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Some Results of Generalized Weighted Norlund-Euler-λ Statistical Convergence in Non-Archimedean Fields

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Abstract Non-Archimedean analysis is the study of fields that satisfy the stronger triangular inequality, also known as ultrametric property. The theory of summability has many uses throughout analysis and applied mathematics. The origin of summability methods developed with the study of convergent and divergent series by Euler, Gauss, Cauchy and Abel. There is a good number of special methods of summability such as Abel, Borel, Euler, Taylor, Norlund, Hausdroff in classical Analysis.
Norlund, Euler, Taylor and weighted mean methods in Non-Archimedan Analysis have been investigated in detail by Natarajan and Srinivasan. Schoenberg developed some basic properties of statistical convergence and also studied the concept as a summability method. The relationship between the summability theory and statistical convergence has been introduced by Schoenberg. The concept of weighted statistical convergence and its relations of statistical summability were developed by Karakaya and Chishti. Srinivasan introduced some summability methods namely y -method, Norlund method and Weighted mean method in p -adic Fields. The main objective of this work is to explore some important results on statistical convergence and its related concepts in Non-Archimedean fields using summability methods. In this article, Norlund-Euler-λ statistical convergence, generalized weighted summability using Norlund-Euler-λ method in an Ultrametric field are defined. The relation between Norlund-Euler-λ statistical convergence and Sta-tistical Norlund-Euler-λ summability has been extended to non-Archidemean fields. The notion of Norlund-Euler-λ statistical convergence and inclusion results of Norlund-Euler statistical convergent sequence has been characterized. Further the relation between Norlund-Euler-λ statistical convergence of order α & β has been established.

Keywords Weighted Norlund-Euler Statistical Convergence, NE-statistical Summable, Generalized Weighted Norlund-Euler-λ-statistical Convergence, Non-Archimedean Field, Ultrametric Field

1 Introduction

Fast, H [1], in 1951, introduced the concept of statistical convergence. Steinhaus [2] and Fridy, J.A. [3] were pioneers to analyze statistical convergence though independent of each other. Rath and Tripathy, B.C. [4] studied statistically convergent and statistically Cauchy sequences for reals. Karakaya and Chishti [5]; Kucukaslan [6] discussed weighted statistical convergence. Belene and Mohiuddine [7] studied Generalized Weighted statistical convergence. The theory of p-adic fields was developed by Kurt Hensel[8], in 1908. Suja K, Srinivasan V [9] investigated statistical convergence in Non-Archimedean fields. Also, Suja K, Srinivasan V [10] studied Weighted statistical convergence in Non-Archimedean fields. Since then, Sangeetha. S[11], Uma. J, Srinivasan. V [12],Eunice Jemima. D, Srinivasan. V [13] worked on statistical convergence in Non-archimedean fields. For a general reference on Ultrametric analysis, the book is [14]. V. Loku, and E. Aljimi [15] developed Norlund-Euler-λ statistical convergence and Norlund-Euler statistical convergence concepts in classical analysis. M. Et etal [16] developed weighted Norlund-Euler-λ- statistical convergence for different orders, in paranormed spaces.
In this present article, the relation between Norlund-Euler-λ statistical convergence and statistical Norlund-Euler-λ summability, Norlund-Euler-λ and Norlund-Euler statistical convergence are extended to Non-archimedean fields, based on [15].

Also, inclusion criteria of weighted Norlund-Euler-λ- statistical convergence for the orders of α and β orders are investigated based on [16] in paranormed spaces.

2 Preliminaries

Throughout this article, K represents a non-trivially valued complete non-Archimedean field. The generalized weighted Norlund-Euler statistical convergence was defined by E. Aljimi, V. Loku [17] as below. " Let $\sum_{k=0}^{n} x_k$ be a given infinite series with sequence of its n^{th} partial sum $\{S_n\}(E,q)$ transform defined as

$$E_n^{(E,q)} = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k \tag{1}$$

This summability method is said to be convergent if $E_n^{(E,q)} \to S$, as $n \to \infty$ and we say $\sum_{k=0}^{n} x_k$ is (E,q)-summable to a definite number S, and this is written as $S_n \to S(E,q)$. Let $\{p_n\}$ and $\{q_n\}$ be two sequences of real numbers such that

$$P_n = p_0 + p_1 + \cdots + p_n, P_{-1} = p_{-1} = 0.$$
$$Q_n = q_0 + q_1 + \cdots + q_n, Q_{-1} = q_{-1} = 0.$$

For the given sequences $\{p_n\}$ and $\{q_n\}$, convolution of $p * q$ is defined by

$$R_n = p * q = \sum_{k=0}^{n} p_k q_{n-k} \tag{2}$$

The series $\sum_{k=0}^{n} x_k$ or the sequence $\{S_n\}$ is summable to S by generalized Norlund method and it is denoted by $S_n \to S(N,p,q)$ if the nth term of the sequence Sn,

$$t_n^{(N,p,q)} = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k S_k \tag{3}$$

tends to S, as $n \to \infty$
Consider,

$$\begin{aligned} t_n^{(N,p,q)(E,q)} &= \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k E_n^q \\ &= \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v=0}^{n} \binom{k}{v} q^{k-v} S_v \end{aligned} \tag{4}$$

The series $\sum_{k=0}^{n} x_k$ is said to be summable to S, by Norlund-Euler method if $t_n^{(N,p,q)(E,q)} \to L$, as $n \to \infty$, and it is denoted by $S_n \to S(N,p,q)(E,q)$. A Sequence $x = \{x_n\}$ is said to be generalized weighted Norlund-Euler-statistically convergent (or S_{NE}^q-convergent), to L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{R_n} \left| \left\{ k \le R_n : p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v=0}^{n} \binom{k}{v} q^{k-v} |x_v - L| \ge \varepsilon \right\} \right| = 0. \tag{5}$$

In this case, we write $L = S_{NE}^q(st) - \lim x$. [17] "

Definition 1 "Let K be a complete field with a non-trivially valued non-Archimedean valuation $|.|$ defined. A valuation is a function $|| : X \to \mathscr{R}$ satisfying the following axioms
i. $|x| >= 0, |x| = 0$ iff $x = 0.\ \forall\, x \in X.$
ii. $|xy| = |x|\,|y|\ \forall\, x, y \in X.$
iii. $|x + y| = max\{|x|, |y|\}\ \forall\, x, y \in X.$
"

Definition 2 [9] "A sequence $\{x_n\}$ in K is said to be statistically convergent to a limit ℓ if for any $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : |x_n - \ell| \ge \varepsilon \right\} \right| = 0.$$

Symbolically, we write

$$Stat - \lim_{n\to\infty} x_n = \ell or\ x_n \xrightarrow{\text{stat}} \ell \tag{6}$$

"

Definition 3 " Let $\{P_n\}$ be a sequence of real numbers such that $\liminf_n p_n > 0$, $P_n = p_1 + p_2 + p_3 + \cdots + p_n$ for all $n \in N$.
A Sequence $\{x_n\}$ is said to be weighted statistically convergent of order 'α' $(0 < \alpha < 1)$ to L, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{P_n^\alpha} \left| \left\{ k \le P_n : p_k |x_n - L| \ge \varepsilon \right\} \right| = 0,$$

where $P_n^\alpha = (P_n)^\alpha$." [16]

Definition 4 " Let $\lambda = \{\lambda_n\}$ be non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$. The collection of such a sequence will be denoted by Δ. Let $\{p_n\}$ and $\{q_n\}$ be two sequences of non-zero real numbers such that,

$$P_{\lambda_n} = \sum_{k\in I_n} p_k, P_{-1} = p_{-1} = 0,$$
$$Q_{\lambda_n} = \sum_{k\in I_n} q_k, Q_{-1} = q_{-1} = 0.$$

For the given sequences $\{p_n\}$ and $\{q_n\}$, convolution of $p * q$ is defined by

$$R_{\lambda_n} = p * q = \sum_{k\in I_n} p_k q_{n-k} \tag{7}$$

and

$$t_n^{(N_\lambda,p,q)(E_\lambda,q)} = \frac{1}{R_{\lambda_n}} \sum_{k\in I_n} p_n q_{n-k} \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} S_v \tag{8}$$

where $I_n = [n - \lambda_n + 1, n]$. If $t_n^{(N_\lambda,p,q)(E_\lambda,q)} \to S$, as $n \to \infty$, we say $\sum x_n$ is summable to S by generalized weighted Norlund-Euler-λ method and it is denoted by
$S_n \to t_n^{(N_\lambda,p,q)(E_\lambda,q)}$. [7]"

We now define Norlund-Euler-λ statistical convergence, based on work by Loku and Aljimi [15].

Definition 5 " A sequence $x = \{x_n\}$ of elements of K, is said to be generalized weighted Norlund-Euler-λ statistically convergent (or $S^q_{NE_\lambda}$-convergent), to L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{R_{\lambda_n}} \left|\left\{k \le R_{\lambda_n} : p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v=0}^{n} \binom{k}{v} q^{k-v}|x_v - L| \ge \varepsilon\right\}\right| = 0. \quad (9)$$

In this case, we write $L = S^q_{NE\lambda}(st) - \lim x$. [15] "

Definition 6 " A Sequence $x = \{x_n\}$ is said to be generalized weighted summable to L by Norlund-Euler-λ method or $(N_\lambda, p, q)(E_\lambda, q)$ summable to L if

$$\frac{1}{R_{\lambda_n}} \sum_{k\in I_n} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}|x_v - L| = 0 \quad (10)$$

In this case, we write $x_n \to L[(N_\lambda, p, q)(E_\lambda, q)]$ and $[(N_\lambda, p, q)(E_\lambda, q)]$ denote the set of all $(N_\lambda, p, q)(E_\lambda, q)$ summable sequences " [15]

Definition 7 " A Sequence $x = \{x_n\}$ is said to be $(N_\lambda, p, q)(E_\lambda, q)$ statistically summable to L if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \le n : |t_n^{(N_\lambda,p,q)(E_\lambda,q)} - L| \ge \varepsilon\right\}\right| = 0.$$

$$ie., \lim_{n\to\infty} \frac{1}{n} \left|\left\{k \le n : \frac{1}{R_{\lambda_n}} \sum_{k\in I_n} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}|x_v - L| \ge \varepsilon\right\}\right| = 0.$$

Symbolically, we write $stat - t_n^{(N_\lambda,p,q)(E_\lambda,q)} = L \; or$ $x_n \overset{stat(N_\lambda,p,q)(E_\lambda,q)}{\longrightarrow} \ell$ "[15]

Definition 8 A sequence $x = \{x_n\}$ is said to be statistically convergent of order 'α' ($0 < \alpha \le 1$) by generalized weighted Norlund-Euler-λ method to L if

$$\lim_{n\to\infty} \frac{1}{R^\alpha_{\lambda_n}} \left|\left\{k \le R_{\lambda_n} : p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}|x_v - L| \ge \varepsilon\right\}\right| = 0. \quad (11)$$

When $\alpha = 1$, we get weighted statistical convergence by Norlund-Euler-λ method for the sequence $x = \{x_n\}$.
When $\alpha = 1$, $\lambda_n = n$, we get statistical convergence by Norlund-Euler-λ method for the sequence $x = \{x_n\}$.

3 Main Results

Theorem 1 Let $\{x_n\}$ be a sequence of elements from K - a non-trivially valued, non-Archimedean field. Let $p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}|x_v - L| \le M$, for all k. If a sequence $x = \{x_n\}$ is S_{NE_λ} convergent to L then it is $(N_\lambda, p, q)(E_\lambda, q)$ statistically summable to L.

Proof
Given,
$p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}|x_v - L| \le M$,
for all k and $x = \{x_n\}$ is S_{NE_λ} convergent to L.
i.e., Given $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{R_{\lambda_n}} |\{k \le R_{\lambda_n} : p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}|x_k - L| \ge \varepsilon\}| = 0. \quad (12)$$

Consider,

$$\begin{aligned}
&\left|t_n^{(N_\lambda,p,q)(E_\lambda,q)} - L\right| \\
&= \left|\frac{1}{R_{\lambda_n}} \sum_{k\in I_n} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}x_v - L\right| \\
&\le \left|\frac{1}{R_{\lambda_n}} \sum_{k\in I_n} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}(x_v - L)\right|
\end{aligned}$$

Fix an $m \le n$ such that

$$p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_m} \binom{k}{v} q^{k-v}|x_v - x_m| \le M$$

$$\begin{aligned}
&\Rightarrow \left|t_n^{(N_\lambda,p,q)(E_\lambda,q)} - L\right| \\
&= \left|\frac{1}{R_{\lambda_n}} \sum_{k=0}^{n} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}(x_v - x_m + x_m - L)\right| \\
&= \left|\frac{1}{R_{\lambda_n}} \left(\sum_{k=0}^{m} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}(x_v - x_m) + \sum_{k=m+1}^{n} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}(x_m - L)\right)\right| \\
&\le \max\left\{\left|\frac{1}{R_{\lambda_n}} \sum_{k=0}^{m} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}(x_v - x_m)\right|, \left|\frac{1}{R_{\lambda_n}} \sum_{k=m+1}^{n} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}(x_m - L)\right|\right\} \\
&\le \max\left\{\frac{1}{R_{\lambda_n}} \sum_{k=0}^{m} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}|x_v - x_m|, \frac{1}{R_{\lambda_n}} \sum_{k=m+1}^{n} p_{n-k}q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v}|x_m - L|\right\} \\
&\le \max\left\{\frac{M}{R_{\lambda_n}}, \varepsilon\right\}
\end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : |t_n^{(N_\lambda,p,q)(E_\lambda,q)} - L| \ge \varepsilon\right\}\right| = 0.$$

$$\Rightarrow \lim_{n\to\infty} Stat - t_n^{(N_\lambda,p,q)(E_\lambda,q)} = L.$$

$$\Rightarrow x_n \xrightarrow{stat(N_\lambda,p,q)(E_\lambda,q)} \ell.$$

Hence, proved.

Example

Consider any prime p, and construct a sequence as below

$$x_n = \begin{cases} x_k, & \text{if } k \text{ is divisible by } p \\ 0, & \text{otherwise} \end{cases}$$

Clearly $\{x_n\}$ is S_{NE_λ} convergent to L, which is $(N_\lambda, p, q)(E_\lambda, q)$ statistically summable to L.

Theorem 2 If $x_n \to L[(N_\lambda, p, q)(E_\lambda, q)]$, then $S_{NE_\lambda} - \lim x_n = L$. Infact, In non-Archimedean field, Norlund-Euler-λ statistical convergence and statistical Norlund-Euler-λ summability are equivalent.

Proof

Let $x_n \to L[(N_\lambda, p, q)(E_\lambda, q)]$.
For a given $\varepsilon > 0$, Define

$$K_{R_{\lambda_n}}(\varepsilon) = \left\{k \le R_{\lambda_n} : p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L| \ge \varepsilon\right\} \quad (13)$$

Consider,

$$\frac{1}{R_{\lambda_n}} \sum_{k\in I_n} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L|$$
$$= \frac{1}{R_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\in K_{R_{\lambda_n}}(\varepsilon)}} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L|$$
$$+ \frac{1}{R_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\notin K_{R_{\lambda_n}}(\varepsilon)}} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L|$$

Since, $x_n \to L[(N_\lambda, p, q)(E_\lambda, q)]$, we know that,

$$\lim_{n\to\infty} \frac{1}{R_{\lambda_n}} \sum_{k\in I_n} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L| = 0.$$

$\Rightarrow$ Each term in the R.H.S equals to zero,
and in particular,

$$\frac{1}{R_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\notin K_{R_{\lambda_n}}(\varepsilon)}} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L| = 0.$$

$$\text{i.e., } \lim_{n\to\infty} \frac{1}{R_{\lambda_n}} \left|\left\{k \le R_{\lambda_n} : p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L| \ge \varepsilon\right\}\right| = 0.$$

$$\Rightarrow S_{(NE)_\lambda} - \lim x = L$$

i.e., statistical summability of a sequence implies NE_λ-statistical convergence.
i.e., $\{x_n\}$ is statistically summable implies $\{x_n\}$ is Norlund-Euler-λ-statistically convergent.
To prove the converse,
i.e., NE_λ-statistical convergence implies statistical summability.
For a given $\varepsilon > 0$, Let

$$K_{R_{\lambda_n}}(\varepsilon) = \{k \le R_{\lambda_n} : p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L| \ge \varepsilon\} \quad (14)$$

Consider,

$$\frac{1}{R_{\lambda_n}} \sum_{k\in I_n} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L|$$

$$= \frac{1}{R_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\in K_{R_{\lambda_n}}(\varepsilon)}} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L| + \frac{1}{R_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\notin K_{R_{\lambda_n}}(\varepsilon)}} \binom{k}{v} q^{k-v} |x_v - L| \quad (15)$$

Since $\{x_n\}$ is S_{NE_λ} convergent, we have

$$\lim_{n\to\infty} \frac{1}{R_{\lambda_n}} |\{k \le R_{\lambda_n} : p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L| \ge \varepsilon\}| = 0. \quad (16)$$

Define

$$M = \sup_{k\in K_{R_{\lambda_n}}(\varepsilon)} \left\{\left|p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{v\in I_k} \binom{k}{v} q^{k-v} |x_v - L| \ge \varepsilon\right|\right\} \quad (17)$$

Taking valuation on both sides in (14),

$$\left|\frac{1}{R_{\lambda_n}}\sum_{k\in I_n}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\right|$$
$$=\left|\frac{1}{R_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K_{R_{\lambda_n}}(\varepsilon)}}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\right.$$
$$\left.+\frac{1}{R_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\notin K_{R_{\lambda_n}}(\varepsilon)}}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\right|$$
$$\leq \max\left\{\left|\frac{1}{R_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K_{R_{\lambda_n}}(\varepsilon)}}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\right|,\right.$$
$$\left.\left|\frac{1}{R_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\notin K_{R_{\lambda_n}}(\varepsilon)}}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\right|\right\}$$
$$=\frac{1}{R_{\lambda_n}}\max\{|M|,\varepsilon\}$$
$$=\max\left\{\frac{|M|}{R_{\lambda_n}},\varepsilon\right\}$$

Taking limit as $n\to\infty$, we get

$$\lim_{n\to\infty}\left|\frac{1}{R_{\lambda_n}}\sum_{k\in I_n}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\right|$$
$$\leq \lim_{n\to\infty}\max\left\{\frac{|M|}{R_{\lambda_n}},\varepsilon\right\}=0.$$
$$\Rightarrow \lim_{n\to\infty}\left|\frac{1}{R_{\lambda_n}}\sum_{k\in I_n}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\right|=0.$$
$$\Rightarrow x_n\to L[(N_\lambda,p,q)(E_\lambda,q)]$$

i.e., In a Non-Archimedean field, Norlund-Euler-λ statistical convergence implies Norlund-Euler-λ statistical summability.

Theorem 3 If $Sup\left(\frac{R_n}{R_{\lambda_n}}\right)=1$. Then for a sequence $\{x_n\}$ Norlund-Euler statistical convergence implies Norlund-Euler-λ statistical convergence.

Proof Given $\{x_n\}$ is Norlund-Euler statistical convergent.

$$\text{i.e., }\lim_{n\to\infty}\frac{1}{R_n}\left|\left\{k\leq R_n: p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v=0}^{n}\binom{k}{v}q^{k-v}|x_v-L|\geq\varepsilon\right\}\right|=0. \quad (18)$$

Consider,

$$\frac{1}{R_{\lambda_n}}\left|\left\{k\leq R_{\lambda_n}: p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v=0}^{n}\binom{k}{v}q^{k-v}|x_v-L|\geq\varepsilon\right\}\right|$$
$$=\frac{1}{R_{\lambda_n}}\frac{R_n}{R_n}\left|\left\{k\leq R_{\lambda_n}: p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v=0}^{n}\binom{k}{v}q^{k-v}|x_v-L|\geq\varepsilon\right\}\right|$$
$$\leq\frac{1}{R_n}\frac{R_n}{R_{\lambda_n}}\left|\left\{k\leq R_n: p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v=0}^{n}\binom{k}{v}q^{k-v}|x_v-L|\geq\varepsilon\right\}\right|$$
$$\leq\frac{1}{R_n}\left|\left\{k\leq R_n: p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v=0}^{n}\binom{k}{v}q^{k-v}|x_v-L|\geq\varepsilon\right\}\right|$$
$$\left\{\text{Since } Sup\left(\frac{R_n}{R_{\lambda_n}}\right)=1\right\}$$

using (18), we get

$$\frac{1}{R_{\lambda_n}}\left|\left\{k\leq R_{\lambda_n}: p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v=0}^{n}\binom{k}{v}q^{k-v}|x_v-L|\geq\varepsilon\right\}\right|$$
$$\to 0,\ \text{as } n\to\infty$$
$$\Rightarrow \{x_n\} \text{ is } S_{NE_\lambda}\text{-convergent.}$$

Theorem 4 Let α and β be fixed real numbers such that $0<\alpha\leq\beta\leq 1$. Then the inclusion $S_{\frac{\alpha}{(NE)_\lambda}}\subseteq S_{\frac{\beta}{(NE)_\lambda}}$ is strict for some α and β when $\alpha<\beta$.

Proof Clearly $\alpha<\beta\Rightarrow R^{\alpha}_{\lambda_n}<R^{\beta}_{\lambda_n}\Rightarrow\frac{1}{R^{\beta}_{\lambda_n}}<\frac{1}{R^{\alpha}_{\lambda_n}}$

we have $\frac{1}{R^{\beta}_{\lambda_n}}\left|\left\{k\leq R_{\lambda_n}: p_k|x_k-L|\geq\varepsilon\right|\right\}$

$\leq\frac{1}{R^{\alpha}_{\lambda_n}}\left|\left\{k\leq R_{\lambda_n}: p_k|x_k-L|\geq\varepsilon\right|\right\}$

Extending this observation to Norlund-Euler-λ statistical convergence, we get

$$\frac{1}{R^{\beta}_{\lambda_n}}\left|\left\{k\leq R_{\lambda_n}: p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\geq\varepsilon\right\}\right|$$
$$\leq\frac{1}{R^{\alpha}_{\lambda_n}}\left|\left\{k\leq R_{\lambda_n}: p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{v\in I_k}\binom{k}{v}q^{k-v}|x_v-L|\geq\varepsilon\right\}\right|$$
$$\Rightarrow S_{\frac{\alpha}{(NE)_\lambda}}\subset S_{\frac{\beta}{(NE)_\lambda}}$$

4 Conclusions

In this paper, we have verified Norlund-Euler-λ statistical convergence, generalized weighted summability using Norlund-Euler-λ method in an Ultrametric field. Further established the relation between Norlund-Euler-λ statistical convergence and Statistical Norlund-Euler-λ summability, and further discussed the relation between Norlund-Euler statistical convergence, and Norlund-Euler-λ statistical convergence. Also, we have investigated the Inclusion criteria between Norlund-Euler-λ statistical convergence of orders α & β.

REFERENCES

[1] H. Fast. Sur la convergence statistique, Colloq Math, Vol. 2, No. 3-4, 241-244, 1951.

[2] H. Steinhaus. Sur la convergence ordinaire et la convergence asymptotique, Colloq Math, Vol. 2, 73-74, 1951.

[3] J.A. Friday. On statistical convergence, Analysis, Vol. 5, 301-313, 1985.

[4] D. Rath, B.C. Tripathy. On statistically convergent and statistically cauchy sequences, Indian Journal of Pure and Applied Mathematics, Vol. 25, No. 4, 381-386, 1994.

[5] V. Karayaka, T.A. Chishti. Weighted statistical convergence, Iranian Journal of Science Technology, Transaction A, Vol. 33, No. A3, 219-223, 2009.

[6] M. Kucukaslan. Weighted statistical convergence, International Journal of Science and Technology, Vol. 2, No. 10, 733-737, 2012.

[7] C. Belen, S. A. Mohiuddine. Generalized weighted statistical convergence and application, Applied Mathematics and Computation, Vol. 219, No.18, 9821-9826, 2013.

[8] K. Hensel. Theorie der Algebraischen Zahlen, Erster Band., Leipzig and Berlin, Teubner, 1908.

[9] K. Suja, V. Srinivasan. On statistically convergent and statistically cauchy sequences in non-Archimedean fields, Journal of Advances in Mathematics, Vol. 6, No. 3, 1038-1043, 1951.

[10] K. Suja, V. Srinivasan. Weighted statistical convergence in ultrametric fields, International Journal of Pure and Applied Mathematics, Vol.116, No. 4, 813-817, 2017.

[11] S. Sangeetha. I-convergence and I-core of a double sequence in Non-Archimedean fields, International Journal of Innovation in Science and Mathematics, Vol. 6, No. 2, 84-87, 2018.

[12] J. Uma, V. Srinivasan. Statistical convergence in generalized difference sequence spaces over non-Archimedean fields, International Journal of Pure and Applied Mathematics, Vol. 113, No. 10, 280-289, 2017.

[13] D. Eunice Jemima, V. Srinivasan. Norlund statistical convergence and tauberian conditions for statistical convergence from statistical summability using Norlund means in non-Archimedean fields, J. Math. Computer Sci., Vol. 24, 299-307, 2022.

[14] G. Bachman. Introduction to P-adic Numbers and Valuation Theory. Academic Press, New York/London, 1964.

[15] V. Loku, E. Aljimi. Weighted Norlund-Euler-λ-statistical convergence and application, Journal of Mathematical Analysis, Vol. 9, No. 6, 95-105, 2018.

[16] M. Et, M. Karakas, M. Cinar. Weighted statistical convergence of order α in paranormed spaces, Turkic World Mathematical Society Journal of Applied and Engineering Mathematics, Vol. 10 (Special Issue), 47-52, 2020.

[17] Ekrem A. Aljimi, Elida Hoxha, Valdete Loku. Some Results of Weighted Norlund-Euler statistical Convergence, International Mathematical Forum, Vol. 8, No. 37, 1797-1812, 2013.