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Inclusion Results of a Generalized Mittag-Leffler-Type Poisson Distribution in the k-Uniformly Janowski Starlike and the k-Janowski Convex Functions

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Abstract Due to the Mittag-Leffler function's crucial contribution to solving the fractional integral and differential equations, academics have begun to pay more attention to this function. The Mittag-Leffler function naturally appears in the solutions of fractional-order differential and integral equations, particularly in the studies of fractional generalization of kinetic equations, random walks, Levy flights, super-diffusive transport, and complex systems. As an example, it is possible to find certain properties of the Mittag-Leffler functions and generalized Mittag-Leffler functions [4,5]. We consider an additional generalization in this study, $E_{\alpha,\beta}^{\theta}(z)$, given by Prabhakar [6,7]. We normalize the later to deduce $\mathbb{E}_{\alpha,\beta}^{\theta}(z)$ in order to explore the inclusion results in a well-known class of analytic functions, namely $k-\mathcal{ST}[A,B]$ and $k-\mathcal{UCV}[A,B]$, k-uniformly Janowski starlike and k-Janowski convex functions, respectively. Recently, researches on the theory of univalent functions emphasize the crucial role of implementing distributions of random variables such as the negative binomial distribution, the geometric distribution, the hypergeometric distribution, and in this study, the focus is on the Poisson distribution associated with the convolution (Hadamard product) that is applied to define and explore the inclusion results of the followings: $\mathrm{I}_{\alpha,\beta}^{m,\theta}(z)$, $\mathcal{J}_{\alpha,\beta}^{m}f$ and the integral operator $\mathcal{G}_{\alpha,\beta}^{m,\theta}$. Furthermore, some results of special cases will be also investigated.

Keywords k-Uniformly Janowski Star-like, k-Janowski Convex Functions, Mittag-Leffler Function

Classification of Mathematics (2010): 30C45.

1. Introduction

In recent years, there has been a lot of interest in random variable distributions. In statistics and probability theory, the real variable x and the complex variable z's probability density functions been crucial. The distributions have so been thoroughly investigated. Many different types of distributions, including the negative geometric distribution, hypergeometric distribution, Poisson distribution, and binomial distribution, have been developed as a result of real-world events.

If a random variable's function of probability density is given by, then the variable x has a Poisson distribution:

$$f(x)=\frac{e^{-m}}{x!}m^{x}, x=0,1,2,\dots \tag{1.1}$$

For the parameter of the distribution *m,* the Poisson distribution started receiving interest in the theory of univalent functions, firstly by Porwal [8] and then later by Porwal and Dixit [9] who provided moments and moments' generating functions with the Mittag-Leffler Poisson distribution.

We indicate by $\mathcal{A}$ the well-known type of the form normalized functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.2)$$

Functions that in the open unit disk analyzers $\mathbb{U} = \text{i}\{z \in \mathbb{C}: |z| < 1\}$.

We also let $\mathcal{T}$ a sub-class of $\mathcal{A}$ that includes operations of the form

$$(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,\ z \in \mathbb{U}. \quad (1.3)$$

Now, we recall the definitions of the classes $k-\mathcal{ST}[A,B]$ and $k-\mathcal{UCV}[A,B]$ that were introduced and studied by Noor and Malik [4].

A function $f \in \mathcal{A}$ is considered to be a member of the class of k -Janowski star-like functions. $k-\mathcal{ST}[A,B], k \geq 0, -1 \leq B < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)} - 1\right|. \quad (1.4)$$

Further, a function $f \in \mathcal{A}$ is said to be in the class k-Janowski convex functions $\mathcal{UCV}[A,B], k \geq 0, -1 \leq B\text{i} < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)} - 1\right|, (1.5)$$

clearly

$$f(z) \in k-\mathcal{UCV}[A,B] \Leftrightarrow zf'(z) \in k-\mathcal{ST}[A,B].$$

The above are generalizations of the following special cases:

(1) $k-\mathcal{ST}[1,-1] = k-\mathcal{ST}\text{i}$ and $k-\mathcal{UCV}[1,-1] = k-\mathcal{UCV}$, the well-known classes of k starlike and k-uniformly convex functions respectively, introduced by Kanas and Wisniowska [6,7 and also 1]

(2) $k-\mathcal{ST}[1-2\gamma,-1] = k-\mathcal{SD}[k,\gamma]$ and $k-\mathcal{UCV}[1-2\gamma,-1] = k-\mathcal{KD}[k,\gamma]$, the classes introduced by Shams et al. in [10].

(3) $0-\mathcal{ST}[A,B] = \text{S}^*[A,B]$ and $0-\mathcal{UCV}[A,B] = \mathcal{C}[A,B]$ the well-known classes of Janowski starlike and Janowski convex functions respectively, introduced by Janowski [12].

(4) $0-\mathcal{ST}[1-2\gamma,-1] = \mathcal{S}^*(\gamma)$ and $0-\mathcal{UCV}[1-2\gamma,-1] = \mathcal{C}(\gamma)$, the well-known classes of starlike functions of order $\gamma(0 \leq \gamma < 1)$ and convex functions of order $\gamma(0 \leq \gamma < 1)$ respectively, (see [3]).

If $f(z) \in k-\mathcal{ST}[A,B]$ then

$$w = \frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)}$$

takes all values from the domain $\Omega_k, k \geq 0$ as

$$\begin{aligned}\Omega_k &= \{w: \Re w > k|w-1|\}\\ &= \left\{u+iv: u > k\sqrt{(u-1)^2+v^2}\right\}\end{aligned}$$

The domain Ω_k represents the right half plane for $k = 0$; a hyperbola for $0 < k < 1$; a parabola for $k = 1$ and an ellipse for $k > 1$, (see [4]).

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(C,D), \tau \in \mathbb{C} \setminus \text{i}\{0\}, -1 \leq D < C \leq 1$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(C-D)\tau - D[f'(z)-1]}\right| < 1,\ z \in \mathbb{U}$$

The class above was introduced by Dixit and Pal [13] providing the below results

Lemma 1.1. [13] If $f \in \mathcal{R}^\tau(C,D)$ is of the form (1.2), then

$$|a_n| \leq (C-D)\frac{|\tau|}{n},\ n \in \mathbb{N} \setminus \{1\}$$

The result is sharp for the function

$$f(z) = \int_0^z \left(1 + \frac{(C-D)|\tau|t^{n-1}}{1+Dt^{n-1}}\right)dt,\ (z \in \mathbb{U}; n \in \mathbb{N} \setminus \{1\}).$$

Mittag-Leffler function $E_\alpha(z)$ is studied by Mittag-Leffler [2] and given by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},\ (z \in \mathbb{C}, \Re(\alpha) > 0).$$

Prabhakar [5, 11] has generalized the Mittag – Leffler function as follows

$$E_{\alpha,\beta}^{\theta}(z) := \sum_{n=0}^{\infty} \frac{(\theta)_n}{\Gamma(\alpha n + \beta)}.\frac{z^n}{n!},\quad z,\beta,\theta \in \mathbb{C};\ \Re(\alpha) > 0,$$

here; $(\theta)_v$ denotes the familiar Pochhammer symbol defined as

$$(\theta)_v := \frac{\Gamma(\theta+v)}{\Gamma(\theta)} = \begin{cases}1, & if\ v = 0,\quad \theta \in \mathbb{C}\setminus\{0\}\\ \theta(\theta+1)\ldots(\theta+n-1), & if\ v = n \in N, \theta \in \mathbb{C}\end{cases}$$

$$(1)_n = n!,\qquad n \in N_0, N_0 = N \cup \{0\},\qquad N = \{1,2,3,\ldots\}.$$

Since the generalized Mittag-Leffler function $E^{\theta}_{\alpha,\beta}(z)$ doesn't belong to the family $\mathcal{A}$. Let us consider the following normalization of the Mittag-Leffler function

$$\mathbb{E}^{\theta}_{\alpha,\beta}(z) = \Gamma(\beta) z E^{\theta}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta)}{n!\Gamma(\alpha(n-1)+\beta)} z^n \tag{1.6}$$

where $z, \alpha, \beta \in \mathbb{C}; \beta \neq 0, -1, -2, \cdots$ and $\Re(\beta) > 0, \Re(\alpha) > 0$.

Our attention in this paper is only to the cases; where α, β are real-valued and $z \in \mathbb{U}$.

The generalized Mittag-Leffler-type Poisson distribution's probability mass function is then given by

$$P(x=r) = \frac{m^r}{\Gamma(\alpha k + \beta)\mathbb{E}^{\theta}_{\alpha,\beta}(m)}, \; r = 0,1,2,3,\cdots,$$

in where m>0, α>0, and β>0. One can introduce a power series whose coefficients are probabilities of the generalized Mittag-Leffler-type Poisson distribution series using the normalized version of the Mittag-Leffler function in (1.6), as follows:

$$H^{m,\theta}_{\alpha,\beta}(z) := z + \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}^{\theta}_{\alpha,\beta}(m)} z^n, \; z \in \mathbb{U}$$

To serve our purpose, we also need to define the series

$$I^{m,\theta}_{\alpha,\beta}(z) := 2z - H^{m}_{\alpha,\beta}(z) = z - \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n!\Gamma(\alpha(n-1)+\beta)\mathbb{E}^{\theta}_{\alpha,\beta}(m)} z^n, \; z \in \mathbb{U} \tag{1.7}$$

Finally, and by the means of the convolution, we deduce the following operator:

$$\mathcal{J}^{m,\theta}_{\alpha,\beta} f(z) = H^{m,\theta}_{\alpha,\beta}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}^{\theta}_{\alpha,\beta}(m)} a_n z^n, \; z \in \mathbb{U},$$

2. Inclusion Results of $I^{m,\theta}_{\alpha,\beta}(z)$

To establish our primary findings, we shall require the below given lemmas.

Lemma 2.1. *[4] A function f of the form (1.2) is in the class $k - \mathcal{ST}[A,B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} [2(k+1)(n-1) + |n(B+1) - (A+1)|]|a_n| \leq |B - A| \tag{2.1}$$

where $-1 \leq B < A \leq 1$ and $k \geq 0$.

Lemma 2.2. *[4] A function f of the form (1.2) is in the class $k - \mathcal{UCV}[A,B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n[2(k+1)(n-1) + |n(B+1) - (A+1)|]|a_n| \leq |B - A \tag{2.2}$$

where $-1 \leq B < A \leq 1$ and $k \geq 0$.

In this study, we will assume that until otherwise stated that $\alpha, m > 0, k \geq 0$ and $-1 \leq B < A \leq 1$.

Theorem 2.3. *Let $\beta > 1$. Then $I^{m,\theta}_{\alpha,\beta} \in k - \mathcal{ST}[A,B]$ if*

$$\begin{aligned} &\frac{(\theta)_n \Gamma(\beta)}{n!\mathbb{E}^{\theta}_{\alpha,\beta}(m)} \Big[\frac{2k+B+3}{\alpha}\Big(E_{\alpha,\beta-1}(m) - \frac{1}{\Gamma(\beta-1)}\Big) \\ &+ \Big[\Big(\frac{2k+B+3}{\alpha}\Big)(1-\beta) + (B+A+2)\Big]\Big(E^{\theta}_{\alpha,\beta}(m) - \frac{n!}{(\theta)_n \Gamma(\beta)}\Big)\Big] \\ \leq \quad & |B - A| \end{aligned} \tag{2.3}$$

Proof. Given Lemma 2.1 and (2.1), it is sufficient to demonstrate that

$$J_1 := \sum_{n=2}^{\infty} [2(k+1)(n-1) + |n(B+1) - (A+1)|] \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} \leq |B - A|$$

We have

$$J_1 \leq \sum_{n=2}^{\infty} [2(k+1)(n-1) + n(B+1) + (A+1)] \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}^{\theta}_{\alpha,\beta}(m)}$$

$$= \sum_{n=2}^{\infty} [(2k+B+3)n + (A-2k-1)] \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n!\, \Gamma(\alpha(n-1)+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$
$$= \sum_{n=1}^{\infty} [(2k+B+3)(n+1) + (A-2k-1)] \frac{(\theta)_n \Gamma(\beta) m^{n}}{n!\, \Gamma(\alpha n+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$
$$= \sum_{n=1}^{\infty} [(2k+B+3)n + (B+A+2)] \frac{(\theta)_n \Gamma(\beta) m^{n}}{n!\, \Gamma(\alpha n+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$
$$= \left(\frac{2k+B+3}{\alpha}\right) \sum_{n=1}^{\infty} [(\alpha n+\beta-1) + (1-\beta)] \frac{(\theta)_n \Gamma(\beta) m^{n}}{n!\, \Gamma(\alpha n+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$
$$+(B+A+2) \sum_{n=1}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n}}{n!\, \Gamma(\alpha n+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$
$$= \left(\frac{2k+B+3}{\alpha}\right) \sum_{n=1}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n}}{n!\, \Gamma(\alpha n+\beta-1) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$
$$+\left[\left(\frac{2k+B+3}{\alpha}\right)(1-\beta) + (B+A+2)\right] \sum_{n=1}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n}}{n!\, \Gamma(\alpha n+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$
$$= \frac{(\theta)_n \Gamma(\beta)}{n!\, \mathbb{E}_{\alpha,\beta}^{\theta}(m)} \left[\frac{2k+B+3}{\alpha}\left(E_{\alpha,\beta-1}^{\theta}(m) - \frac{1}{\Gamma(\beta-1)}\right)\right.$$
$$\left. + \left[\left(\frac{2k+B+3}{\alpha}\right)(1-\beta) + (B+A+2)\right]\left(E_{\alpha,\beta}^{\theta}(m) - \frac{n!}{(\theta)_n \Gamma(\beta)}\right)\right]$$
$$\leq |B-A|,$$

This completes the evidence for Theorem 2.3.

Theorem 2.4. *Let* $\beta > 2$. *Then* $I_{\alpha,\beta}^{m,\theta} \in k - \mathcal{UCV}[A,B]$ *if*

$$\frac{(\theta)_n \Gamma(\beta)}{n!\, \mathbb{E}_{\alpha,\beta}^{\theta}(m)} \left[\frac{2k+B+3}{\alpha^2}\left(E_{\alpha,\beta-2}^{\theta}(m) - \frac{1}{\Gamma(\beta-2)}\right)\right.$$
$$+\left(\frac{(2k+B+3)(3-2\beta) + \alpha(2B+A+2k+5)}{\alpha^2}\right)\left(E_{\alpha,\beta-1}^{\theta}(m) - \frac{1}{\Gamma(\beta-1)}\right)$$
$$\left.+\left(\frac{(2k+B+3)(1-\beta)^2}{\alpha^2} + \frac{(2B+A+2k+5)(1-\beta)}{\alpha} + (B+A+2)\right)\left(E_{\alpha,\beta}^{\theta}(m) - \frac{n!}{(\theta)_n \Gamma(\beta)}\right)\right]$$
$$\leq |B-A|$$

Proof. We consider the same approach of Theorem 2.3 by the means of Lemma 2.2 and (2.2). Here we let

$$J_2 := \sum_{n=2}^{\infty} n[2(k+1)(n-1) + |n(B+1) - (A+1)|] \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n!\, \Gamma(\alpha(n-1)+\beta)\, \mathbb{E}_{\alpha,\beta}^{\theta}(m)} \leq |B-A|.$$

3. Inclusion Results of $\mathcal{J}_{\alpha,\beta}^{m} f$

Theorem 3.1. *Let* $\beta > 1$. *If* $f \in \mathcal{R}^{\tau}(C,D)$, *then* $\mathcal{J}_{\alpha,\beta}^{m,\theta} f \in k - \mathcal{UCV}[A,B]$ *if*

$$\begin{aligned} &\frac{(C-D)|\tau|(\theta)_n \Gamma(\beta)}{n!\mathbb{E}_{\alpha,\beta}^{\theta}(m)}\left[\frac{2k+B+3}{\alpha}\left(E_{\alpha,\beta-1}(m) - \frac{1}{\Gamma(\beta-1)}\right)\right. \\ &\left.+\left[\left(\frac{2k+B+3}{\alpha}\right)(1-\beta) + (B+A+2)\right]\left(E_{\alpha,\beta}(m) - \frac{n!}{(\theta)_n \Gamma(\beta)}\right)\right] \\ \leq\ & |B-A| \end{aligned} \tag{3.1}$$

Proof. Using Lemma 2.2 and (2.1) it is enough to verify that

$$\sum_{n=2}^{\infty} n[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{(\theta)_n\Gamma(\beta)m^{n-1}}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}^{\theta}(m)}|a_n| \leq |B-A|$$

Now, since $f \in \mathcal{R}^{\tau}(C,D)$, in view of Lemma 1.1 the coefficients bound is

$$|a_n| \leq \frac{(C-D)|\tau|}{n}, n \in \mathbb{N} \setminus \{1\}$$

Thus, it is sufficient to show that

$$(C-D)|\tau|\left[\sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{(\theta)_n\Gamma(\beta)m^{n-1}}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}\right]$$
$$\leq \quad |B-A|.$$

Which is the same approach of the proof of Theorem 2.3, we conclude that $\mathcal{I}_{\alpha,\beta}^{m}f \in k-\mathcal{UCV}[A,B]$ if (3.1) holds true.

4. Inclusion Results of the Integral Operator $\mathcal{G}_{\alpha,\beta}^{m,\theta}$

Following the same previous methods, we can readily deduce the next result

Theorem 4.1. *If* $\beta > 1$, *the integ iral operator follows*

$$\mathcal{G}_{\alpha,\beta}^{m,\theta}(z) := \int_0^z \frac{I_{\alpha,\beta}^{m,\theta}(t)}{t}dt, z \in \mathbb{U},$$

is in k-$\mathcal{UCV}[A,B]$ *if the condition of inequality (2.3) is met.*

Proof. By the assumption (1.7) we have

$$\mathcal{G}_{\alpha,\beta}^{m,\theta}(z) = z - \sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)m^{n-1}}{(\theta)_n\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}^{\theta}(m)}\frac{z^n}{n}.$$

Now, using (2.1) and Lemma 2.2, the integral operator; $\mathcal{G}_{\alpha,\beta}^{m}(z)$ belongs to $k-\mathcal{UCV}[A,B]$; if

$$\sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{(\theta)_n\Gamma(\beta)m^{n-1}}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}^{\theta}(m)} \leq |B-A|$$

we conclude that $\mathcal{G}_{\alpha,\beta}^{m,\theta} \in k-\mathcal{UCV}[A,B]$ if (2.3) holds true.

5. Special Cases

Let $A = 1-2\gamma$, and $B = -1$ with $0 \leq \gamma < 1$ in the above theorems, we receive the following special cases:

Corollary 5.1. *Let* $\beta > 1$. *Then* $I_{\alpha,\beta}^{m,\theta} \in k-\mathcal{SD}[k,\gamma]$ *if*

$$\frac{(\theta)_n\Gamma(\beta)}{n!\,\mathbb{E}_{\alpha,\beta}^{\theta}(m)}\left[\frac{k+1}{\alpha}\left(E_{\alpha,\beta-1}^{\theta}(m)\mathrm{i}-\frac{1}{\Gamma(\beta-1\mathrm{i})}\right)\right.$$
$$\left.+\left[\left(\frac{k+1}{\alpha}\right)\mathrm{i}(1-\beta)+1-\gamma\mathrm{i}\right]\left(E_{\alpha,\beta}^{\theta}(m)-\frac{n!}{(\theta)_n\Gamma(\beta)}\right)\right]$$
$$\leq \quad 1-\gamma.$$

Corollary 5.2. *Let* $\beta > 2$. *Then* $I_{\alpha,\beta}^{m,\theta} \in k-\mathcal{KD}[k,\gamma]$ *if*

$$\frac{(\theta)_n\Gamma(\beta)}{n!\,\mathbb{E}_{\alpha,\beta}(m)}\Bigg[\frac{k+1}{\alpha^2}\left(E^{\theta}_{\alpha,\beta-2}(m)-\frac{1}{\Gamma(\beta-2)}\right) + \left(\frac{(k+1)(3-2\beta)+\alpha(2-\gamma+k)}{\alpha^2}\mathrm{i}\right)\left(E^{\theta}_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right) + \left(\frac{(k+1)(1-\beta)^2}{\alpha^2}+\frac{(2-\gamma+k)(1-\beta)}{\alpha}+(1-\alpha)\right)\left(E^{\theta}_{\alpha,\beta}(m)-\frac{n!}{(\theta)_n\Gamma(\beta)}\right)\Bigg]$$

$$1-\gamma$$

Corollary 5.3. *Let* $\beta > 1$. *If* $f \in \mathcal{R}^{\tau}(C,D)$, *then* $\mathcal{I}^{m,\theta}_{\alpha,\beta} f \in k-\mathcal{KD}[k,\gamma]$ *if*

$$\frac{(C-D)|\tau|(\theta)_n\Gamma(\beta)}{n!\,\mathbb{E}_{\alpha,\beta}(m)}\Bigg[\frac{k+1}{\alpha}\left(E^{\theta}_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right) + \left[\left(\frac{k+1}{\alpha}\right)\mathrm{i}(1-\beta)+1-\gamma\right]\left(E^{\theta}_{\alpha,\beta}(m)-\frac{n!}{(\theta)_n\Gamma(\beta)}\right)\Bigg] \leq 1-\gamma$$

Corollary 5.4. *Let* $\beta > 1$. *The component operator provided by (4.1) is then in class k;* $\mathcal{KD}[k,\gamma]$ *if the inequality in Corollary 5.1 holds true.*

6. Conclusions

The generalized Mittag-Leffler function has been investigated by the means of Poisson distribution. A normalized form $\mathbb{E}^{\theta}_{\alpha,\beta}(z)$ has been studied in terms of its inclusion in the well know subclasses of analytic functions, here we have considered $k-\mathcal{ST}[A,B]$ and $k-\mathcal{UCV}[A,B]$. Sufficient conditions are derived for $\mathrm{I}^{m,\theta}_{\alpha,\beta}(z)$, $\mathcal{I}^{m}_{\alpha,\beta} f$ and the integral operator $\mathcal{G}^{m,\theta}_{\alpha,\beta}$ to belong to k-Janowski convex and k-uniformly star-like functions. Lastly, given some A and B parameter values, special cases are discussed.

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