Mathematics and Statistics 11(1): 13-21, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110102

Linear Stability of Double-sided Symmetric Thin Liquid Film by Integral-theory

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Received September 16, 2022; Revised November 24, 2022; Accepted December 22, 2022

Cite This Paper in the Following Citation Styles

(a): *[1] Ibrahim S. Hamad , "Linear Stability of Double-sided Symmetric Thin Liquid Film by Integral-theory," Mathematics and Statistics, Vol. 11, No. 1, pp. 13 - 21, 2023. DOI: 10.13189/ms.2023.110102.*

(b): *Ibrahim S. Hamad (2023). Linear Stability of Double-sided Symmetric Thin Liquid Film by Integral-theory. Mathematics and Statistics, 11(1), 13 - 21. DOI: 10.13189/ms.2023.110102.*

Abstract The Integral Theory approach is used to explore the stability and dynamics of a free double-sided symmetric thin liquid film. For a Newtonian liquid with non-variable density and moving viscosity, the flowing in a thinning liquid layer is analyzed in two dimensions. To construct an equation that governs such flow, the Navier and Stokes formulas are utilized with proper boundary conditions of zero shear stress conjointly of normal stress on the bounding free surfaces with dimensionless variables. After that, the equations that are a non-linear evolution structure of layer thickness, local stream rate, and the unknown functions can be solved by using straight stability investigation, and the normal mode strategy can moreover be connected to these conditions to reveal the critical condition. The characteristic equation for the growth rate and wave number can be analyzed by using MATLAM programming to show the region of stable and unstable films. As a result of our research, we are able to demonstrate that the effect of a thin, free, double-sided liquid layer is an unstable component.

Keywords Thinning Liquid Layers, Navier and Stokes Equations, Continuity-formulas

1. Introduction

This research is to study the linear stability analysis of thinning liquid symmetrical double-sided layers. A two-dimensional flow of a Newtonian liquid with constant density and viscosity could be used to describe the film. The local surface tension variation is taken into consideration while solving the equations of continuity and motion with proper boundary conditions. When those governing equations are approximated using Integral Theory, the result is an equation for the film's evolution as a function of time and space. In many applications, the steadiness and flow of thin liquid layers are important [1]. The stability analysis of an inclined free thin liquid film is considered, with hydrostatic pressure and surface tension impacting the equation [2]. In applications such as coatings, photographic films, microelectronic devices, and insulating layers, the stability of thin fluid films is essential [3]. The stability analysis considers, which takes into account the critical number, wave length, and most extreme growth rate of the foremost unstable disturbance [4]. The strategy for analyzing nonlinear stability takes advantage of the truth that is intuitive between nonlinear streams is restricted by the material science inherent within the nonlinearity. It demonstrates how nonlinear steady-state investigation issues can be surrounded by increased possibility and optimization issues based on Lyapunov lattice imbalances, as well as a set of quadratic constraints that characterize nonlinear stream physics [15]. A thin, pseudo-plastic fluid layer is flowing down on a vertical wall. The long-wave perturbation strategy is utilized to fathom the generalized nonlinear kinematic condition with a free film interface. The straight stability arrangement for the film stream is calculated through the normal mode strategy [16]. This work studies the stream in a double-sided horizontal thin liquid layer. Using the Navier-Stokes equation, we can derive the equation that regulates such a flow without an inertia term. Since the flow is predominantly in the x-direction, the Integral Theory is used to explore the stability analysis.

2. Formulations and Governing Equations

In two dimensions, the stream of a viscous liquid inside a horizontal double-sided symmetrical thinning liquid layer with zero-shear exertion on its boundary surface is considered. The $\bar{x}$-axis is the axis of symmetrical, and the flow is largely in the x direction, hence the Cartesian Coordinates x and z are being used. The $\bar{z}$-axis, on the other hand, is perpendicular to the film's plane, as illustrated in Figure (1), and the formula for the boundary surfaces layer is $\bar{z} = \pm\bar{h}(\bar{x}, \bar{t})$ where $\bar{t}$ is the time.

Let the velocity of the fluid be denoted by $\bar{q} = (\bar{u}, \bar{w})$, where $\bar{u}$ and $\bar{w}$ are the velocity components of the velocity field $\bar{q}$ in $\bar{x}$ and $\bar{z}$ directions respectively. The Navier and Stokes formulas in the $\bar{x}$ and $\bar{z}$ directions, which are the longitudinal and transverse motion conditions of unstable stream and are provided in the differentiation form correspondingly, are assumed to govern the two-dimensional streams of an incompressible in this situation as :

$$\rho\left[\frac{\partial\bar{u}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{u}}{\partial\bar{z}}\right] = -\frac{\partial\bar{p}}{\partial\bar{x}} + \mu\left[\frac{\partial^2\bar{u}}{\partial\bar{x}^2} + \frac{\partial^2\bar{u}}{\partial\bar{z}^2}\right] \quad (1)$$

and

$$\rho\left[\frac{\partial\bar{w}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{w}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{w}}{\partial\bar{z}}\right] = -\frac{\partial\bar{p}}{\partial\bar{z}} + \mu\left[\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + \frac{\partial^2\bar{w}}{\partial\bar{z}^2}\right] \quad (2)$$

when $\bar{t}$ is the period of time, $\bar{p}$ is the applied pressure, μ is the liquid's dynamic viscosity and ρ is the fluid's density.

The continuity formula is provided as

$$\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{w}}{\partial\bar{z}} = 0 \quad (3)$$

The inertia terms in lubrication theory can be ignored, resulting in the following Navier-Stokes equations (1) and (2):

$$\frac{\partial\bar{p}}{\partial\bar{x}} = \mu\left[\frac{\partial^2\bar{u}}{\partial\bar{x}^2} + \frac{\partial^2\bar{u}}{\partial\bar{z}^2}\right] \quad (4)$$

and

$$\frac{\partial\bar{p}}{\partial\bar{z}} = \mu\left[\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + \frac{\partial^2\bar{w}}{\partial\bar{z}^2}\right] \quad (5)$$

Now, because of mass conservation and the fact that the free interface could be a stream path, the derivative (the materials or considerable derivatives) $\frac{DF}{D\bar{t}} = 0$ must disappear on $\bar{z} = \bar{h}(\bar{x}, \bar{t})$, [5], and we obtain

$$\frac{DF}{D\bar{t}} = \frac{\partial F}{\partial\bar{t}} + \bar{u}\frac{\partial F}{\partial\bar{x}} + \bar{w}\frac{\partial F}{\partial\bar{z}} = 0 \quad (6)$$

when

$$F(\bar{x}, \bar{z}, \bar{t}) = \bar{z} - \bar{h}(\bar{x}, \bar{t}) \quad (7)$$

From equations (6) and (7), we obtain

$$\frac{\partial\bar{h}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{h}}{\partial\bar{x}} - \bar{w} = 0 \quad (8)$$

The perturbations of the two surfaces are symmetric in this squeezing mode [6], and thus:

$$+\bar{h}(\bar{x}, \bar{t}) = -\bar{h}(\bar{x}, \bar{t}).$$

Here after, we write the dimensional thickness as $+\bar{h}(\bar{x}, \bar{t}) = -\bar{h}(\bar{x}, \bar{t})$ and will consider only the upper part of the symmetric film $0 \leq \bar{z} \leq \bar{h}(\bar{x}, \bar{t})$.

The shear-stress condition and the normal-stress condition by [7] respectively given as:

$$\frac{\partial\bar{u}}{\partial\bar{z}} + \frac{\partial\bar{w}}{\partial\bar{x}} - 4\frac{\partial\bar{h}}{\partial\bar{x}}\frac{\partial\bar{u}}{\partial\bar{x}} = 0 \quad (9)$$

And

$$\bar{p} + 2\frac{\partial\bar{u}}{\partial\bar{x}} + 2\frac{\partial\bar{h}}{\partial\bar{x}}\left[\frac{\partial\bar{u}}{\partial\bar{z}} + \frac{\partial\bar{w}}{\partial\bar{x}}\right] = -\frac{\partial^2\bar{h}}{\partial\bar{x}^2} \quad (10)$$

The conditions for the squeezing mode [6,8] of the free film are then:

$$\bar{w} = 0$$

at

$$\bar{z} = 0 \quad (11)$$

3. Dimensional Analysis

The following non-dimensional variables are introduced to represent the Navier-Stokes formulas, the continuity-formula and the related boundaries conditions in the dimensionless shape [7]:

$$x = \frac{k\varepsilon\bar{x}}{H}$$
$$z = \frac{\bar{z}}{H}$$
$$h = \frac{\bar{h}}{H}$$
$$u = \frac{\bar{u}}{U}$$
$$w = \frac{\bar{w}}{\varepsilon U}$$
$$p = \frac{H\bar{p}}{\varepsilon^2\sigma}$$
$$t = \frac{\varepsilon U\bar{t}}{H} \quad (12)$$

Where H and U are the parameters. The Capillary-number is characterized as

$$\varepsilon = \frac{\mu U}{\sigma} \quad (13)$$

The Reynolds number is also known as

$$k\varepsilon = \frac{\rho U H}{\mu} \quad (14)$$

Equation (13) is substituted into formula (14), and the result is

$$k = \frac{\rho\sigma H}{\mu^2} \quad (15)$$

Equations contain dimensionless variables that can be used to (12) into equations (3), (4) and (5), the continuity equation gives

$$\frac{kU\varepsilon}{H}\frac{\partial u}{\partial x}+\frac{U\varepsilon}{H}\frac{\partial w}{\partial z}=0 \quad (16)$$

and the Navier-Stoke equations in x-direction gives

$$\frac{k\varepsilon^3\sigma}{H^2}\frac{\partial p}{\partial x}=\mu\left[\frac{k^2\varepsilon^2 U}{H^2}\frac{\partial^2 u}{\partial x^2}+\frac{U}{H^2}\frac{\partial^2 u}{\partial z^2}\right] \quad (17)$$

Also, the Navier-Stoke equations in z -direction give

$$\frac{\varepsilon^2\sigma}{H^2}\frac{\partial p}{\partial z}=\mu\left[\frac{k^2\varepsilon^3 U}{H^2}\frac{\partial^2 w}{\partial x^2}+\frac{\varepsilon U}{H^2}\frac{\partial^2 w}{\partial z^2}\right] \quad (18)$$

After simplifying equations (16)-(18), and Equations (13) through (15) are used to generate the results:

$$k\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \quad (19)$$

$$\frac{\partial^2 u}{\partial z^2}=\varepsilon^2\left[k\frac{\partial p}{\partial x}-k^2\frac{\partial^2 u}{\partial x^2}\right] \quad (20)$$

$$\frac{\partial^2 w}{\partial z^2}=\left[\frac{\partial p}{\partial z}-k^2\varepsilon^2\frac{\partial^2 w}{\partial x^2}\right] \quad (21)$$

and equation (8) in non-dimensional gives the form

$$w=\frac{\partial h}{\partial t}+ku\frac{\partial h}{\partial x} \quad (22)$$

The shear-stress condition and the normal-stress condition in non-dimensional form respectively given as:

$$\frac{\partial u}{\partial z}+\varepsilon^2 k\frac{\partial w}{\partial x}-4\varepsilon^2 k^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}=0 \quad (23)$$

$$p+2\frac{\partial u}{\partial x}+2\frac{\partial h}{\partial x}\left[\frac{\partial u}{\partial z}+\varepsilon^2\frac{\partial w}{\partial x}\right]=-\frac{\partial^2 h}{\partial x^2} \quad (24)$$

and the equation (11) in non-dimensional gives the form

$$w=0$$

at

$$z=0 \quad (25)$$

4. Mathematical Formulation

Now, using the following concepts, the technique of multiple spatial scales may be utilized to think about nonlinear stability [9]:

$$u=O(1)$$
$$w=O(\Upsilon)$$
$$p=O(\Upsilon^{-1})$$
$$x=O(\Upsilon^{-1})$$
$$z=O(1)$$
$$t=O(\Upsilon^{-1}) \quad (26)$$

The other non-dimensional variables' ordering are as follows:

$$\varepsilon=O(1)$$
$$k=O(\Upsilon^3) \quad (27)$$

Introducing equations (26) and (27) in equation (19)-(25), we can formulate the equation of continuity and Navier-stokes equations as:

$$\Upsilon^4 k\frac{\partial u}{\partial x}+\Upsilon\frac{\partial w}{\partial z}=0 \quad (28)$$

$$\frac{\partial^2 u}{\partial z^2}=\varepsilon^2\Upsilon^3 k\frac{\partial p}{\partial x}-\varepsilon^2\Upsilon^8 k^2\frac{\partial^2 u}{\partial x^2} \quad (29)$$

and

$$\frac{\partial^2 w}{\partial z^2}=\frac{1}{\Upsilon^2}\frac{\partial p}{\partial z}-\varepsilon^2\Upsilon^8 k^2\frac{\partial w}{\partial x^2} \quad (30)$$

At the interface $z=h$, we have:

$$w=\frac{\partial h}{\partial t}+\Upsilon^3 ku\frac{\partial h}{\partial x} \quad (31)$$

and

$$\frac{\partial u}{\partial z}+\varepsilon^2\Upsilon^5 k\frac{\partial w}{\partial x}-4\varepsilon^2\Upsilon^8 k^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}=0 \quad (32)$$

and

$$p+2\Upsilon^2\frac{\partial u}{\partial x}+2\Upsilon^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial z}+2\varepsilon^2\Upsilon^4\frac{\partial h}{\partial x}\frac{\partial w}{\partial x}=-\Upsilon^3\frac{\partial^2 h}{\partial x^2} \quad (33)$$

Now, at $z=0$, we have:

$$w=0$$

at

$$z=0 \quad (34)$$

Neglecting the terms higher than $O(\Upsilon^6)$ with $\Upsilon=O\left(\Upsilon^{\frac{3}{2}}\right)$, the reduced conditions of movement and the relevant boundary equations can be inferred as follows:

$$\frac{\partial^2 u}{\partial z^2}=\varepsilon^2\left[k\frac{\partial p}{\partial x}-k^2\frac{\partial^2 u}{\partial x^2}\right] \quad (35)$$

$$\frac{\partial^2 w}{\partial z^2}=\left[\frac{\partial p}{\partial z}-k^2\varepsilon^2\frac{\partial^2 w}{\partial x^2}\right] \quad (36)$$

And the continuity equation:

$$k\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \quad (37)$$

and at the surface, $z=h$:

$$w=\frac{\partial h}{\partial t}+ku\frac{\partial h}{\partial x} \quad (38)$$

$$\frac{\partial u}{\partial z}+\varepsilon^2 k\frac{\partial w}{\partial x}-4\varepsilon^2 k^2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x}=0 \quad (39)$$

and

$$p+2\frac{\partial u}{\partial x}+2\frac{\partial h}{\partial x}\left[\frac{\partial u}{\partial z}+\varepsilon^2\frac{\partial w}{\partial x}\right]=\frac{-\partial^2 h}{\partial x^2} \quad (40)$$

Also, the boundary condition at $z=0$ becomes:

$$w=0$$

at

$$z=0 \quad (41)$$

The high arrange viscous dissemination terms incorporate the normal stretch in the x-momentum equation, the shear stretch in the z-momentum equation and the interfacial shear and normal stress within the right-hand-side of the dynamic boundary conditions (39) and (40), which are caused by substantial deformation. When the rupture processes reach the nonlinear stage, the over-high-order terms get to be essential.

Integrate equation (35) over the film thickness with respect to z.

We have the following:

$$\int_0^h \frac{\partial^2 u}{\partial z^2} dz = \int_0^h \varepsilon^2 k \frac{\partial p}{\partial x} dz - \int_0^h \varepsilon^2 k^2 \frac{\partial^2 u}{\partial x^2} dz \quad (42)$$

and integrating equations (36) and (37) with respect to z , we get:

$$\int_0^h \frac{\partial^2 w}{\partial z^2} dz = \int_0^h \frac{\partial p}{\partial z} dz - \int_0^h k^2 \varepsilon^2 \frac{\partial^2 w}{\partial x^2} dz \quad (43)$$

and

$$\int_0^h k \frac{\partial u}{\partial x} dz + \int_0^h \frac{\partial w}{\partial z} dz = 0 \quad (44)$$

Using $w|_h$ given by (41) and $w|_0$ from (38), defining the local Instantaneous flow rate [9] as:

$$q(x,t) = \int_0^{h(x,t)} u(x,z,t)\, dz \quad (45)$$

Through the integral condition, we have:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (46)$$

Where q is the local flow rate.

In this integral boundary-layer hypothesis, a specific profile must now be enforced. A level profile is commonly accepted for highly turbulent streams. A parabolic profile determined experimentally by [10] is better appropriate for the flow of interest here. As a result, we define u's second-order self-similar profile [9,11] as follows:

$$u = a(x,t)z^2 + b(x,t)z + C(x,t) \quad (47)$$

This velocity profile must meet the taking after boundary- conditions:

$$\frac{\partial u}{\partial z}|_{z=0} = 0 \quad (48)$$

$$\frac{\partial u}{\partial z}|_{z=h} = g(x,t) \quad (49)$$

Where $g(x,t)$ is an indefinite function. Because of the shear-stress equilibrium condition, condition (39), we know that the esteem of $\frac{\partial u}{\partial z}$ does not disappear at interfacing. Differentiating equation (47) with respect to , we have:

$$\frac{\partial u}{\partial z} = 2A(x,t)z + B(x,t),$$

Using equation (48), we obtain:

$$B(x,t) = 0$$

Substituting the value of $B(x,t)$ into equation (47), we get that:

$$u = A(x,t)z^2 + C(x,t) \quad (50)$$

Differentiating equation (50) with regard to z, we get:

$$\frac{\partial u}{\partial z} = 2A(x,t)z$$

And from boundary condition (49), we find that:

$$A(x,t) = \frac{1}{2} g h^{-1} \quad (51)$$

Inserting equation (51) into equation (50), we obtain that:

$$u = \left(\frac{1}{2} g h^{-1}\right) z^2 + C(x,t) \quad (52)$$

Integrating equation (52) with regard to y over the layer thickness, we obtain:

$$\int_0^h u\, dz = \int_0^h \left(\left(\frac{1}{2} g\, h^{-1}\right) z^2 + C(x,t)\right) dz$$

Now from the description of the local flowing rate equation (45), one can relatethe indefinite function $C(x,t)$ to q , g and h as :

$$q = \int_0^h \left(\left(\frac{1}{2} g h^{-1}\right) z^2 + C(x,t)\right) dz$$

$$q = \frac{1}{6} g h^{-1} h^3 + Ch$$

$$C(x,t) = q h^{-1} - \frac{1}{6} g\, h \quad (53)$$

Therefore, by substituting the equation (53) into equation (52), the velocity profile can be rewritten as:

$$u = \frac{q}{h} - \frac{g\, h}{6}\left[1 - 3\left(\frac{z}{h}\right)^2\right] \quad (54)$$

Furthermore, we can formulate the velocity profile of the form:

$$u = M + N z^2 \quad (55)$$

$$M(x,t) = q h^{-1} - \frac{1}{6} g\, h \quad (56)$$

$$N(x,t) = \frac{1}{2} g\, h^{-1} \quad (57)$$

This velocity profile satisfies boundary condition at the center of film and $\frac{\partial u}{\partial z} = g(x,t)$ at free surface.

Differentiating equation (55) with regard to x and substitute it into equation (37), we obtain:

$$\frac{\partial u}{\partial x} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial x} z^2$$

$$\Rightarrow k^{-1} \frac{\partial w}{\partial z} + \frac{\partial M}{\partial x} + \frac{\partial N}{\partial x} z^2 = 0 \quad (58)$$

Integrating formula (58) with regard to z, we could define the value of w as follows:

$$w = -k \frac{\partial M}{\partial x} z - k \frac{1}{3} \frac{\partial N}{\partial x} z^3 + d(x,t)$$

Applying the boundary condition $w|_{z=0} = 0$, we obtain:

$$w = -k \frac{\partial M}{\partial x} z - k \frac{1}{3} \frac{\partial N}{\partial x} z^3 \quad (59)$$

Applying the equation (55), and equation (59), the shear-stress balance boundary condition at free interface as follows as:

$$g = \varepsilon^2 k \left(-k \frac{\partial^2 M}{\partial x^2} h - k \frac{1}{3} \frac{\partial^2 N}{\partial x^2} h^3\right)$$

$$-4\varepsilon^2 k^2 \frac{\partial h}{\partial x}\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial x} h^2\right) = 0$$

$$g = 4\varepsilon^2k^2 \frac{\partial M}{\partial x}\frac{\partial h}{\partial x} + 4\varepsilon^2k^2 \frac{\partial N}{\partial x} h^2 \frac{\partial h}{\partial x} + \varepsilon^2k^2 \frac{\partial^2 M}{\partial x^2} h + \frac{1}{3}\varepsilon^2k^2 \frac{\partial^2 N}{\partial x^2} h^3 \tag{60}$$

This equation represents the nonlinear shear stress equilibrium relation at the interface and the unknown function g now is correlated to h and q in the rupture process.

Now, we can rewrite equation (42), by using equation (55) as:

$$-\int_0^h \varepsilon^2 \frac{\partial p}{\partial x} dz + \varepsilon^2k^2 \frac{\partial^2 M}{\partial x^2} h + \frac{1}{3}\varepsilon^2k^2 \frac{\partial^2 N}{\partial x^2} h^3 + g = 0 \tag{61}$$

From the result of integration of equation (43), we find the value of p

$$p = \int_0^h k^2\varepsilon^2 \frac{\partial^2 w}{\partial x^2} dz + \frac{\partial w}{\partial z} + c(x,t) \tag{62}$$

From equation (59) and equation (62), we obtain:

$$p = \int_0^h k^2\varepsilon^2 \left(-k\frac{\partial^3 M}{\partial x^3} - \frac{1}{3}k\frac{\partial^3 N}{\partial x^3} z^3\right)\partial z + (-k\frac{\partial M}{\partial x} - k\frac{\partial N}{\partial x} z^2) + c(x,t)$$

$$p = -k^3\varepsilon^2 \frac{\partial^3 M}{\partial x^3} z - \frac{1}{12}k^3\varepsilon^2 \frac{\partial^3 N}{\partial x^3} z^4 - k\frac{\partial M}{\partial x} - k\frac{\partial N}{\partial x} z^2 + c(x,t) \tag{63}$$

From the boundary condition (40), we have

$$p + 2\frac{\partial u}{\partial x} + 2\frac{\partial h}{\partial x}\left[\frac{\partial u}{\partial z} + \varepsilon^2 \frac{\partial w}{\partial x}\right] = -\frac{\partial^2 h}{\partial x^2}$$

$$p = -\frac{\partial^2 h}{\partial x^2} - \frac{\partial u}{\partial x} - 2\frac{\partial h}{\partial x}\frac{\partial u}{\partial z} - 2\varepsilon^2 \frac{\partial h}{\partial x}\frac{\partial w}{\partial x} \tag{64}$$

Now, from equation (63) and (64), we have at $z = h$

$$\therefore c(x,t) = -\frac{\partial^2 h}{\partial x^2} - 2\frac{\partial u}{\partial x}|_h + k^3\varepsilon^2 \frac{\partial^3 M}{\partial x^3} h + \frac{1}{12}k^3\varepsilon^2 \frac{\partial^3 N}{\partial x^3} h^4 + k\frac{\partial M}{\partial x} + k\frac{\partial N}{\partial x} h^2 \tag{65}$$

Then, we can rewrite the value of p in equation (63)

$$p = \left(-\frac{\partial^2 h}{\partial x^2} - 2\frac{\partial u}{\partial x}\right)|_h - k^3\varepsilon^2 \frac{\partial^3 M}{\partial x^3} z - \frac{1}{12}k^3\varepsilon^2 \frac{\partial^3 N}{\partial x^3} z^4 - k\frac{\partial N}{\partial x} z^2 - k^3\varepsilon^2 \frac{\partial^3 M}{\partial x^3} h + \frac{1}{12}k^3\varepsilon^2 \frac{\partial^3 N}{\partial x^3} h^4 + k\frac{\partial N}{\partial x} h^2 \tag{66}$$

Now, by integrating equation (66) with respect to z at $|_{z=0}^h$ and substituting the equation (55) in it, we get

$$\int_0^h p dz = -\frac{\partial^2 h}{\partial x^2} h - 2\frac{\partial M}{\partial x} h - \frac{2}{3}\frac{\partial N}{\partial x} h^3 - \frac{3}{2}k^3\varepsilon^2 \frac{\partial^3 M}{\partial x^3} h^2 + \frac{1}{15}k^3\varepsilon^2 \frac{\partial^3 N}{\partial x^3} h^5 + \frac{2}{3}k\frac{\partial N}{\partial x} h^3 \tag{67}$$

From equation (61) and equation (67), we obtain:

$$k\frac{\partial^2 M}{\partial x^2} h + \frac{1}{3}k\frac{\partial^2 N}{\partial x^2} h^3 + \varepsilon^{-2}k^{-1} g = \left(-\frac{\partial^2 h}{\partial x^2} h - 2\frac{\partial M}{\partial x} h - \frac{2}{3}\frac{\partial N}{\partial x} h^3 - \frac{3}{2}k^3\varepsilon^2 \frac{\partial^3 M}{\partial x^3} h^2 + \frac{1}{15}k^3\varepsilon^2 \frac{\partial^3 N}{\partial x^3} h^5 + \frac{2}{3}k\frac{\partial N}{\partial x} h^3\right)_x \tag{68}$$

Equations (46), (60) and (68) are non-linear evolution structures of layer thickness h, local stream rate q, and the unknown function g.

5. Linear Stability-analysis

The normal form strategy may moreover be utilized to conditions (46), (60) and (68) to reveal the critical condition. Represent the overall flow as [11]:

$$h = h_0 + h'$$

$$g = g_0 + g'$$

$$q = q_0 + q' \tag{69}$$

The equilibrium states [9] of equations (46), (60) and (68) are:

$$(h_0, q_0, g_0) = \left(\frac{1}{2}, 0, 0\right) \tag{70}$$

Putting the equilibrium states into equation (46), (57), (58), (60) and (68), we get the following:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{71}$$

$$M(x,t) = 2q - \frac{1}{12} g \tag{72}$$

$$N(x,t) = f \tag{73}$$

The shear stress boundary necessity at the interface layer can be revised as

$$g = 4\varepsilon^2 k^2 \frac{\partial M}{\partial x}\frac{\partial h}{\partial x} + \varepsilon^2 k^2 \frac{\partial N}{\partial x}\frac{\partial h}{\partial x} + \frac{1}{2}\varepsilon^2 k^2 \frac{\partial^2 M}{\partial x^2} + \frac{1}{24}\varepsilon^2 k^2 \frac{\partial^2 N}{\partial x^2} \tag{74}$$

Additionally, the averaged x-momentum formula (68) has the following form:

$$\frac{1}{2}k\frac{\partial^2 M}{\partial x^2} + \frac{1}{24}k\frac{\partial^2 N}{\partial x^2} + \varepsilon^2 k^{-1} g = \frac{\partial}{\partial x}\left(\frac{-1}{2}\frac{\partial^2 h}{\partial x^2} - \frac{\partial M}{\partial x} - \frac{1}{12}\frac{\partial N}{\partial x} - \frac{3}{8}k^3\varepsilon^2\frac{\partial^3 M}{\partial x^3} + \frac{1}{480}k^3\varepsilon^2\frac{\partial^3 N}{\partial x^3} + \frac{1}{12}k\frac{\partial N}{\partial x}\right) \tag{75}$$

Hence, substituting equations (72) and (73) into equation (74) and (75), we obtain the following system:

$$g = 4\varepsilon^2 k^2 \frac{\partial}{\partial x}\left(2q - \frac{1}{12}g\right)\frac{\partial h}{\partial x} + \varepsilon^2 k^2 \frac{\partial g}{\partial x}\frac{\partial h}{\partial x} + \frac{1}{2}\varepsilon^2 k^2 \frac{\partial^2}{\partial x^2}\left(2q - \frac{1}{12}g\right) + \frac{1}{24}\varepsilon^2 k^2 \frac{\partial^2 g}{\partial x^2}$$

$$g = 8\varepsilon^2 k^2 \frac{\partial q}{\partial x}\frac{\partial h}{\partial x} + \frac{2}{3}\varepsilon^2 k^2 \frac{\partial g}{\partial x}\frac{\partial h}{\partial x} + \varepsilon^2 k^2 \frac{\partial^2 q}{\partial x^2} - \frac{1}{12}\varepsilon^2 k^2 \frac{\partial^2 g}{\partial x^2} \tag{76}$$

And

$$\frac{1}{2}k\frac{\partial}{\partial x}\left(2q - \frac{1}{12}g\right) + \frac{1}{24}k\frac{\partial^2 g}{\partial x^2} + \varepsilon^2 k^{-1} g =$$

$$\frac{\partial}{\partial x}\left(\frac{-1}{2}\frac{\partial^2 h}{\partial x^2} - \frac{\partial}{\partial x}\left(2q - \frac{1}{12}g\right) - \frac{1}{12}\frac{\partial g}{\partial x} - \frac{3}{8}\varepsilon^2 k^3 \frac{\partial^3}{\partial x^3}\left(2q - \frac{1}{12}g\right) + \frac{1}{480}\varepsilon^2 k^3 \frac{\partial^3 g}{\partial x^3} + \frac{1}{12}k\frac{\partial g}{\partial x}\right)$$

$$k\frac{\partial q}{\partial x} - \frac{1}{24}k\frac{\partial g}{\partial x} + \frac{1}{24}k\frac{\partial^2 g}{\partial x^2} + \varepsilon^2 k^{-1} g = \frac{\partial}{\partial x}\left[\frac{-1}{2}\frac{\partial^2 h}{\partial x^2} - 2\frac{\partial q}{\partial x} - \frac{3}{4}k^3\varepsilon^2\frac{\partial^3 q}{\partial x^3} - \frac{219}{480}k^3\varepsilon^2\frac{\partial^3 g}{\partial x^3} + \frac{1}{12}k\frac{\partial g}{\partial x}\right] \tag{77}$$

Thereafter, the disturbances of those three variables can be assumed to be [12,13]:

$$h' = h - \frac{1}{2}$$
$$q' = q - 0$$
$$g' = g - 0 \tag{78}$$

Substituting the disturbances into the evolution equation (71), (70) and (77), the linearized equations can be written as:

$$\frac{\partial h'}{\partial t} + \frac{\partial q'}{\partial x} = 0 \tag{79}$$

$$g' = 8\varepsilon^2 k^2 \frac{\partial q'}{\partial x}\frac{\partial h'}{\partial x} + \frac{2}{3}\varepsilon^2 k^2 \frac{\partial g'}{\partial x}\frac{\partial h'}{\partial x} + \varepsilon^2 k^2 \frac{\partial^2 q'}{\partial x^2} - \frac{1}{12}\varepsilon^2 k^2 \frac{\partial^2 g'}{\partial x^2} \tag{80}$$

and

$$k\frac{\partial q'}{\partial x} - \frac{1}{24}k\frac{\partial g'}{\partial x} + \frac{1}{24}k\frac{\partial^2 g'}{\partial x^2} + \varepsilon^2 k^{-1} g' =$$
$$\frac{\partial}{\partial x}\left[\frac{-1}{2}\frac{\partial^2 h'}{\partial x^2} - 2\frac{\partial q'}{\partial x} - \frac{3}{4}k^3\varepsilon^2\frac{\partial^3 q'}{\partial x^3} - \frac{219}{480}k^3\varepsilon^2\frac{\partial^3 g'}{\partial x^3} + \frac{1}{12}k\frac{\partial g'}{\partial x}\right] \tag{81}$$

As the nonlinear terms of equations (79)-(81) are neglected, the linearized equation is gotten and has the following form:

$$\frac{\partial h'}{\partial t} + \frac{\partial q'}{\partial x} = 0 \tag{82}$$

$$g' = \varepsilon^2 k^2 \frac{\partial^2 q'}{\partial x^2} - \frac{1}{12}\varepsilon^2 k^2 \frac{\partial^2 g'}{\partial x^2} \tag{83}$$

$$k\frac{\partial q'}{\partial x} = \frac{1}{24}k\frac{\partial g'}{\partial x} - \frac{1}{24}k\frac{\partial^2 g'}{\partial x^2} - \varepsilon^2 k^{-1} g' \tag{84}$$

Following [14], the results of those disturbances are supposed to be:

$$(h', q', g') = (H_0 Q_0 G_0)\exp(wt + i\varphi x) \tag{85}$$

Where H_0, Q_0 and G_0 denote the non-dimensional initial amplitude of the disturbances. By substituting equation (85) into equations (82)-(84), the characteristic equation can be determined to give:

$$H_0 w e^{(wt+i\varphi x)} + Q_0 i\varphi e^{(wt+i\varphi x)} = 0$$

$$H_0 w + Q_0 i\varphi = 0 \tag{86}$$

$$G_0 = -\varepsilon^2 k^2 \varphi^2 Q_0 + \frac{1}{12}\varepsilon^2 k^2 \varphi^2 G_0 \tag{87}$$

$$Q_0 w = \frac{1}{24}G_0 i\varphi + \frac{1}{24}G_0\varphi^2 - \varepsilon^2 k^{-2} G_0 \tag{88}$$

We insert equations (87) into equation (88), we can discover the taking after characteristic condition for the development rate w:

$$w = \frac{-i\varepsilon^2 k^2 \varphi^3}{24(1-\frac{1}{2}\varepsilon^2 k^2 \varphi^2)} - \frac{\varepsilon^2 k^2 \varphi^4}{24\left(1-\frac{1}{2}\varepsilon^2 k^2 \varphi^2\right)} + \frac{\varepsilon^4 \varphi^2}{1-\frac{1}{2}\varepsilon^2 k^2 \varphi^2} \tag{89}$$

Here, we discover that the thin layer is unstable in case $w > 0$ as it were, when $\varphi < \varphi_c$, where φ_c is critical 'cut-off' wave constant, and this is obviously shown in Figure (2). Moreover, for natural stable wave ($w = 0$), then φ_c as followed by

$$\varphi_c^2 \varepsilon^2 k^2 + i\varepsilon^2 k^2 \varphi_c + 24\varepsilon^4 = 0 \tag{90}$$

The greatest increase rate w_m of straight waves happens for the fastest growing wave number, φ_m which is gotten by setting $\frac{\partial w}{\partial \varphi} = 0$.

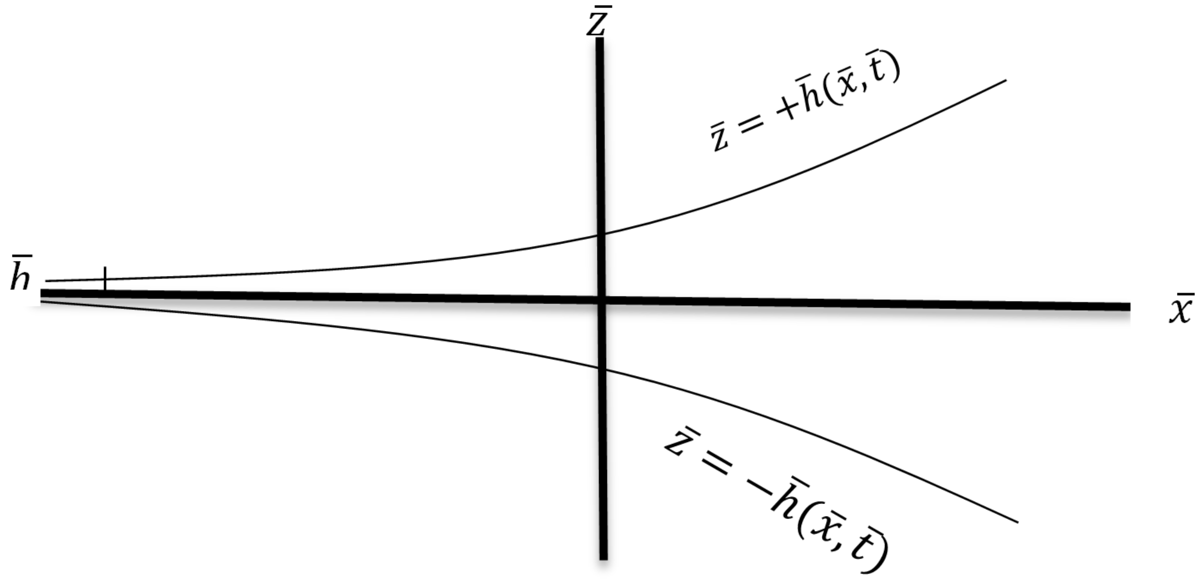

Figure 1. A horizontal, symmetrical, thin liquid layer in cross-section

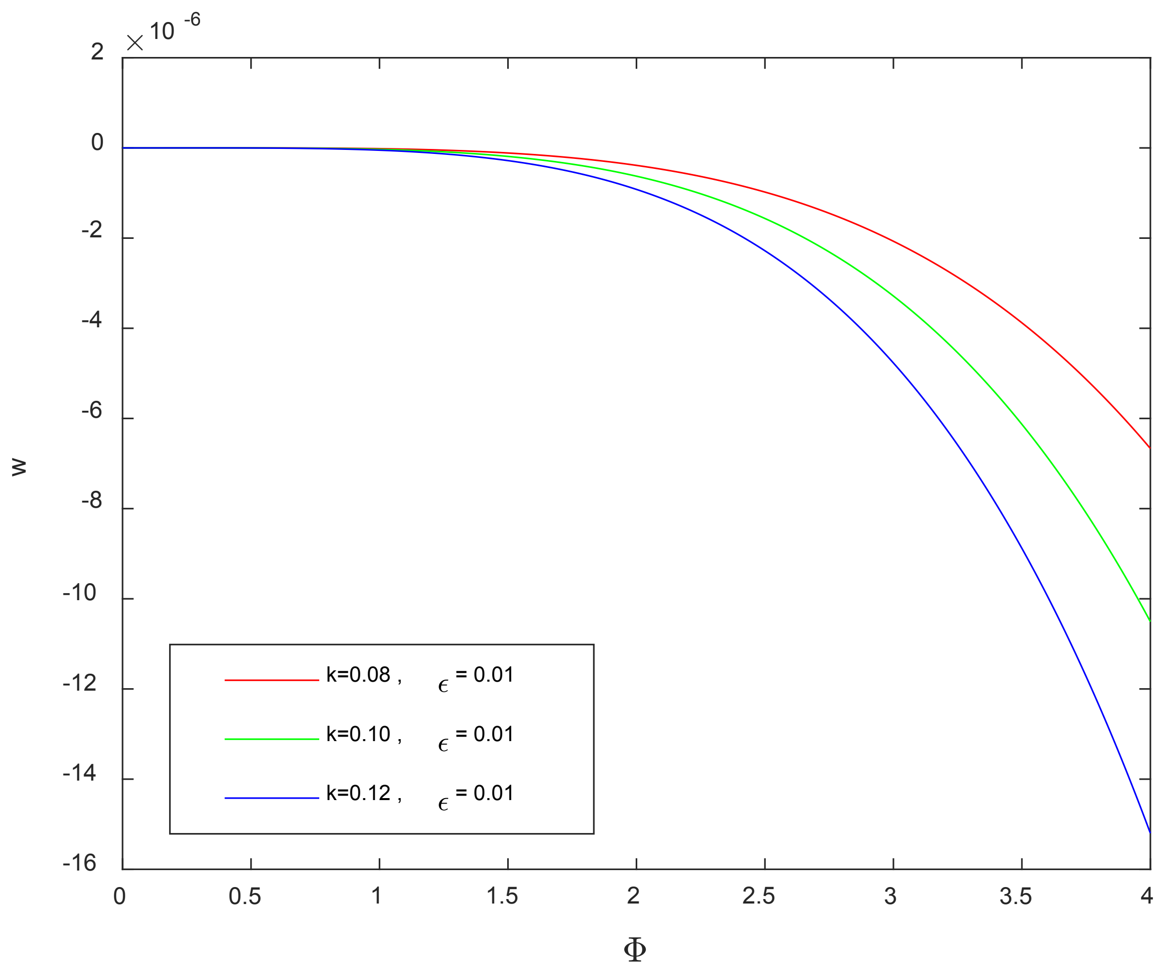

Figure 2. wave number φ vs. the increase rate w plotted from equation (89), for $\epsilon = 0.01$ and different values of k-number: $k = 0.08$, $k = 0.10$, $k = 0.12$

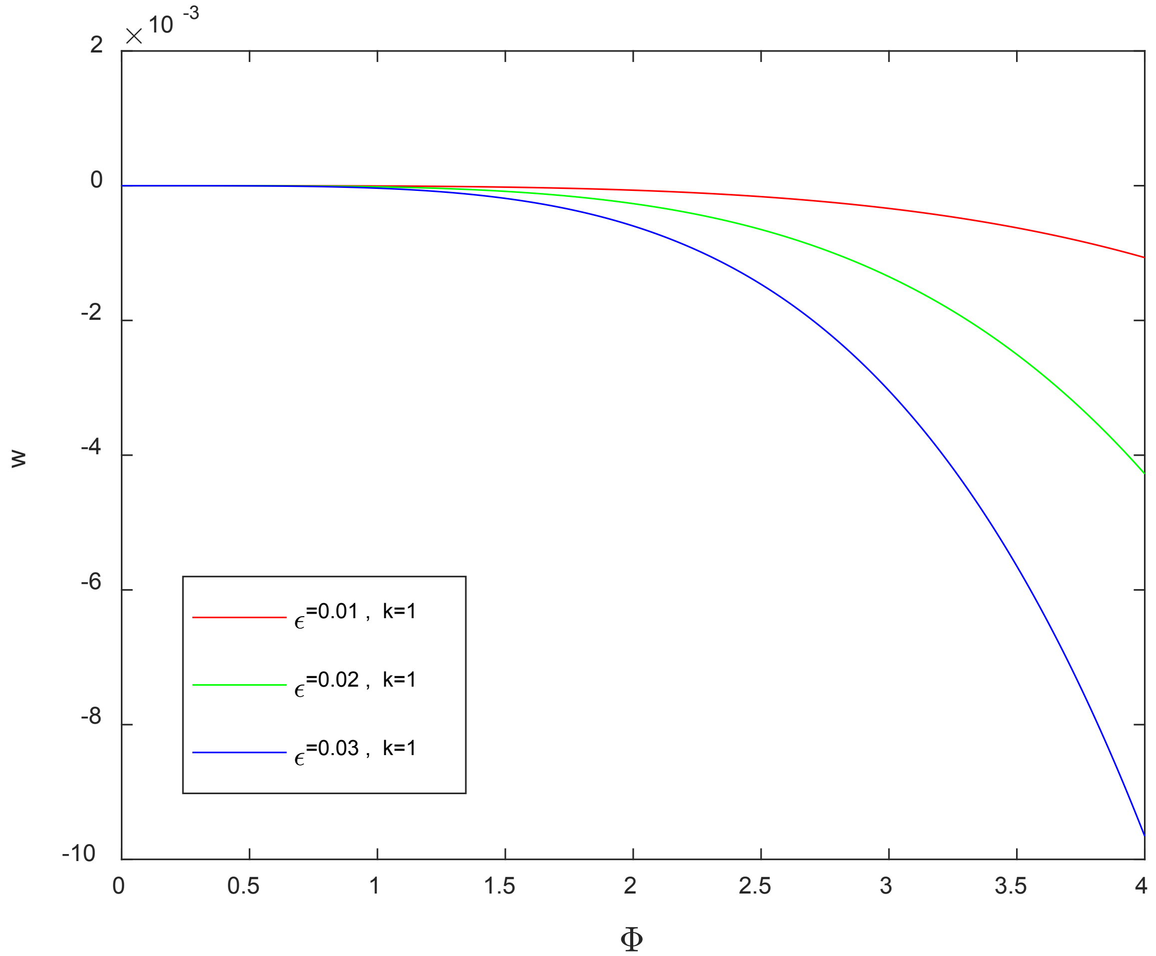

Figure 3. wave number φ vs. the increase rate w plotted from equation (89), for $k = 1$, and different values of the Capillary-number $\epsilon = 0.01,\ \epsilon = 0.02,\ \epsilon = 0.03$

6. Conclusion

The stability of thin liquid symmetric double-sided film modeled in a two-dimensional flow is constructed and it is seen from equation (89) that the film gets to be stable to short wave-perturbation in case $\varphi > \varphi_c$ and unstable to long-wave –perturbation when $\varphi < \varphi_c$. Furthermore when $= 0$, the film becomes neutrally stable wave and also the critical wave number φ_c and the maximum growth rate w_m of linear waves. Figure (2) shows that the increasing dimensionless k-number is the main effect that reduces the region of stability. In Figure (3), that represents the wave number φ vs. the increase rate w with increasing the Capillary-number. It is also the main effect that increases the region of stability.

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