Mathematics and Statistics 11(1): 144-148, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110116

A Formal Solution of Quadruple Series Equations

A.K. Awasthi*, **Rachna**, **Rohit**

Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, 144411, India

Received September 23, 2022; Revised December 10, 2022; Accepted December 22, 2022

Cite This Paper in the following Citation Styles

***(a)**: [1] A.K. Awasthi, Rachna, Rohit, "A Formal Solution of Quadruple Series Equations," Mathematics and Statistics, Vol.11, No.1, pp. 144-148, 2023. DOI: 10.13189/ms.2023.110116*

***(b)**: A.K. Awasthi, Rachna, Rohit, (2023). A Formal Solution of Quadruple Series Equations. Mathematics and Statistics, 11(1), 144-148. DOI: 10.13189/ms.2023.110116*

Abstract It cannot be overstated how significant Series Equations are to the fields of pure and applied mathematics respectively. The majority of mathematical topics revolve around the use of series. Virtually, in every subject of mathematics, series play an important role. Series solutions play a major role in the solution of mixed boundary value problems. Dual, triple, and quadruple series equations are useful in finding the solution of four part boundary value problems of electrostatics, elasticity and other fields of Mathematical physics. Cooke devised a method for finding the solution of quadruple series equations involving Fourier-Bessel series and obtained the solution using operator theory. Several workers have devoted considerable attention to the solutions of various equations involving for instance, trigonometric series, The Fourier-Bessel series, The Fourier Legendre series, The Dini series, series of Jacobi and Laguerre polynomials and series equations involving Bateman K-functions. Indeed, many of these problems arise in the investigation of certain classes of mixed boundary value problems in potential theory. There has been less work on quadruple series equations involving various polynomials and functions. In light of the significance of quadruple series solutions, proposed work examines quadruple series equations that include the product of r generalised Bateman K functions. Solution is formal, and there has been no attempt made to rationalise many restricting processes that have been encountered.

Keywords Series Equations, Integral Equations, Quadruple Series Equations, Fredholm Equations

1 Introduction

Series play an important role in the solution of mixed boundary value problems [1]. Series equations were first investigated by Cooke[8]. In the past five decades various authors Srivastava [2], Narain, Singh and Lal [3], Awasthi and Rachna[4,6] and Narain and Lal [5] have worked on different dual and triple series equations. Apart from dual, triple series solutions, Dwivedi and Trivedi have worked on quadruple series equations which are very useful for four part boundary value problems. Awasthi and Rachna [7] also worked on variety of integral and series equations to solve cracks. In view of importance of quadruple series solutions, the following paper considers the quadruple series equations involving the product of 'r' generalized Bateman K - functions. This solution is formal and no attempts have been made to justify various limiting processes.

1.1 Problem

Here, consider the following quadruple series equations involving the product of "r" generalized Bateman K-functions

$$\sum_{n_1,n_2.......n_r=0}^{\infty} A_{n_1,n_2,....,n_r} \prod_{i=1}^{r} K_{2(n_i+\alpha_i)}^{2(\alpha_i+\sigma_i)}(x_i) = f_1(x_1,x_2,......x_r), \quad 0 \le x_i < a_i \quad (i = 1........r) \tag{1}$$

$$\sum_{n_1,n_2.......n_r=0}^{\infty} A_{n_1,n_2,....,n_r} \prod_{i=1}^{r} K_{2(n_i+\beta_i)}^{2(\beta_i+\sigma_i)}(x_i) = f_2(x_1,x_2,......x_r), \quad a_i < x_i < b_i \quad (i = 1........r) \tag{2}$$

$$\sum_{n_1,n_2.......n_r=0}^{\infty} A_{n_1,n_2,....,n_r} \prod_{i=1}^{r} K_{2(n_i+\alpha_i)}^{2(\alpha_i+\sigma_i)}(x_i) = f_3(x_1,x_2,......x_r), \quad b_i < x_i < c_i \quad (i = 1......r) \tag{3}$$

$$\sum_{n_1,n_2.......n_r=0}^{\infty} A_{n_1,n_2,....,n_r} \prod_{i=1}^{r} K_{2(n_i+\beta_i)}^{2(\beta_i+\sigma_i)}(x_i) = f_4(x_1,x_2,......x_r), \quad c_i < x_i < \infty \quad (i = 1......r) \tag{4}$$

where $f_1(x_1,x_2,......x_r)$ to $f_4(x_1,x_2,......x_r)$ are prescribed functions, $A_{n_1,n_2,....,n_r}$ is to be determined.

1.2 Some useful Results

In the following, given some useful results that help us in solving the problem defined above.

$$\int_0^\infty x^{-2\alpha-2\sigma-1} K_{2(m+\alpha)}^{2(\alpha+\sigma)}(x) K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x)dx = \frac{2^{2\alpha+2\sigma}\Gamma(n-\sigma)}{\Gamma(2\alpha+\sigma+n+1)}\delta_{mn} \tag{5}$$

Where $\alpha+\sigma+1>0, \sigma+1\leq 0$ and δ_{mn} is the kronecker delta. The value of the series is given below.

$$S(r,x) = \sum_{n=0}^{\infty} \frac{\Gamma(2v+\sigma+n+1)}{2^{2\beta+2\sigma}\Gamma(n-\sigma)} K_{2(n+\beta)}^{2(\beta+\sigma)}(r) K_{2(n+\beta)}^{2(\alpha+\sigma)}(x) \tag{6}$$

$$= \frac{\exp^{-x} 2^{2\alpha-2v}}{\Gamma(2\alpha-2v)\Gamma(2\beta-2v)} \int_0^t E(\varepsilon)(x-\varepsilon)^{2\alpha-2v-1}(r-\varepsilon)^{2\beta-2v-1} d\varepsilon \tag{7}$$

$$= \frac{\exp^{-x} 2^{2\alpha-2v}}{\Gamma(2\alpha-2v)\Gamma(2\beta-2v)} S_t(r,x) \tag{8}$$

$$= \frac{\exp^{-x} 2^{2\alpha-2v}}{\Gamma(2\alpha-2v)\Gamma(2\beta-2v)} \int_0^t E(\varepsilon)(x-\varepsilon)^{2\alpha-2v-1}(r-\varepsilon)^{2\beta-2v-1} d\varepsilon \tag{9}$$

where,

$$E(\varepsilon) = \exp^{2\varepsilon} \varepsilon^{2v+2\sigma+1}, t = min(r,x)$$

The solutions of the abel integral equations of r-dimensions:

$$F(x_1,x_2,.....x_r) = \int_{a_1}^{x_1}\int_{a_2}^{x_2} .....\int_{a_r}^{x_r} \frac{F(y_1,y_2.....y_r)dy_1,dy_2,...dy_r}{(x_1-y_1)^{\sigma_1}(x_2-y_2)^{\sigma_2}....(x_r-y_r)^{\sigma_r}} \tag{10}$$

and

$$F(x_1,x_2,....x_r) = \int_{x_1}^{b_1}\int_{x_2}^{b_2} ......\int_{x_r}^{b_r} \frac{F(y_1,y_2.....y_r)dy_1,dy_2,...dy_r}{(y_1-x_1)^{\sigma_1}(y_2-x_2)^{\sigma_2}....(y_r-x_r)^{\sigma_r}} \tag{11}$$

are given by

$$F(y_1,y_2,....y_r) = \frac{\sin\sigma_1\pi\sin\sigma_2\pi.....\sin\sigma_r\pi}{\pi^r} x \frac{d^r}{dy_1,dy_2,...dy_r} x[\int_{y_1}^{b_1}\int_{y_2}^{b_2}....\int_{y_r}^{b_r} \frac{F(x_1,x_2.....x_r)dx_1,dx_2,...dx_r}{(x_1-y_1)^{1-\sigma_1}(x_2-y_2)^{1-\sigma_2}.....(x_r-y_r)^{1-\sigma_r}}] \tag{12}$$

and

$$F(y_1,y_2,....y_r) = (-1)^r\frac{\sin\sigma_1\pi\sin\sigma_2\pi.......\sin\sigma_r\pi}{\pi^r} x \frac{d^r}{dy_1,dy_2,...dy_r} x[\int_{y_1}^{b_1}\int_{y_2}^{b_2}......\int_{y_r}^{b_r} \frac{F(x_1,x_2.....x_r)dx_1,dx_2,...dx_r}{(x_1-y_1)^{1-\sigma_1}(x_2-y_2)^{1-\sigma_2}......(x_r-y_r)^{1-\sigma_r}}] \tag{13}$$

respectively.

2 Solutions

Let us assume

$$\sum_{n_1,n_2.......n_r=0}^{\infty} A_{n_1,n_2,....,n_r} \prod_{i=1}^{r} K_{2(n_i+\beta_i)}^{2(\beta_i+\sigma_i)}(x_i) = \begin{pmatrix} h(x_1,x_2,....x_r), 0<x_i<a_i \\ g(x_1,x_2,....x_r), b_i<x_i<c_i \end{pmatrix} \tag{14}$$

making use of orthogonality relation (5) and from (2), (4), and (14)

$$A_{n_1,n_2,....,n_r} = \prod_{i=1}^{r} \frac{\Gamma(2v_i+\sigma_i+n_i+1)\Gamma(2\beta_i+\sigma_i+n_i+1)}{2^{2\beta_i+2\sigma_i}\Gamma(n_i-\sigma_i)} \Big[\int_0^{a_i} x^{-2\beta_i-2\sigma_i-1} K_{2(n_i+\beta_i)}^{2(\beta_i+\sigma_i)}(x_i)h(x_1,x_2.....x_r)dx_i + \int_{a_i}^{b_i} x^{-2\beta_i-2\sigma_i-1} K_{2(n_i+\beta_i)}^{2(\beta_i+\sigma_i)}(x_i)f_2(x_1,x_2.....x_r)dx_i + \int_{b_i}^{c_i} x^{-2\beta_i-2\sigma_i-1} K_{2(n_i+\beta_i)}^{2(\beta_i+\sigma_i)}(x_i)g(x_1,x_2.....x_r)dx_i + \int_{c_i}^{\infty} x^{-2\beta_i-2\sigma_i-1} K_{2(n_i+\beta_i)}^{2(\beta_i+\sigma_i)}(x_i)h(x_1,x_2.....x_r)dx_i\Big] \tag{15}$$

Substituting this expression for $A_{n_1,n_2....n_r}$ from (15) in (1), and (4) and interchanging the order of integration, the following equation is obtained

$$\prod_{i=1}^{r}\Big[\int_0^{a_i} u_i^{-2\beta_i-2\sigma_i-1} S(u_i,x_i)h(u_1,u_2,....u_r)du_i + \int_{b_i}^{c_i} u_i^{-2\beta_i-2\sigma_i-1} S(u_i,x_i)g(u_1,u_2,....u_r)du_i\Big] = M(x_1,x_2,....x_r), \quad 0<x_i<a_i(i=1.......r) \tag{16}$$

$$\prod_{i=1}^{r}\Big[\int_0^{a_i} u_i^{-2\beta_i-2\sigma_i-1} S(u_i,x_i) h(u_1,u_2,....u_r) du_i + \int_{b_i}^{c_i} u_i^{-2\beta_i-2\sigma_i-1} S(u_i,x_i) g(u_1,u_2,....u_r) du_i\Big] = N(x_1,x_2,....x_r), \quad b_i < x_i < c_i \tag{17}$$

Where

$$M(x_1,x_2,....x_r) = f_1(x_1,x_2,....x_r) - \prod_{i=1}^{r}\Big[\int_{a_i}^{b_i} u_i^{-2\beta_i-2\sigma_i-1} S(u_i,x_i) f_2(x_1,x_2,....x_r) dx_i - \int_{c_i}^{\infty} u_i^{-2\beta_i-2\sigma_i-1} S(u_i,x_i) f_4(x_1,x_2,....x_r) dx_i\Big] \tag{18}$$

$$N(x_1,x_2,....x_r) = f_3(x_1,x_2,....x_r) - \prod_{i=1}^{r}\Big[\int_{a_i}^{b_i} u_i^{-2\beta_i-2\sigma_i-1} S(u_i,x_i) f_2(x_1,x_2,....x_r) dx_i - \int_{c_i}^{\infty} u_i^{-2\beta_i-2\sigma_i-1} S(u_i,x_i) f_4(x_1,x_2,....x_r) dx_i\Big] \tag{19}$$

Equation (16) can be written with the help of (7) as

$$\prod_{i=1}^{r}\int_0^{x_i} \frac{E(\varepsilon_i) d(\varepsilon_i)}{(x_i-\varepsilon_i)^{1-2\alpha_i+2v_i}} \int_i^{\alpha_i} \frac{u_i^{-2\beta_i-2\sigma_i-1} h(u_1,u_2,....u_r) du_i}{(u_i-\varepsilon_i)^{1-2\beta_i+2v_i}} = \prod_{i=1}^{r}\frac{\Gamma(2\alpha_i-2v_i)\Gamma(2\beta_i-2v_i)}{2^{(2\alpha_i-2v_i)}} M(x_1,x_2,....x_r) - \int_{b_i}^{c_i} u_i^{-2\beta_i-2\sigma_i-1} g(u_1,u_2,....u_r) du_i \int_0^{x_i} \frac{E(\varepsilon_i) d(\varepsilon_i)}{(x_i-\varepsilon_i)^{1-2\alpha_i+2v_i}(u_i-\varepsilon_i)^{1-2\beta_i+2v_i}} \tag{20}$$

with the help of (12), and from (20) concluded that

$$\prod_{i=1}^{r} E(\varepsilon_i)\int_i^{\alpha_i} \frac{u_i^{-2\beta_i-2\sigma_i-1} h(u_1,u_2,....u_r) du_i}{(u_i-\varepsilon_i)^{1-2\beta_i+2v_i}} = \frac{1}{\pi^r}.\prod_{i=1}^{r}\frac{\Gamma(2\alpha_i-2v_i)\Gamma(2\beta_i-2v_i)}{2^{(2\alpha_i-2v_i)}} \sin(1-2\alpha_i+2v_i)\pi \frac{d}{d\varepsilon_i}\int_0^{\varepsilon_i}\frac{M(x_1,x_2,....x_r)dx_i}{\exp^{-x}(\varepsilon_i-x_i)^{2\alpha_i-2v_i}} - \frac{1}{\pi^r}.\prod_{i=1}^{r}[\sin(1-2\alpha_i+2v_i)\pi \frac{d}{d\varepsilon_i}\int_0^{\varepsilon_i}\frac{dx_i}{(\varepsilon_i-x_i)^{2\alpha_i-2v_i}} \int_{b_i}^{c_i} u_i^{-2\beta_i-2\sigma_i-1} g(u_1,u_2,....u_r) du_i \, x\int_0^{x_i}\frac{E((t_i)d(t_i)}{(x_i-t_i)^{1-2\alpha_i+2v_i}(u_i-t_i)^{1-2\beta_i+2v_i}} \tag{21}$$

The second integral of (21) can be written in the form

$$-\frac{1}{\pi^r}.\prod_{i=1}^{r}\Big[\sin(1-2\alpha_i+2v_i)\pi \int_{b_i}^{c_i} u_i^{-2\beta_i-2\sigma_i-1} g(u_1,u_2,....u_r) du_i\Big] x\frac{d}{d\varepsilon_i}\int_0^{x_i}\frac{E((t_i)d(t_i)}{(u_i-t_i)^{1-2\beta_i+2v_i}} \int_{t_i}^{\varepsilon_i}\frac{d(x_i)}{(\varepsilon_i-x_i)^{2\alpha_i-2v_i}(x_i-t_i)^{1-2\alpha_i+2v_i}} = -\prod_{i=1}^{r}\int_{b_i}^{c_i}\frac{g(u_1,u_2,....u_r)E(\varepsilon_i)u_i^{-2\beta_i-2\sigma_i-1}}{(u_i-\varepsilon_i)^{1-2\beta_i+2v_i}} \tag{22}$$

Where

$$\prod_{i=1}^{r}\int_{t_i}^{\varepsilon_i}\frac{d(x_i)}{(\varepsilon_i-x_i)^{2\alpha_i-2v_i}(x_i-t_i)^{1-2\alpha_i+2v_i}} = \frac{\pi^r}{\prod_{i=1}^{r}\sin(1-2\alpha_i+2v_i)\pi} \tag{23}$$

Hence,

$$\prod_{i=1}^{r}\int\frac{h(u_1,u_2,....u_r)u_i^{-2\beta_i-2\sigma_i-1}}{(u_i-\varepsilon_i)^{1-2\beta_i+2v_i}} = \frac{M_1(\varepsilon_1,\varepsilon_2,....\varepsilon_r}{\prod_{i=1}^{r}E(\varepsilon_i)} - \prod_{i=1}^{r}\int_{b_i}^{c_i}\frac{g(u_1,u_2,....u_r)E(\varepsilon_i)d(u_i)u_i^{-2\beta_i-2\sigma_i-1}d(u_i)}{(u_i-\varepsilon_i)^{1-2\beta_i+2v_i}} \tag{24}$$

Where,

$$M_1(\varepsilon_1,\varepsilon_2,....\varepsilon_r) = \frac{1}{\pi^r}.\prod_{i=1}^{r}\frac{\Gamma(2\alpha_i-2v_i)\Gamma(2\beta_i-2v_i)}{2^{(2\alpha_i-2v_i)}} \sin(1-2\alpha_i+2v_i)\pi\frac{d}{d\varepsilon_i}\int_0^{\varepsilon_i}\frac{M(x_1,x_2,....x_r)dx_i}{\exp^{-x}(\varepsilon_i-x_i)^{2\alpha_i-2v_i}} \tag{25}$$

Equation (24) is of the form of abel integral equation of r diameter so applying (13), obtain from (24).

$$\prod_{i=1}^{r} h(u_1,u_2,....u_r)u_i^{-2\beta_i-2\sigma_i-1} = \Big(-\frac{1}{\pi}\Big)^r\prod_{i=1}^{r}\sin(1-2\beta_i+2v_i)\pi \frac{d}{du_i}\int_{u_i}^{a_i}\frac{M_1(\varepsilon_1,\varepsilon_2,....\varepsilon_r)d\varepsilon_i}{E(\varepsilon_i)(\varepsilon_i-u_i)^{2\beta_i-2v_i}} - \Big(-\frac{1}{\pi}\Big)^r\prod_{i=1}^{r}\sin(1-2\beta_i+2v_i) \pi\frac{d}{du_i}\int_{u_i}^{a_i}\frac{d\varepsilon_i}{(\varepsilon_i-u_i)^{2\beta_i+2v_i}}. \int_{b_i}^{c_i}\frac{g(S_1,S_2,....S_r)S_i^{-2\beta_i-2\sigma_i-1}d(S_i)}{(S_i-\varepsilon_i)^{1-2\beta_i+2v_i}} \tag{26}$$

Now using the result,

$$\prod_{i=1}^{r}\frac{d}{du_i}\int_{u_i}^{a_i}\frac{d\varepsilon_i}{(\varepsilon_i-u_i)^{1-\beta_i}(S_i-\varepsilon_i)^{\beta_i}} = (-1)^r\prod_{i=1}^{r}\frac{(\alpha_i-u_i)^{\beta_i-1}}{(S_i-a_i)^{\beta_i}(S_i-u_i)},\quad 0<u_i<a_i \tag{27}$$

concluded that,

$$h(u_1,u_2,....u_r)=M_2(u_1,u_2,....u_r) - \frac{1}{\pi^r}\prod_{i=1}^{r}\frac{\sin(1-2\beta_i+2v_i)\pi}{(a_i-u_i)^{2\beta_i-2v_i}}u_i^{2\beta_i+2\sigma_i+1} \int_{b_i}^{c_i}\frac{(S_i-a_i)^{2\beta_i-2v_i}g(S_1,S_2,....S_r)S_i^{-2\beta_i-2\sigma_i-1}}{S_i-u_i},0<u_i<a_i \tag{28}$$

Where,

$$M_2(u_1,u_2,....u_r)=-\frac{1}{\pi^r}\prod_{i=1}^{r}\sin(1-2\beta_i+2v_i)\pi u_i^{2\beta_i+2\sigma_i-1} \frac{d}{du_i}\int_{u_i}^{a_i}\frac{M_1(\varepsilon_1,\varepsilon_2,....\varepsilon_r)d\varepsilon_i}{E(\varepsilon_i)(\varepsilon_i-u_i)^{2\beta_i-2v_i}} \tag{29}$$

Using (16) and (7), concluded that

$$\begin{aligned}\prod_{r=1}^{r}\int_{b_i}^{x_i}\frac{E(\varepsilon_i)G(\varepsilon_1,\varepsilon_2,....\varepsilon_r)d\varepsilon_i}{(x_i-\varepsilon_i)^{1-2\alpha_i+2v_i}} &= \prod_{i=1}^{r}\frac{\Gamma(2\alpha_i-2v_i)\Gamma(2\beta_i-2v_i)M(x_1,x_2,....x_r)}{\exp^{-x_i}2^{(2\alpha_i-2v_i)}} \\ &- \prod_{i=1}^{r}\int_{a_i}^{b_i}u_i^{-2\beta_i-2\sigma_i-1}h(u_1,u_2,....u_r)du_i \\ &x\int_0^{u_i}\frac{E(\varepsilon_i)d(\varepsilon_i)}{(x_i-\varepsilon_i)^{1-2\alpha_i+2v_i}(u_i-\varepsilon_i)^{1-2\beta_i+2v_i}} \\ &- \prod_{i=1}^{r}\int_0^{b_i}\frac{E(\varepsilon_i)d(\varepsilon_i)}{(x_i-\varepsilon_i)^{1-2\alpha_i+2v_i}} \int_{b_i}^{c_i}\frac{g(u_1,u_2,....u_r)u_i^{-2\beta_i-2\sigma_i-1}d(u_i)}{(u_i-\varepsilon_i)^{1-2\beta_i+2v_i}}\end{aligned} \tag{30}$$

where,

$$G(\varepsilon_1,\varepsilon_2,....\varepsilon_r) = \prod_{i=1}^{r}\int_{\varepsilon_i}^{c_i}\frac{g(u_1,u_2,....u_r)u_i^{-2\beta_i-2\sigma_i-1}d(u_i)}{(u_i-\varepsilon_i)^{1-2\beta_i+2v_i}} \tag{31}$$

From (31) and (13), concluded that

$$\prod_{i=1}^{r}g(u_1,u_2,....u_r)u_i^{-2\beta_i-2\sigma_i-1} = \left(-\frac{1}{\pi}\right)^r\prod_{i=1}^{r}\sin(1-2\beta_i+2v_i)\pi\frac{d}{du_i}\int_{u_i}^{c_i}\frac{G(\varepsilon_1,\varepsilon_2,....\varepsilon_r)d\varepsilon_i}{(u_i-\varepsilon_i)^{1-2\beta_i+2v_i}} \tag{32}$$

The equation (30) has the form (10), then using the equation (30) can be written in the form

$$\prod_{i=1}^{r}E(\varepsilon_i)G(\varepsilon_1,\varepsilon_2,....\varepsilon_r)=N_1(\varepsilon_1,\varepsilon_2,....\varepsilon_r)+I_1+I_2 \qquad b_i<\varepsilon_i<c_i \tag{33}$$

Where,

$$N_1(\varepsilon_1,\varepsilon_2,....\varepsilon_r)=\frac{1}{\pi^r}\prod_{i=1}^{r}\frac{\Gamma(2\alpha_i-2v_i)\Gamma(2\beta_i-2v_i)}{2^{(2\alpha_i-2v_i)}} \sin(1-2\alpha_i+2v_i)\pi\int_{b_i}^{\varepsilon_i}\frac{N(x_1,x_2,....x_r)dx_i}{(\varepsilon_i-x_i)^{2\alpha_i-2v_i}} \tag{34}$$

$$\begin{aligned}I_1 &= \frac{1}{\pi^r}\prod_{i=1}^{r}\sin(1-2\alpha_i+2v_i)\pi\frac{d}{d\varepsilon_i} \int_{b_i}^{\varepsilon_i}\frac{dx_i}{(\varepsilon_i-x_i)^{2\alpha_i-2v_i}} \\ &\int_0^{u_i}h(u_1,u_2,....u_r)u_i^{-2\beta_i-2\sigma_i-1}d(u_i) \\ &\int_{b_i}^{c_i}u_i^{-2\beta_i-2\sigma_i-1}g(u_1,u_2,....u_r)du_i \\ &x\int_0^{x_i}\frac{d(t_i)}{(x_i-t_i)^{1-2\alpha_i+2v_i}(u_i-t_i)^{1-2\beta_i+2v_i}}\end{aligned} \tag{35}$$

$$I_2=\frac{1}{\pi^r}\prod_{i=1}^{r}\sin(1-2\alpha_i+2v_i)\pi\frac{d}{d\varepsilon_i}\int_{b_i}^{\varepsilon_i}\frac{dx_i}{(\varepsilon_i-x_i)^{2\alpha_i-2v_i}} \int_0^{b_i}\frac{E(t_i)d(t_i)}{(x_i-t_i)^{1-2\alpha_i+2v_i}}\int_{b_i}^{c_i}\frac{u_i^{-2\beta_i-2\sigma_i-1}g(u_1,u_2,....u_r)du_i}{(u_i-t_i)^{1-2\beta_i+2v_i}} \tag{36}$$

After some manipulation, get

$$I_1=-\frac{1}{\pi^r}\prod_{i=1}^{r}\frac{\sin(1-2\alpha_i+2v_i)\pi}{(\varepsilon_i-b_i)^{2\alpha_i-2v_i}} \int_0^{a_i}\frac{(b_i-t_i)^{2\alpha_i-2v_i}E(t_i)d(t_i)}{(\varepsilon_i-t_i)} x\int_{t_i}^{a_i}\frac{u_i^{-2\beta_i-2\sigma_i-1}h(u_1,u_2,....u_r)du_i}{(u_i-t_i)^{1-2\beta_i+2v_i}} \tag{37}$$

Putting the value of last integral in the above equation, from (24), obtained

$$I_1=M_3(\varepsilon_1,\varepsilon_2,....\varepsilon_r)+\frac{1}{\pi^r}\prod_{i=1}^{r}\frac{\sin(1-2\alpha_i+2v_i)\pi}{(\varepsilon_i-b_i)^{2\alpha_i-2v_i}} x\int_{b_i}^{c_i}u_i^{-2\beta_i-2\sigma_i-1}g(u_1,u_2,....u_r)du_i \; x\int_0^{a_i}\frac{(b_i-t_i)^{2\alpha_i-2v_i}E(t_i)d(t_i)}{(\varepsilon_i-t_i)(u_i-t_i)^{1-2\beta_i+2v_i}} \tag{38}$$

Where

$$M_3(\varepsilon_1,\varepsilon_2,....\varepsilon_r)=-\frac{1}{\pi^r}\prod_{i=1}^{r}\frac{\sin(1-2\alpha_i+2v_i)\pi}{(\varepsilon_i-b_i)^{2\alpha_i-2v_i}} x\int_0^{a_i}\frac{M_1(t_1,t_2,....t_r)(b_i-t_i)^{2\alpha_i-2v_i}d(t_i)}{(\varepsilon_i-t_i)} \tag{39}$$

after some manipulation, (36) can be written in the form

$$I_2 = \frac{1}{\pi^r}\prod_{i=1}^{r}\frac{\sin(1-2\sigma_i+2v_i)\pi}{(\varepsilon_i-b_i)^{2\alpha_i-2v_i}} \int_{b_i}^{c_i} u_i^{-2\beta_i-2\sigma_i-1} g(u_1,u_2,....u_r)du_i \; x\int_{a_i}^{b_i}\frac{(b_i-t_i)^{2\alpha_i-2v_i}E(t_i)d(t_i)}{(\varepsilon_i-t_i)(u_i-t_i)^{1-2\beta_i+2v_i}} \tag{40}$$

Hence,

$$I_1 + I_2 = M_3(\varepsilon_1,\varepsilon_2,....\varepsilon_r) - \frac{1}{\pi^r}\prod_{i=1}^{r}\frac{\sin(1-2\alpha_i+2v_i)\pi}{(\varepsilon_i-b_i)^{2\alpha_i-2v_i}} \int_{b_i}^{c_i} u_i^{-2\beta_i-2\sigma_i-1} g(u_1,u_2,....u_r)du_i \; x\int_{a_i}^{b_i}\frac{(b_i-t_i)^{2\alpha_i-2v_i}E(t_i)d(t_i)}{(\varepsilon_i-t_i)(u_i-t_i)^{1-2\beta_i+2v_i}}$$
$$= M_3(\varepsilon_1,\varepsilon_2,....\varepsilon_r) - \frac{1}{\pi^r}\prod_{i=1}^{r}\frac{\sin(1-2\sigma_i+2v_i)\pi}{(\varepsilon_i-b_i)^{2\alpha_i-2v_i}} \; x\int_{a_i}^{b_i}\frac{(b_i-t_i)^{2\alpha_i-2v_i}Rd(t_i)}{(\varepsilon_i-t_i)} \tag{41}$$

Where,

$$R = \prod_{i=1}^{r}\frac{u_i^{-2\beta_i-2\sigma_i-1}g(u_1,u_2,....u_r)du_i}{(u_i-t_i)^{1-2\beta_i+2v_i}} \tag{42}$$

By using a method similar to that of Cooke [1963], it can be easily shown that

$$R = \frac{1}{\pi^r}\prod_{i=1}^{r}\sin(1-2\alpha_i+2v_i)\Pi(b_i-t_i)^{2\beta_i-2v_i} \int_{b_i}^{c_i}\frac{g(u_1,u_2,....u_r)du_i}{(u_i-t_i)^{2\beta_i+2v_i}} \tag{43}$$

Using equation (41) and (43), the equation (33) can be written in the form

$$\prod_{i=1}^{r}P(\varepsilon_i)G(\varepsilon_1,\varepsilon_2,....\varepsilon_r) = M_3(\varepsilon_1,\varepsilon_2,....\varepsilon_r) \; N_1(\varepsilon_1,\varepsilon_2,....\varepsilon_r) + \prod_{i=1}^{r}\int_{b_i}^{c_i} G(u_1,u_2,....u_r) \; K(u_i,\varepsilon_i)du_i, b_i < \varepsilon_i < c_i(i=1,2,....r) \tag{44}$$

Where,

$$\prod_{i=1}^{r}K(u_i,\varepsilon_i)du_i = \frac{1}{\pi^{2r}}\frac{\sin(1-2\beta_i+2v_i)\pi\sin(1-2\alpha_i+2v_i)\pi}{(\varepsilon_i-x_i)^{2\alpha_i-2v_i}(u_i-b_i)^{2\beta_i-2v_i}} \; x\int_{a_i}^{b_i}\frac{E(t_i)(b_i-t_i)^{2\alpha_i+\beta_i-4v_i}dt_i}{(\varepsilon_i-t_i)(u_i-t_i)} \tag{45}$$

3 Conclusion

Equation (44) is a Fredholm integral equation of the second kind. This is a standard equation. From this equation, determine the value of $G(S_1, S_2, ..S_r)$. Knowing $G(S_1, S_2, ...S_r)$, $g(I_1, I_2, ...I_r)$ and $h(I_1, I_2, ...I_r)$ can be easily obtained from (28) and (32) and hence the coefficients $A_(n_1, n_2....n_r)$ can be determined.

REFERENCES

[1] Sneddon I.N., “Mixed Boundary value Problem in Potential theory”, in North-Holland, Amsterdam, 1966.

[2] Srivastava H. M., “A pair of dual series equations involving generalized Bateman K-functions”, Nederl.Akad.Wetensch.Proc.Ser.A.indag.Math., vol 34, pp 53-61, 1972.

[3] Narain K, Singh, V.B. and Lal M, “Triple series equations involving generalized Bateman K-functions”, Indian J.PureAppl.Math.,vol 15, no. 4, pp 435-440, 1982.

[4] Awasthi A.K., Rachna, “Fourier Series and Fourier Integral Equations and Their Applications in Elasticity”, J. Phys.: Conf. Ser., Vol. 2267 012158, 2022.

[5] Narain K., Lal M., “On the solution of simultaneous dual series equations involving Laguerre Polynomials”, Acta Ciencia Indica, Vol. 11, pp 222-224, 1985.

[6] Awasthi A.K., Rachna, “The Solution of Simultaneous Dual Integral Equations Involving Fox H-Function”, AIP Conference Series, 2022.

[7] Awasthi A.K., Rachna, Kaur H. “A Griffith Crack at the Interface of an Isotropic and orthotropic half space Bonded together”, Mathematics and Statistics, vol. 10, pp 166-175, 2022. DOI: 10.13189/ms.2022.100115

[8] Cooke, J.C. “The solution of some integral equations and their connection with dual integral equations and series”, Glasgow Mathematical Journal, vol 11, pp 9-20, 1970.