathcal{L}}_{\iota}(\mathcal{Z})-\frac{\iota}{\iota+1}\bar{\mathcal{L}}_{\iota-1}(\mathcal{Z}),\quad \iota=1,2,...,\tag{8}$$

in which $P_0(\tau)=1$ and $P_1(\tau)=2\tau-1$. Thus, the analytically form of SLPs $P_\iota(x)$ of degree ι is obtained as given below

$$P_\iota(\tau)=\sum_{\kappa=0}^{\iota}(-1)^{\iota+\kappa}\frac{(\iota+\kappa)!}{(\iota-\kappa)!}\frac{\tau^\kappa}{(\kappa!)^2}.\tag{9}$$

Here, the $P_\iota(0)=(-1)^\iota$ and $P_\iota(1)=1$. Therefore, the orthogonal condition is

$$\int_0^1 P_\iota(\tau)P_j(\tau)d\tau=\begin{cases}\frac{1}{2\iota+1}, & for \quad \iota=j,\\ 0, & for \quad \iota\neq j.\end{cases}\tag{10}$$

A function given by $\mathfrak{u}(\tau)$, resembling a square-integrable in the integral from $[0,1]$, can be defined by shifted Legendre polynomials (SLPs) given as follows:

$$\mathfrak{u}(\tau)=\sum_{j=0}^{\infty}c_jP_j(\tau),\tag{11}$$

in which the coefficients c_j are acquired as expressed below

$$c_j=(2j+1)\int_0^1 u(\tau)P_j(\tau)d\tau,\quad j=1,2,...\tag{12}$$

When $(\mathcal{H}+1)$ SLPs terms are taken into consideration, we then obtain

$$\mathfrak{u}(\tau)=\sum_{j=0}^{\infty}c_jP_j(\tau)=C^T\phi(\tau).\tag{13}$$

in which the shifted Legendre coefficient vector C including the shifted Legendre vector $\phi(\tau)$ is given by

$$C^T=[c_0,...,c_m],\quad \phi(\tau)=[P_0(\tau),P_1(\tau),...,P_m(\tau)]^T.\tag{14}$$

The $\phi(\tau)$ derivative can be expressed as follows

$$\frac{d\phi(\tau)}{d\tau}=\mathfrak{D}^{(1)}\phi(\tau),\tag{15}$$

in which $\mathfrak{D}^{(1)}$ denotes the operational matrix $(\mathcal{H}+1)\times(\mathcal{H}+1)$ of derivative that is expressed as follows

$$\mathfrak{D}^{(1)}=(d_{\iota j})=\begin{cases}2(2j+1), & for \quad j=\iota-\kappa, \begin{cases} for \quad \kappa=1,3,...,m \quad if \quad m \quad odd\\ for \quad \kappa=1,3,...,m-1 \quad if \quad m \quad even\end{cases}\\ 0, & otherwise,\end{cases}$$

Eq.(15) obviously indicates the following

$$\frac{d^n\phi(\tau)}{d\tau^n}=(\mathfrak{D}^{(1)})^n\phi(\tau),\tag{16}$$

in which $n\in\mathbb{N}$, as well as $\mathfrak{D}^{(1)}$ are expressed as matrix powers. Therefore

$$(\mathfrak{D}^{(1)})^n=\mathfrak{D}^{(n)},\qquad for \quad n=1,2,...\tag{17}$$

3 OMM for solve FFABC derivative

This section applies the operational matrix method (OMM) to solve fractional differential equations (FDEs). Moreover, the application of the OMM for non-linear and linear FDEs may be found in [42]. The differential equation is written as follows

$$^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau)=f(\tau,\mathfrak{u}).\tag{18}$$

Since the fractional integral may be differentiated, Eq.(18) may be expressed in [42, 43]. Hence, the Atangana-Baleanu Caputo (ABC) fractal-fractional derivative corresponding to the ABC sense is expressed below.

$$^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau)=q\tau^{q-1}f(\tau,\mathfrak{u})=\ ^{ABC}\mathfrak{D}^{\nu}\mathfrak{u}(\tau).\tag{19}$$

Theorem 3.1. Assume $\phi(\tau)$ vector is SLPs as expressed in Eq.(14). We assume that $\nu>0$, $q>0$, and the LHS for Eq.(19) gives

$$^{FFABC}\mathfrak{D}^{\nu,q}\phi(\tau)\simeq\ ^{ABC}\mathfrak{D}^{\nu}\phi(\tau)\simeq\ ^{ABC}\mathfrak{D}^{(\nu)}\phi(\tau),\tag{20}$$

in which $\mathfrak{D}^{\nu}$ denotes the operational matrix $(\mathcal{H}+1)\times(\mathcal{H}+1)$ that may be expressed as follows:

$$\mathfrak{D}^{(\nu)}=\begin{pmatrix}0 & 0 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ \sum_{\kappa=\lceil\nu\rceil}^{\lceil\nu\rceil}\theta_{\lceil\nu\rceil,0,\kappa} & \sum_{\kappa=\lceil\nu\rceil}^{\lceil\nu\rceil}\theta_{\lceil\nu\rceil,1,\kappa} & \dots & \sum_{\kappa=\lceil\nu\rceil}^{\lceil\nu\rceil}\theta_{\lceil\nu\rceil,m,\kappa}\\ \vdots & \vdots & \vdots & \vdots\\ \sum_{\kappa=\lceil\nu\rceil}^{\iota}\theta_{\iota,0,\kappa} & \sum_{\kappa=\lceil\nu\rceil}^{\iota}\theta_{\iota,1,\kappa} & \dots & \sum_{\kappa=\lceil\nu\rceil}^{\iota}\theta_{\iota,m,\kappa}\\ \vdots & \vdots & \vdots & \vdots\\ \sum_{\kappa=\lceil\nu\rceil}^{m}\theta_{m,0,\kappa} & \sum_{\kappa=\lceil\nu\rceil}^{m}\theta_{m,1,\kappa} & \dots & \sum_{\kappa=\lceil\nu\rceil}^{m}\theta_{m,m,\kappa}\end{pmatrix}\tag{21}$$

in which $\theta_{\iota,j,\kappa}$ is expressed as below

$$\theta_{\iota,j,\kappa} = \frac{M(\nu)}{(1-\nu)} \sum_{l=0}^{j} (-1)^{\iota+j} \frac{(\iota+\kappa)!}{(\iota-\kappa)!\kappa!\Gamma(\kappa)} b_{\kappa,j} \quad (22)$$

Proof. Using (3), (5), and (9), we now have

$$^{FFABC}\mathfrak{D}^{\nu}\tau^{\sigma} = \frac{M(\nu)}{1-\nu}\int_0^{\tau} u^{(n+1)}(s)E_{\nu}\left[\frac{-\nu(\tau-s)^{\nu}}{1-\nu}\right]ds,$$

$$\sigma > 1, \sigma \geqslant \lceil \nu \rceil$$

$$= \frac{M(\nu)}{1-\nu}\int_0^{\tau} \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-n)} S^{\sigma-n-1} E_{\nu}\left[\frac{-\nu(\tau-s)^{\nu}}{1-\nu}\right]ds.$$

$Assume \quad 0<\nu<1, \quad \lfloor \nu \rfloor = n = 0, and \quad \psi(\tau)$

$is \quad solved \quad for \int_0^{\tau} S^{\sigma-1}E_{\nu}\left[\frac{-\nu(\tau-s)^{\nu}}{1-\nu}\right]ds.$

$$^{FFABC}\mathfrak{D}^{\nu}p_{\iota}(\tau) = \sum_{\kappa=0}^{\iota} \frac{(-1)^{\iota+\kappa}(\iota+\kappa)!}{(\iota-\kappa)!\kappa!^2} \, {}^{FFABC}\mathfrak{D}^{\nu}\tau^{q}$$

$$= \sum_{\kappa=\lceil \nu \rceil}^{\iota} \frac{(-1)^{\iota+\kappa}(\iota+\kappa)!}{(\iota-\kappa)!\kappa!\Gamma(\kappa)} \frac{M(\nu)}{1-\nu}\psi(\tau)$$

$$\psi(\tau) \simeq \sum_{j=0}^{m} b_{\kappa,j}p_j(\tau)$$

$$b_{\kappa,j} = (2j+1)\sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2}\int_0^1 \psi(\tau)\tau^l d\tau$$

$$^{FFABC}\mathfrak{D}^{\nu}p_{\iota}(\tau) = \sum_{\kappa=\lceil \nu \rceil}^{\iota}\sum_{j=0}^{m} \frac{(-1)^{\iota+\kappa}(\iota+\kappa)!}{(\iota-\kappa)!\kappa!\Gamma(\kappa)} \frac{M(\nu)}{1-\nu} b_{\kappa,j}p_j(\tau)$$

$$= \sum_{j=0}^{m}\left(\sum_{\kappa=\lceil \nu \rceil}^{\iota} \theta_{\iota,j,\kappa}\right)p_j(\tau), \quad \iota = \lceil \nu \rceil, ..., m \quad (23)$$

3.1 Linear FDEs

Consider the linear FDEs given by

$$^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) = b_r\mathfrak{u}(\tau) + b_r q(\tau), \quad for \quad r = 1,2,...$$

The initial conditions are as follows

$$\mathfrak{u}_0^{(i)} = d_i, \quad i = 0,1,2,...,n.$$

To solve the above equations, we present an approximation of the function

$$q(\tau) \simeq \sum_{u=0}^{m} q_j P_j(\mathsf{x}) = Q^T\phi(\tau),$$

$$^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) \simeq C^T \, {}^{FFABC}\mathfrak{D}^{\nu,q}\phi(\tau),$$

where $Q = [q_0, ..., q_m]^T$ is known as a vector. By employing the residual $R_m(\tau)$, we can now write it as follows:

$$R_m(\tau) \simeq \left(C^T \, {}^{FFABC}\mathfrak{D}^{\nu,q} - b_{r+1}C^T - b_{r+2}Q^T\right)\phi(\tau).$$

We generate $\mu - n$ linear equations concerning a typical tau method [40] by employing

$$\langle R_m(\tau), P_i(\tau)\rangle = \int_0^1 R_m(\tau)P_j(\tau)d\tau = 0,$$

$$j = 0,1,...,\mu-n-1.$$

Moreover, we also obtain

$$\mathfrak{u}_0 = C^T\phi(0) = d_0,$$
$$\mathfrak{u}_0^{(1)} = C^T\mathfrak{D}^{(1)}\phi(0) = d_1,$$
$$\vdots$$
$$\mathfrak{u}_0^{(n)} = C^T\mathfrak{D}^{(n)}\phi(0) = d_n.$$

We now have the generation $(\mu - n)$ of linear equations. Here, the linear equations may be solved by employing unknown coefficients with respect to vector C.

3.2 Nonlinear FDEs

Consider the non-linear FDEs

$$^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) = F(\tau, \mathfrak{u}(\tau), ...).$$

The initial conditions are given by

$$\mathfrak{u}_0^{(i)} = d_i, \quad i = 0,...,n.$$

We put in mind that, generally, F may be non-linear.

$$C^T \, {}^{FFABC}\mathscr{D}^{\nu,q}\phi(\tau) \simeq F(\tau, C^T\phi(\tau), ...).$$

We collocate using first $(\mu - n)$ shifted Legendre roots of $\bar{P}_{\mu+1}(\tau)$. The equations given above generate $(\mu+1)$ non-linear equations that can be solved by employing Newton's iterative method.

3.3 Estimation Error

Let non-linear FDEs be given as below:

$$\mathfrak{D}^{\nu}\mathfrak{u}(\tau) = F(\tau, \mathfrak{u}(\tau), \mathfrak{D}^{\beta_1}\mathfrak{u}(\tau), ..., \mathfrak{D}^{q_\kappa}\mathfrak{u}(\tau)). \quad (24)$$

Using [44], the residual correction procedure uses Eq.(24) to estimate the absolute error as follows:

$$F = g(\mathfrak{u}(\tau), \mathfrak{D}^{(q_1)}\mathfrak{u}(\tau), ..., \mathfrak{D}^{(q_\kappa)}\mathfrak{u}(\tau)) + h(\tau).$$

Next, by subtracting and adding the term

$$\mathfrak{D}^{(\nu)}\mathfrak{u}_\mu(\tau) - g\big(\mathfrak{u}_\mu(\tau), \mathfrak{D}^{(q_1)}\mathfrak{u}_\mu(\tau), ..., \mathfrak{D}^{(q_\kappa)}\mathfrak{u}_\mu(\tau)\big) \quad (25)$$

into Eq.(24) as well as obtaining $e_\mu := \mathfrak{u}(\tau) - \mathfrak{u}_\mu(\tau)$ gives

$$e_\mu^{(\nu)} - g\big(e_\mu(\tau), e_\mu^{(\beta_1)}(\tau), ..., e_\mu^{(q_\kappa)}(\tau)\big)$$
$$= h(\tau) + g\big(\mathfrak{u}_\mu(\tau), \mathfrak{u}_\mu^{(\beta_1)}(\tau), ..., \mathfrak{u}_\mu^{(q_\kappa)(\tau)}\big) + R, \quad (26)$$

in which R may be gained from the non-linear g terms. Moreover, the answer for Eq.(24) is obtained in an identical manner as the "Applications of Operational Matrices for Legendre polynomials" section, which depends on the initial conditions given below

$$e(0) = 0.$$

Let $\hat{e}_n$ resemble the rough answer with respect to Eq.(24), which is solved utilising this particular method.
If

$$||e - \hat{e}_n|| < \epsilon, \tag{27}$$

then, the absolute error e can be gained by employing $\hat{e}_n$.

3.4 Error bound

The error bound for the fractal fractional Atangana–Baleanu Caputo (FFABC)-derivative operational matrix is produced in this section. The following theorem [45] is given for this purpose.

Theorem 3.2. The error $|\triangle_\mu| = |{}^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) - {}^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}_\mu(\tau)|$ in approximating ${}^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau)$ having the OMM with respect to the fractional derivative has the following bounds:

$$|\triangle_\mu| \leq \sum_{\iota=\mu+1}^{\infty} |c_\iota| \sum_{j=1}^{\mu} |D_{\iota,j}| \sum_{\kappa=1}^{j} |(-1)^{j+\kappa} \frac{(j+\kappa)!}{(\kappa!)^2(j-\kappa)!}|,$$

in which $\mathfrak{u}_\mu$ resembles the h function approximation depending on the shifted Legendre polynomials (SLPs), c_ι. Here, $\iota = 1, 2, ..., \mu$ resemble the coefficients of this approximation in which

$$\mathfrak{D}_{\iota,j} = \begin{cases} \sum_{\kappa=\lceil\nu\rceil}^{\iota} \theta_{\iota,j,\kappa}, & for \quad \iota = \lceil\nu\rceil, ..., \mu, \quad j = 1, ..., \mu, \\ 0, & for \quad \iota = 1, ..., \lceil\nu\rceil, \quad j = 1, ..., \mu. \end{cases}$$

Proof. We now obtain

$$\mathfrak{u}(\tau) = \sum_{\iota=1}^{\infty} c_\iota P_\iota(\tau).$$

Using Eq.(20), we now have

$${}^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) = \sum_{\iota=1}^{\infty} c_i \sum_{j=1}^{\mu} D_{\iota,j} P_j.$$

Provided that we only consider the infinite series' first m terms given above, we now have

$$\begin{aligned} &{}^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) - \sum_{\iota=1}^{\mu} c_\iota \sum_{j=1}^{\mu} D_{\iota,j} P_j(\tau) \\ &= \sum_{\iota=\mu+1}^{\infty} c_\iota \sum_{j=1}^{\mu} D_{\iota,j} P_j(\tau). \end{aligned} \tag{28}$$

By employing Eq.(13) and Eq.(20), we illustrate that Eq.(28) has a matrix form given below:

$$\begin{aligned} &{}^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) - C^T \mathfrak{D}^{(\nu)} \phi(\tau) \\ &= \sum_{\iota=\mu+1}^{\infty} c_\iota \sum_{j=1}^{\mu} D_{\iota,j} P_j(\tau). \end{aligned}$$

Now, we have:

$$\begin{aligned} &|{}^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) - C^T \mathfrak{D}^{(\nu)} \phi(\tau)| \\ &= |\sum_{\iota=\mu+1}^{\infty} c_\iota \sum_{j=1}^{\mu} \mathfrak{D}_{\iota,j} P_j(\tau)| \leq \sum_{\iota=\mu+1}^{\infty} |c_\iota| \sum_{j=1}^{\mu} |\mathfrak{D}_{\iota,j}||P_j(\tau)|. \end{aligned} \tag{29}$$

The following is the upper bound of SLPs:

$$\begin{aligned} |P_j(\tau)| &= \left| \sum_{\kappa=1}^{j} \frac{(-1)^{j+\kappa}(j+\kappa)!}{(j-\kappa)!(\kappa!)^2} \tau^\kappa \right| \\ &\leq \sum_{\kappa=1}^{j} \left| \frac{(-1)^{j+\kappa}(j+\kappa)!}{(\kappa!)^2(j-\kappa)!} \right| |\tau^\kappa| \leq \sum_{\kappa=1}^{j} \left| \frac{(-1)^{j+\kappa}(j+\kappa)!}{(\kappa!)^2(j-\kappa)!} \right|. \end{aligned} \tag{30}$$

Thus, by substituting Eqs.(29) into (30) gives:

$$\begin{aligned} &|{}^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) - C^T D^{(\nu)} \phi(\tau)| \\ &\leq \sum_{\iota=\mu+1}^{\infty} |c_\iota| \sum_{j=1}^{\mu} |\mathfrak{D}_{\iota,j}| \sum_{\kappa=1}^{j} \left| (-1)^{j+\kappa} \frac{(j+\kappa)!}{(\kappa!)^2(j-\kappa)!} \right|. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &|{}^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) - {}^{FFABC}\mathfrak{D}^{\nu,q}\mathfrak{u}_\mu(\tau)| \\ &\leq \sum_{\iota=\mu+1}^{\infty} |c_\iota| \sum_{j=1}^{\mu} |\mathfrak{D}_{\iota,j}| \sum_{\kappa=1}^{j} |(-1)^{j+\kappa} \frac{(j+\kappa)!}{(\kappa!)^2(j-\kappa)!}|. \end{aligned}$$

Thus, the proof is complete.

3.5 OMM of VOFFABC-derivative

This section can be dedicated to the numerical solution of the problem employing the OMM of VOFFABC-derivative [46], as shown in Eq.(19)

$${}^{VOFFABC}\mathfrak{D}^{\nu(\tau),q(\tau)}\mathfrak{u}(\tau) = q(\tau)\tau^{q(\tau)-1} f(\tau, \mathfrak{u}), \tag{31}$$

as well as for

$$\begin{aligned} \phi(\tau) &= [P_0(\tau), P_1(\tau), ..., P_m(\tau)]^T, \\ \lambda(\tau) &= [1, \tau, \tau^2, ..., \tau^n]^T. \end{aligned} \tag{32}$$

Here, the vector $\phi(\tau)$ may be defined as given below

$$\phi(\tau) = A\lambda(\tau), \tag{33}$$

in which A denotes square matrix $(\mathcal{H}+1) \times (\mathcal{H}+1)$ as follows

$$\begin{aligned} &(a_{\iota,j})_{0 \leq \iota,j \leq n} = \\ &\begin{cases} (-1)^{\iota-j} \frac{\Gamma(\iota+1)\Gamma(\iota+j+1)}{\Gamma(j+1)\Gamma(\iota+1)\Gamma(\iota-j+1)\Gamma(j+1)}, & for \quad \iota \geq j, \\ 0, & for \quad otherwise. \end{cases} \end{aligned} \tag{34}$$

Thus, by employing Eq.(33), we state that

$$\lambda(\tau) = A^{-1}\phi(\tau). \tag{35}$$

The OMM depending on SLPs of VOFFABC operator can be written by ${}^{VOFFABC}\mathfrak{D}^{\nu(\tau)}\phi(\tau)$. By employing Eq.(33), we have

$$\begin{aligned} &{}^{VOFFABC}\mathfrak{D}^{\nu(\tau),q(\tau)}\phi(\tau) \simeq {}^{ABC}\mathfrak{D}^{\nu(\tau)}\phi(\tau) = \\ &{}^{ABC}\mathfrak{D}^{\nu(\tau)}(A\lambda(\tau)) = A\ {}^{ABC}\mathfrak{D}^{\nu(\tau)}[1, \tau, \tau^2, ..., \tau^n]^T. \end{aligned} \tag{36}$$

Eq.(5)'s ABC-derivative with respect to the variable order can be employed here. Following that, we may acquire Eq.(36) in the form given below:

$$\begin{aligned}
{}^{ABC}\mathfrak{D}^{\nu(\tau)}\phi(\tau) &= [0, \frac{\Gamma(2)}{\Gamma(1-\nu(\tau))}\tau\sum_{\kappa=0}^{\infty}\frac{(\frac{-\nu(\tau)}{1-\nu(\tau)}\tau^{\nu(\tau)})^{\kappa}}{\Gamma(\kappa\nu(\tau)+2)}, \frac{\Gamma(3)}{\Gamma(1-\nu(\tau))}\tau^2\sum_{\kappa=0}^{\infty}\frac{(\frac{-\nu(\tau)}{1-\nu(\tau)}\tau^{\nu(\tau)})^{\kappa}}{\Gamma(\kappa\nu(\tau)+3)}, \\
&\qquad ..., \frac{\Gamma(n+1)}{\Gamma(1-\nu(\tau))}\tau^n\sum_{\kappa=0}^{\infty}\frac{(\frac{-\nu(\tau)}{1-\nu(\tau)}\tau^{\nu(\tau)})^{\kappa}}{\Gamma(\kappa\nu(\tau)+n+1)}]^T \\
&= A\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(1-\nu(\tau))}\sum_{\kappa=0}^{\infty}\frac{(\frac{-\nu(\tau)}{1-\nu(\tau)}\tau^{\nu(\tau)})^{\kappa}}{\Gamma(\kappa\nu(\tau)+2)} & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(1-\nu(\tau))}\sum_{\kappa=0}^{\infty}\frac{(\frac{-\nu(\tau)}{1-\nu(\tau)}\tau^{\nu(\tau)})^{\kappa}}{\Gamma(\kappa\nu(\tau)+3)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma(n+1)}{\Gamma(1-\nu(\tau))}\sum_{\kappa=0}^{\infty}\frac{(\frac{-\nu(\tau)}{1-\nu(\tau)}\tau^{\nu(\tau)})^{\kappa}}{\Gamma(\kappa\nu(\tau)+n+1)} \end{pmatrix}\begin{pmatrix} 1 \\ \tau \\ \tau^2 \\ \vdots \\ \tau^n \end{pmatrix} \\
&= A \quad B(\tau)\lambda(\tau).
\end{aligned} \tag{37}$$

Substituting Eq.(35) into Eq.(37), we now have

$$ {}^{ABC}\mathfrak{D}^{\nu(\tau)}\phi(\tau) = AB(\tau)A^{-1}\phi(\tau). \tag{38}$$

The approximate solution may be expressed as

$$\begin{aligned} {}^{ABC}\mathfrak{D}^{\nu(\tau)}\mathfrak{u}(\tau) &\simeq {}^{ABC}\mathfrak{D}^{\nu(\tau)}(C^T\phi(\tau)) \\ &= C^T \; {}^{ABC}\mathfrak{D}^{\nu(\tau)}\phi(\tau) = C^T AB(\tau)A^{-1}\phi(\tau) \end{aligned} \tag{39}$$

$$\begin{aligned} &C^T AB(\tau)A^{-1}\phi(\tau) = F[\tau, C^T\phi(\tau), C^T A \; {}^{ABC}\mathfrak{D}^{(1)}A^{-1}\phi(\tau), \\ &..., C^T A \; {}^{ABC}\mathfrak{D}^{(n)}A^{-1}\phi(\tau)], \\ &\quad 0 \leq \tau \leq 1. \end{aligned} \tag{40}$$

Moreover, the collocation points, $z_\iota = \frac{2\iota+1}{2n+2}, \iota = 0, 1, ..., n$ are utilized to switch the equations system in Eq.(40) to an algebraic equations system in the form given below:

$$\begin{aligned} &C^T AB(\tau_\iota)A^{-1}\phi(\tau_\iota) = F[\tau_\iota, C^T\phi(\tau_\iota), C^T A \; {}^{ABC}\mathfrak{D}^{(1)}A^{-1}\phi(\tau_\iota), \\ &..., C^T A \; {}^{ABC}\mathfrak{D}^{(n)}A^{-1}\phi(\tau_i)], \\ &C^T\phi(0) = \tau_0. \end{aligned} \tag{41}$$

Eventually, by solving the system algebraic equations provided in Eq.(41), unknown vector C provided in Eq.(12) may be acquired.

4 Numerical examples

The numerical examples of non-linear and linear VOFFABC-derivatives will be solved in this section. In our computational findings, the absolute error will be employed to quantify the difference between the approximate and exact solutions. When the results achieved by the current approach are compared to those acquired by other approaches, it is clear that the current approach is both convenient and effective. MATLAB R2020b is utilised to code and run all of the numerical programmes.

- **LOMM** Legendre Operational Matrix Method derived in this study.
- **PRCO** Predictor-Corrector method as provided in [38].
- **MTSLP** Mixture Two Step Lagrange Polynomial, as well as the fundamental theorem of fractional calculus as provided in [39].

Example 4.1. By considering the fractal fractional differential equations (FDEs) given below having the generalised Mittag-Leffler kernel as:

$$\mathfrak{D}^{\nu,q}\mathfrak{u}(\tau) = \tau, \quad \mathfrak{u}(0) = 0$$

The exact solution is given by

$$\mathfrak{u}(\tau) = \frac{q\tau^q\Gamma(1+q)}{AB(\nu)}\left[\frac{1-\nu}{\Gamma(1+q)} + \frac{\nu\tau^\nu}{\Gamma(\nu+q+1)}\right].$$

Tables 1, 2, and 3 demonstrate the absolute error for the two methods with respect to $\nu = 0.85, 0.9, 0.95, \quad q = 0.85, 0.9, 0.95$ as well as $\mu = 6$. Utilising shifted Legendre polynomials (SLPs), Legendre operational matrix method (LOMM) can produce a good approximate solution equivalent to the exact solution.

Example 4.2. Let the variable order fractal fractional problem be written below as

$$\begin{aligned} &\mathfrak{D}^{\nu(\tau),q}\mathfrak{u}(\tau) + u(\tau) = f(\tau), \\ &\mathfrak{u}(0) = 0, \\ &inwhich \quad f(\tau) = \frac{\Gamma(4)\tau^{3-\nu(\tau)}}{\Gamma(4-\nu(\tau))} + \frac{\Gamma(3)\tau^{2-\nu(\tau)}}{\Gamma(3-\nu(\tau))}. \end{aligned}$$

The exact solution is given by $\mathfrak{u}(\tau) = \tau^2 + \tau^3$. Here, the comparison with respect to absolute error for the two methods for

q	Method	$\tau = 1\times10^{-1}$	$\tau = 3\times10^{-1}$	$\tau = 5\times10^{-1}$	$\tau = 7\times10^{-1}$	$\tau = 9\times10^{-1}$
0.85	LOMM	2.06313×10^{-4}	1.13294×10^{-4}	5.82273×10^{-5}	1.55907×10^{-4}	1.67169×10^{-4}
	PRCO [38]	6.39359×10^{-4}	6.51022×10^{-4}	6.48747×10^{-4}	6.44036×10^{-4}	6.39035×10^{-4}
	MTSLP [39]	2.93578×10^{-2}	1.96203×10^{-2}	1.76722×10^{-2}	1.65664×10^{-2}	1.58088×10^{-2}
0.9	LOMM	1.03475×10^{-4}	5.71677×10^{-5}	1.67782×10^{-5}	6.04172×10^{-5}	6.49549×10^{-5}
	PRCO [38]	3.92143×10^{-4}	4.09983×10^{-4}	4.13114×10^{-4}	4.13023×10^{-4}	4.11936×10^{-4}
	MTSLP [39]	2.74931×10^{-2}	2.03492×10^{-2}	1.88290×10^{-2}	1.79523×10^{-2}	1.73447×10^{-2}
0.95	LOMM	2.58108×10^{-5}	1.45461×10^{-5}	1.55983×10^{-5}	1.33557×10^{-5}	1.39378×10^{-5}
	PRCO [38]	1.79949×10^{-4}	1.93451×10^{-4}	1.97280×10^{-4}	1.98763×10^{-4}	1.99369×10^{-4}
	MTSLP [39]	2.56756×10^{-2}	2.10144×10^{-2}	1.99903×10^{-2}	1.93944×10^{-2}	1.89783×10^{-2}

Table 1. Absolute error gained for $\nu = 0.85$ employing distinct q values as well as comparing with other methods following Example 4.1.

q	Method	$\tau = 1\times10^{-1}$	$\tau = 3\times10^{-1}$	$\tau = 5\times10^{-1}$	$\tau = 7\times10^{-1}$	$\tau = 9\times10^{-1}$
0.85	LOMM	1.22633×10^{-4}	6.16777×10^{-5}	3.33070×10^{-5}	7.81926×10^{-5}	7.52988×10^{-5}
	PRCO [38]	5.87586×10^{-4}	6.42649×10^{-4}	6.59114×10^{-4}	6.66523×10^{-4}	6.70413×10^{-4}
	MTSLP [39]	2.12319×10^{-2}	1.45256×10^{-2}	1.31305×10^{-2}	1.23360×10^{-2}	1.17902×10^{-2}
0.9	LOMM	6.19779×10^{-5}	3.21822×10^{-5}	6.81177×10^{-6}	2.67806×10^{-5}	2.56285×10^{-5}
	PRCO [38]	3.60234×10^{-4}	4.04217×10^{-4}	4.19149×10^{-4}	4.26841×10^{-4}	4.31539×10^{-4}
	MTSLP [39]	1.98418×10^{-2}	1.49861×10^{-2}	1.39043×10^{-2}	1.32766×10^{-2}	1.28400×10^{-2}
0.95	LOMM	1.64624×10^{-5}	9.88356×10^{-6}	1.31106×10^{-5}	1.19212×10^{-5}	1.17213×10^{-5}
	PRCO [38]	1.65237×10^{-4}	1.90491×10^{-4}	1.99879×10^{-4}	2.05108×10^{-4}	2.08537×10^{-4}
	MTSLP [39]	1.84928×10^{-2}	1.54068×10^{-2}	1.46869×10^{-2}	1.42635×10^{-2}	1.39660×10^{-2}

Table 2. For $\nu = 0.9$ the absolute error gained employing distinct q values as well as comparing with other methods as given in Example 4.1.

$\nu(\tau) = \sin(\tau)$, $\nu(\tau) = 0.5\tau + 0.2$, $q = 0.97, 0.98, 0.99$, $q(\tau) = e^{\tau}/2$ and $\mu = 4$ is illustrated in Tables 4, 5, as well as Figures 1, 2, and 3. Furthermore, the operation matrix depending on SLPs, is capable of gaining excellent approximate solution equivalent with respect to the exact solution.

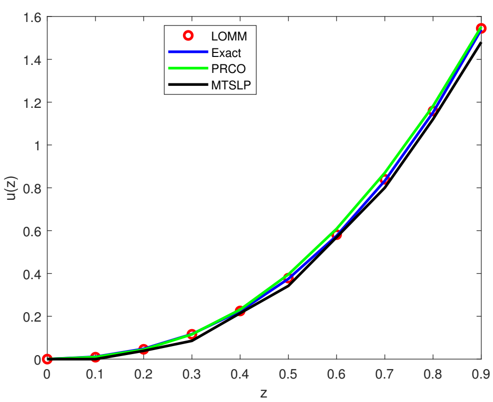

Figure 1. The comparison of MTSLP, PRCO, LOMM as well as an exact solution for $\nu(\tau) = \sin(\tau), q = e^{\tau}/2$ following Example 4.2

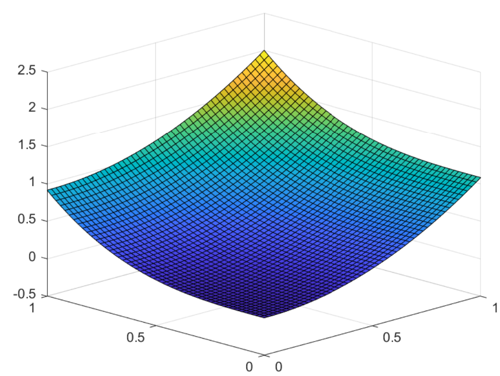

Figure 2. Approximate solution with respect to $\nu = \sin(\tau), q(\tau) = e^{\tau}/2$ in Example 4.2

Example 4.3. We now assume that the non-linear variable order fractal fractional problem is given as follows

$$\mathfrak{D}^{\nu(\tau),q}\mathfrak{u}(\tau) + \mathfrak{u}^2(\tau) = \tau^2 + \tau^4 + \frac{2\tau^{2-\sin(\tau)}}{\Gamma(3-\sin(\tau))}.$$

Here, the exact solution is given by $\mathfrak{u}(\tau) = \tau^2$ and $\mathfrak{u}(0) = 0$ is the initial condition. Here, the approximation of $\nu(\tau) = 0.5\tau + 0.6$, $q = 0.7, 0.8, 0.9$, $q(\tau) = \tau\sin(\tau) + 0.5$ and $\mu = 4$ is shown in Figures 4, 5 and 6. Moreover, an operation matrix depending on SLPs can provide a better approximate solution in comparison to the exact solution.

Example 4.4. We now let the linear variable-order fractal FDE be given below:

$$\mathfrak{D}^{\nu(\tau),q(\tau)}\mathfrak{u}(\tau) + \mathfrak{u}(\tau) = \frac{4.6}{4.6-\nu}\tau^{3.6-\nu} + \tau^{3.6}.$$

The exact solution is expressed by $\mathfrak{u}(\tau) = \tau^{3.6}$. Note that, the approximate solution of $\nu(\tau) = 0.5\tau + 0.1$, $\nu(\tau) = \tau\sin\tau$,

q	Method	$\tau = 1\times10^{-1}$	$\tau = 3\times10^{-1}$	$\tau = 5\times10^{-1}$	$\tau = 7\times10^{-1}$	$\tau = 9\times10^{-1}$
0.85	LOMM	4.69591×10^{-5}	2.34872×10^{-5}	1.58735×10^{-6}	8.76677×10^{-6}	6.14577×10^{-6}
	PRCO [38]	5.36313×10^{-4}	6.29988×10^{-4}	6.65003×10^{-4}	6.85000×10^{-4}	6.98432×10^{-4}
	MTSLP [39]	1.36163×10^{-2}	9.67696×10^{-3}	8.82369×10^{-3}	8.33592×10^{-3}	7.99986×10^{-3}
0.9	LOMM	2.34937×10^{-5}	1.34421×10^{-5}	9.08186×10^{-6}	4.20723×10^{-6}	3.78702×10^{-6}
	PRCO [38]	3.28661×10^{-4}	3.95797×10^{-4}	4.22355×10^{-4}	4.38091×10^{-4}	4.48964×10^{-4}
	MTSLP [39]	1.26720×10^{-2}	9.88623×10^{-3}	9.23544×10^{-3}	8.85550×10^{-3}	8.58997×10^{-3}
0.95	LOMM	6.23697×10^{-6}	5.70149×10^{-6}	1.32362×10^{-5}	1.23700×10^{-5}	9.97319×10^{-6}
	PRCO [38]	1.50693×10^{-4}	1.86301×10^{-4}	2.01139×10^{-4}	2.10219×10^{-4}	2.16645×10^{-4}
	MTSLP [39]	1.17628×10^{-2}	1.00781×10^{-2}	9.66034×10^{-3}	9.41185×10^{-3}	9.23593×10^{-3}

Table 3. For $\nu = 0.95$ Absolute error gained employing distinct values of q as well as comparing with other methods following Example 4.1.

q	Method	$\tau = 1\times10^{-1}$	$\tau = 3\times10^{-1}$	$\tau = 5\times10^{-1}$	$\tau = 7\times10^{-1}$	$\tau = 9\times10^{-1}$
0.97	LOMM	3.97115×10^{-3}	1.06860×10^{-3}	1.12966×10^{-2}	1.21676×10^{-2}	1.08637×10^{-2}
	PRCO [38]	2.97940×10^{-3}	1.18081×10^{-3}	3.17167×10^{-2}	2.47609×10^{-2}	1.50330×10^{-1}
	MTSLP [39]	1.1×10^{-2}	2.50732×10^{-2}	1.28772×10^{-2}	3.95564×10^{-2}	2.14144×10^{-1}
0.98	LOMM	4.01898×10^{-3}	9.66956×10^{-4}	1.19194×10^{-2}	1.51600×10^{-2}	2.98961×10^{-3}
	PRCO [38]	3.02787×10^{-3}	1.28845×10^{-3}	3.23199×10^{-2}	2.75964×10^{-2}	1.43720×10^{-1}
	MTSLP [39]	1.1×10^{-2}	2.51581×10^{-2}	1.22493×10^{-2}	3.69157×10^{-2}	2.08012×10^{-1}
0.99	LOMM	4.06719×10^{-3}	8.59347×10^{-4}	1.25217×10^{-2}	1.81053×10^{-2}	4.79593×10^{-3}
	PRCO [38]	3.07676×10^{-3}	1.40186×10^{-3}	3.29015×10^{-2}	3.03847×10^{-2}	1.37182×10^{-1}
	MTSLP [39]	1.1×10^{-2}	2.52569×10^{-2}	1.16535×10^{-2}	3.43247×10^{-2}	2.01949×10^{-1}

Table 4. Absolute error gained for $\nu(\tau) = \sin(\tau)$ employing distinct q values as well as comparing with other methods as given in Example 4.2.

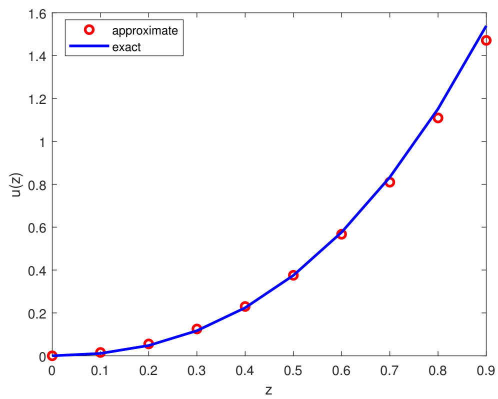

Figure 3. Approximate solution with respect to $\nu(\tau) = 0.5\tau + 0.2, q(\tau) = e^{\tau}/2$ with exact solutions as given in Example 4.2

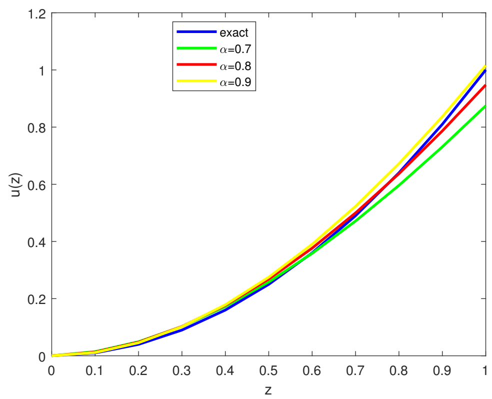

Figure 4. Approximate solution with respect to $q = 0.7, 0.8$, and 0.9 having exact solutions in Example 4.3

$q(\tau) = \tau\sin\tau, \quad q(\tau) = 0.5\tau + 0.1$ and $\mu = 4$ is shown in Figures 7 and 8. OM depending on SLPs can gain a better approximate solution, which is comparable with the exact solution.

Example 4.5. Let the non-linear variable order fractal fractional problem be given below

$$\mathfrak{D}^{\nu(\tau),q(\tau)}\mathfrak{u}(\tau) + \mathfrak{u}(\tau) + \mathfrak{u}^2(\tau) =$$
$$1 + (-2 + 2^{\nu(\tau)})e^{2\tau} + e^{4\tau} - \frac{2^{\nu(\tau)}e^{2\tau}\Gamma(1-\nu(\tau), 2\tau)}{\Gamma(1-\nu(\tau))}.$$

Note that the exact solution is provided by $\mathfrak{u}(\tau) = e^{2\tau} - 1$. Contrarily, the approximate solution of $\nu(\tau) = \tau\sin(\tau) + 0.3, \quad \nu = \sin(\tau), q(\tau) = 0.5\tau + 0.6, \quad q(\tau) = e^{\tau}$ and $\mu = 4$ is shown in Figure 9 as well as Figure 10. Operation matrix depending on SLPs can generate a better approximate solution, which is comparable with the exact solution.

5 Conclusion

With regard to solving fractional-order and VOFFDEs, this paper suggested an operational matrix predicated on shifted Legendre polynomials (SLPs). This derivative was recently

τ	LOMM	PRCO[38]	MTSLP[39]
1×10^{-1}	2.01455×10^{-3}	1.50367×10^{-3}	1.1×10^{-2}
2×10^{-1}	2.21141×10^{-3}	3.48209×10^{-3}	3.57609×10^{-2}
3×10^{-1}	1.08655×10^{-3}	1.34540×10^{-3}	6.52672×10^{-2}
4×10^{-1}	8.64069×10^{-4}	7.20471×10^{-3}	9.13329×10^{-2}
5×10^{-1}	3.14448×10^{-3}	2.03360×10^{-2}	1.07918×10^{-1}
6×10^{-1}	5.25871×10^{-3}	3.25329×10^{-2}	1.14064×10^{-1}
7×10^{-1}	6.71079×10^{-3}	3.71711×10^{-2}	1.11171×10^{-1}
8×10^{-1}	7.00475×10^{-3}	3.00555×10^{-2}	1.01530×10^{-1}
9×10^{-1}	5.64463×10^{-3}	1.28583×10^{-2}	8492968×10^{-2}

Table 5. Absolute error gained for $\nu(\tau) = \sin(\tau)$ and $q(\tau) = e^{\tau}/2$ and contrasted with several other methods for Example 4.2.

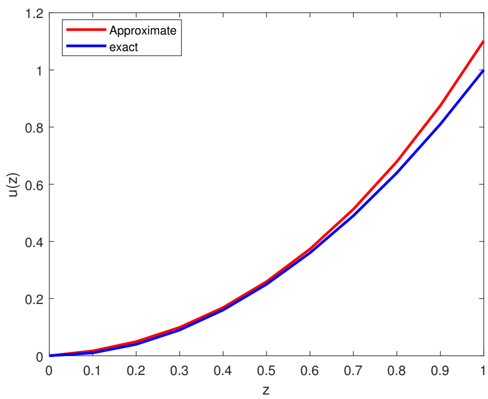

Figure 5. Approximate solution with respect to $q(\tau) = \tau\sin(\tau) + 0.5$ having exact solutions in Example 4.3

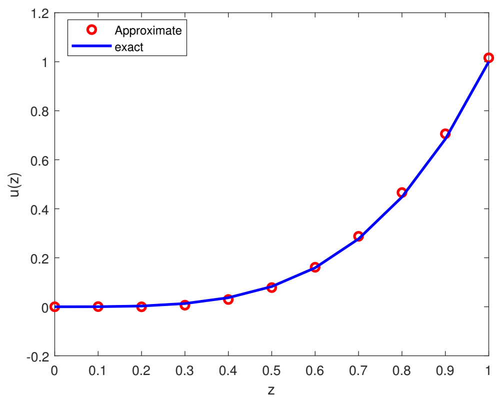

Figure 7. Approximate solution for $\nu(\tau) = 0.5z + 0.1, \quad q(\tau) = \tau\sin\tau$ having exact solutions in Example 4.4

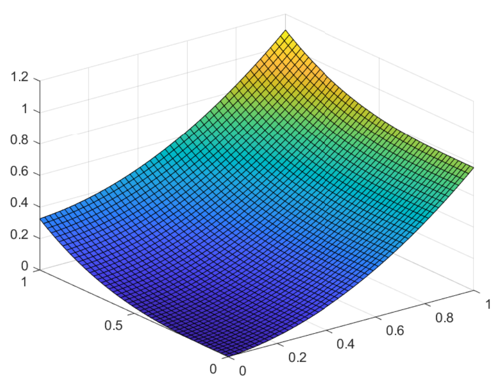

Figure 6. Approximate solution with respect to $\nu(\tau) = 0.5\tau + 0.6, \quad q(\tau) = \tau\sin(\tau) + 0.5$ in Example 4.3

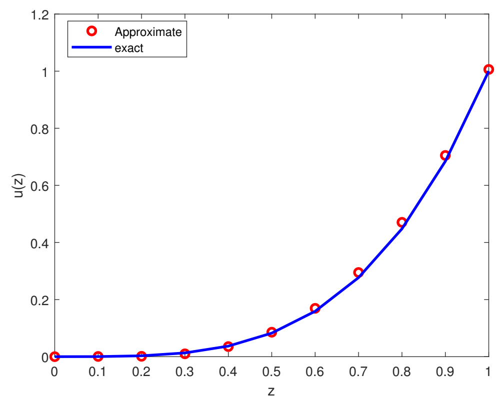

Figure 8. Approximate solution for $\nu(\tau) = \tau\sin\tau, \quad q(\tau) = 0.5\tau + 0.1$ having exact solutions in Example 4.4

proposed and considered a new type of differential equation. By comparing the suggested methods' outcomes with exact solutions to the issues, an estimation error was incorporated. Other than that, several instances were presented and solved to exhibit efficacy and accuracy with regard to the suggested methodologies, which were then contrasted with the findings with several numerical methods currently in use. When the outcomes achieved by the current approach are compared to those acquired by other approaches, it is clear that the current technique is both efficient and convenient. The studies of applying the orthogonal functions based on the fractal Atangana-Baleanu Caputo (ABC) or fractal Caputo-type fractional differentiability have not been completed yet. The research can be extended to the other models of these functions, such as Walsh functions, Haar wavelet polynomials, Bessel series, and Legendre wavelet functions. Moreover, one can extend the results acquired in this thesis under Riemann-Liouville fractional differentiability.

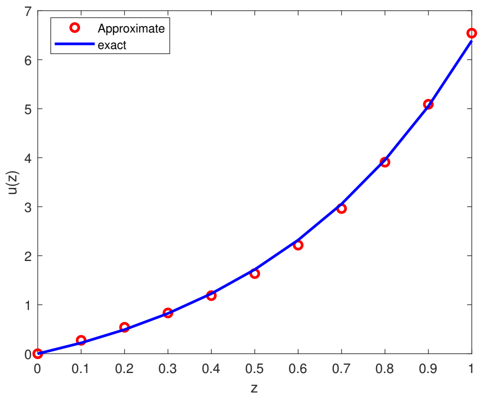

Figure 9. Approximate solution with respect to $\nu(\tau) = \tau\sin(\tau) + 0.3, \quad q(\tau) = 0.5\tau + 0.6$ with exact solutions in Example 4.5

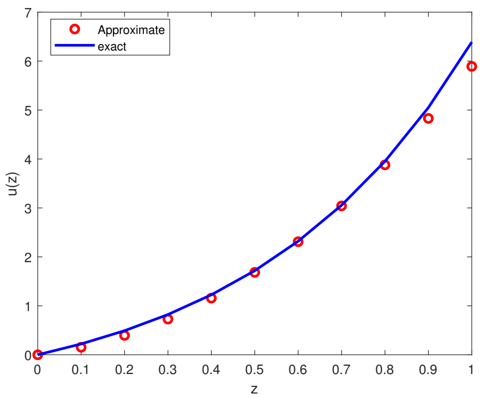

Figure 10. Approximate solution with respect to $\nu = \sin(\tau), \quad q(\tau) = e^{\tau}$ with exact solutions in Example 4.5

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