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A Simple Approach for Explicit Solution of The Neutron Diffusion Kinetic System

Hind K. Al-Jeaid

Department of Mathematical Sciences, Umm Al-Qura University, Makkah, KSA

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Abstract This paper introduces a new approach to directly solve a system of two coupled partial differential equations (PDEs) subjected to physical conditions describing the diffusion kinetic problem with one delayed neutron precursor concentration in Cartesian geometry. In literature, many difficulties arise when dealing with the current model using various numerical/analytical approaches. Normally, mathematicians search for simple but effective methods to solve their physical models. This work aims to introduce a new approach to directly solve the model under investigation. The present approach suggests to transform the given PDEs to a system of linear ordinary differential equations (ODEs). The solution of this system of ODEs is obtained by a simple analytical procedure. In addition, the solution of the original system of PDEs is determined in explicit form. The main advantage of the current approach is that it avoided the use of any natural transformations such as the Laplace transform in the literature. It also gives the solution in a direct manner; hence, the massive computational work of other numerical/analytical approaches is avoided. Hence, the proposed method is effective and simpler than those previously published in the literature. Moreover, the proposed approach can be further extended and applied to solve other kinds of diffusion kinetic problems.

Keywords Partial Differential Equation, Boundary Value Problem, Neutron Diffusion Equation, Closed Solution

1 Introduction

This work considers the coupled PDEs ([1]-[2]):

$$\frac{1}{V}\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial x^2} + \left(-\sum_a + (1-\beta)\nu\sum_f\right)\phi(x,t) + \lambda C(x,t), \tag{1}$$

$$\frac{\partial C}{\partial t} = \beta\nu\sum_f \phi(x,t) - \lambda C(x,t), \tag{2}$$

under the boundary conditions (BCs):

$$\phi(0,t)=0, \qquad \phi(L,t)=0, \qquad t>0, \tag{3}$$

$$\phi(x,0)=\phi_0, \qquad 0<x<L, \tag{4}$$

$$C(x,0)=\frac{\beta\nu\sum_f}{\lambda}\phi_0, \quad 0<x<L, \tag{5}$$

where $\phi(x,t)$ and $C(x,t)$ represent the neutron flux and the delayed neutron concentration, respectively. In the system (1)-(5), the included physical quantities/parameters have been described in Refs. ([1]-[2]) and their values are standard/known for the neutron diffusion system. In addition, this system is of wide applications in particles/nuclear physics which requires accurate approximation for safety considerations. The literature [1-8] is rich of several approaches such as the General Integral Transform Technique (GITT) utilized by Ceolin et al. [1]. The authors [1] imposed artificial auxiliary parameter ϵ into the right hand side of Eq. (2) to deal with the system (1)-(5). Moreover, they considered series expansions for $\phi(x,t)$ and $C(x,t)$ in terms of eigenfunctions.

On the other hand, Khaled [2] obtained explicit expressions for $\phi(x,t)$ and $C(x,t)$ by means of the Laplace transform method (LT) with the help of the residues method. Besides, several authors ([3]-[8]) implemented different analytical and numerical methods to treating the system (1)-(5). The LT method was widely applied to solve various physical models [9-23]. However, it requires massive calculations as seen in Refs. [2], [16], and [20].

Usually, researchers search for simple but effective methods to solve their physical models. Although previous methods ([1]-[8]) were effective to analyze the system (1)-(5), the simplicity of determining the solution was missed. Hence, this work aims to develop a simple approach to deal with the current system.

The paper is constructed as follows. In section 2, a direct approach is presented. Section 3 introduces some theoretical results. Such theoretical results are then invested in section 4 to establish the desired closed-form solution. Conclusion is outlined in section 5.

2 Direct approach

Firstly, let us put the system (1)-(2) in the form:

$$\begin{aligned}\frac{\partial\phi}{\partial t}&=VD\frac{\partial^2\phi}{\partial x^2}+\omega\phi(x,t)+\lambda VC(x,t),\\ \frac{\partial C}{\partial t}&=\alpha\phi(x,t)-\lambda C(x,t),\end{aligned} \tag{6}$$

such that

$$\omega = V\left(-\sum_a + (1-\beta)\nu\sum_f\right), \quad \alpha = \beta\nu\sum_f, \tag{7}$$

and

$$C(x,0) = h\phi_0, \quad \text{where} \quad h = \frac{\beta\nu\sum_f}{\lambda} = \frac{\alpha}{\lambda}. \tag{8}$$

Express $\phi(x,t)$ and $C(x,t)$ as

$$\phi(x,t) = \sum_{n=0}^{\infty} \sin(\gamma_n x) z_1(t), \tag{9}$$

$$C(x,t) = \sum_{n=0}^{\infty} \sin(\gamma_n x) z_2(t), \tag{10}$$

where $\gamma_n = (2n+1)\frac{\pi}{L}$ and $z_1(t)$ and $z_2(t)$ are unknown functions. The assumption (9) satisfies the BCs $\phi(0,t) = 0$ and $\phi(L,t) = 0$ where $\sin(\gamma_n L) = \sin((2n+1)\pi) = 0 \; \forall \; n \in \mathbb{N}$. Based on the assumptions (9) and (10), the ICs (3) and (8) (at $t = 0$) give

$$\sum_{n=0}^{\infty} \sin(\gamma_n x) z_1(0) = \phi_0, \tag{11}$$

$$\sum_{n=0}^{\infty} \sin(\gamma_n x) z_2(0) = h\phi_0. \tag{12}$$

Applying Fourier analysis [24] on Eqs. (11) and (12) yields

$$z_1(0) = \frac{4\phi_0}{\gamma_n L}, \qquad z_2(0) = \frac{4h\phi_0}{\gamma_n L}. \tag{13}$$

Employing Eqs. (9) and (10) into Eqs. (6) implies

$$\sum_{n=0}^{\infty} \sin(\gamma_n x) z_1'(t) = -VD\sum_{n=0}^{\infty} \gamma_n^2 \sin(\gamma_n x) z_1(t) + \omega\sum_{n=0}^{\infty} \sin(\gamma_n x) z_1(t) + \lambda V\sum_{n=0}^{\infty} \sin(\gamma_n x) z_2(t), \tag{14}$$

and

$$\sum_{n=0}^{\infty} \sin(\gamma_n x) z_2'(t) = \alpha\sum_{n=0}^{\infty} \sin(\gamma_n x) z_1(t) - \lambda\sum_{n=0}^{\infty} \sin(\gamma_n x) z_2(t), \tag{15}$$

respectively. This leads to the system:

$$z_1'(t) = \left(\omega - VD\gamma_n^2\right) z_1(t) + \lambda V z_2(t), \tag{16}$$

$$z_2'(t) = \alpha z_1(t) - \lambda z_2(t), \tag{17}$$

which is a linear system of 1st-order ODEs. Such a system is easy to solve as indicated in the next section. Here, it may be noted that the successful solution of the system (16)-(17) completes the target of this paper. By this, the solutions for $z_1(t)$ and $z_2(t)$ are to be substituted into Eqs. (9) and (10) to obtain the explicit forms of $\phi(x,t)$ and $C(x,t)$.

3 Analysis

Theorem 1. *The exact solution of the system:*

$$z_1'(t) = az_1(t) + bz_2(t), \tag{18}$$

$$z_2'(t) = cz_1(t) + dz_2(t), \tag{19}$$

$$z_1(0) = \theta_1, \quad z_2(0) = \theta_2, \tag{20}$$

is

$$z_1(t) = q_1 e^{m_1 t} + q_2 e^{m_2 t}, \tag{21}$$

$$z_2(t) = \frac{q_1}{b}\left(m_1 - a\right) e^{m_1 t} + \frac{q_2}{b}\left(m_2 - a\right) e^{m_2 t}, \tag{22}$$

such that

$$\begin{aligned} q_1 &= \frac{\theta_1(m_2 - a) - b\theta_2}{m_2 - m_1}, \qquad q_2 = \frac{b\theta_2 - \theta_1(m_1 - a)}{m_2 - m_1}, \\ m_1 &= \frac{1}{2}\left(a + d + \sqrt{(a+d)^2 + 4(bc - ad)}\right), \quad m_2 = \frac{1}{2}\left(a + d - \sqrt{(a+d)^2 + 4(bc - ad)}\right), \end{aligned} \tag{23}$$

where a, b, c, d, θ_1, and θ_2 are given constants.

Proof. Differentiating (18) once with respect to t, yields

$$z_1''(t) = az_1'(t) + bz_2'(t) = az_1'(t) + bcz_1(t) + dbz_2(t), \tag{24}$$

where Eq. (19) is implemented. Again, from (18) we have $bz_2(t) = z_1'(t) - az_1(t)$, then by inserting this into (24) we get the $2nd$-order ODE:

$$z_1''(t) - (a + d)z_1'(t) - (bc - ad)z_1(t) = 0. \tag{25}$$

The solution of (25) can be easily obtained as

$$z_1(t) = q_1 e^{m_1 t} + q_2 e^{m_2 t}, \tag{26}$$

where m_1 and m_2 are distinct roots of the algebraic equation $m^2 - (a + d)m - (bc - ad) = 0$, given by

$$\begin{aligned} m_1 &= \frac{1}{2}\left(a + d + \sqrt{(a+d)^2 + 4(bc - ad)}\right), \\ m_2 &= \frac{1}{2}\left(a + d - \sqrt{(a+d)^2 + 4(bc - ad)}\right). \end{aligned} \tag{27}$$

Accordingly, $z_2(t)$ becomes

$$z_2(t) = \frac{q_1}{b}(m_1 - a)e^{m_1 t} + \frac{q_2}{b}(m_2 - a)e^{m_2 t}, \tag{28}$$

where q_1 and q_2 are unknown constants. Applying the initial conditions, it then follows

$$\begin{aligned} &q_1 + q_2 = \theta_1, \\ &\frac{q_1}{b}(m_1 - a) + \frac{q_2}{b}(m_2 - a) = \theta_2. \end{aligned} \tag{29}$$

Solving this system for q_1 and q_2, we obtain

$$q_1 = \frac{\theta_1(m_2 - a) - b\theta_2}{m_2 - m_1}, \quad q_2 = \frac{b\theta_2 - \theta_1(m_1 - a)}{m_2 - m_1}, \tag{30}$$

which completes the proof. □

Theorem 2. *The solutions $z_1(t)$ and $z_2(t)$ are given explicitly by*

$$z_1(t) = \theta_1 e^{\Delta_1 t}\left[\cosh(\Delta_2 t) + \left(\frac{a + hb - \Delta_1}{\Delta_2}\right)\sinh(\Delta_2 t)\right], \tag{31}$$

$$z_2(t) = \theta_1 e^{\Delta_1 t}\left[h\cosh(\Delta_2 t) + \left(\frac{2c + h(d - a)}{2\Delta_2}\right)\sinh(\Delta_2 t)\right], \tag{32}$$

where Δ_1, and Δ_2 are

$$\Delta_1 = \frac{1}{2}(a + d), \qquad \Delta_2 = \frac{1}{2}\sqrt{(a + d)^2 + 4(bc - ad)}. \tag{33}$$

Proof. Let m_1 and m_2 be written as

$$m_1 = \Delta_1 + \Delta_2, \qquad m_2 = \Delta_1 - \Delta_2, \tag{34}$$

where Δ_1 and Δ_2 are defined by Eqs. (33). In view of Eqs. (21) and (22), we can rewrite $z_1(t)$ and $z_2(t)$ as

$$z_1(t) = e^{\Delta_1 t}\left[(q_1 + q_2)\cosh(\Delta_2 t) + (q_1 - q_2)\sinh(\Delta_2 t)\right], \tag{35}$$

and

$$z_2(t) = \frac{e^{\Delta_1 t}}{b}\left[(q_1 m_1 + q_2 m_2 - a(q_1 + q_2))\cosh(\Delta_2 t) + (q_1 m_1 - q_2 m_2 - a(q_1 - q_2))\sinh(\Delta_2 t)\right]. \tag{36}$$

Implementing Eqs. (30) for q_1 and q_2 and Eqs. (34) for m_1 and m_2 we find that

$$\begin{aligned} q_1 + q_2 &= \theta_1, \\ q_1 - q_2 &= \frac{(a - \Delta_1)\,\theta_1 + b\theta_2}{\Delta_2}. \end{aligned} \tag{37}$$

Inserting (37) into (35) we obtain $z_1(t)$ in the form:

$$z_1(t) = e^{\Delta_1 t}\left[\theta_1\cosh(\Delta_2 t) + \left(\frac{(a - \Delta_1)\,\theta_1 + b\theta_2}{\Delta_2}\right)\sinh(\Delta_2 t)\right]. \tag{38}$$

On comparing $z_1(0)$ and $z_2(0)$ in Eqs. (13) and Eqs. (20), we observe that $\theta_2 = h\theta_1$. Hence, $z_1(t)$ becomes

$$z_1(t) = \theta_1 e^{\Delta_1 t}\left[\cosh(\Delta_2 t) + \left(\frac{a + hb - \Delta_1}{\Delta_2}\right)\sinh(\Delta_2 t)\right]. \tag{39}$$

Again, using Eqs. (30) and Eqs. (34) for q_1, q_2, m_1 and m_2 we get

$$q_1 m_1 + q_2 m_2 - a(q_1 + q_2) = b\theta_2 = hb\theta_1, \tag{40}$$

and

$$q_1m_1 - q_2m_2 - a(q_1 - q_2) = \frac{\theta_1}{\Delta_2}\left[(2a + hb)\Delta_1 - m_1m_2 - a(a + hb)\right]. \tag{41}$$

The product m_1m_2 can be evaluated from (23) and given as

$$m_1m_2 = ad - bc. \tag{42}$$

On substituting the value of $\Delta_1 = \frac{1}{2}(a + d)$, given in Eq. (33), along with the product m_1m_2 into Eq. (41), it then follows

$$q_1m_1 - q_2m_2 - a(q_1 - q_2) = \frac{b\theta_1}{2\Delta_2}\left[2c + h(d - a)\right]. \tag{43}$$

substituting the quantities in (40) and (43) into (36) and simplifying, we obtain $z_2(t)$ in the form:

$$z_2(t) = \theta_1 e^{\Delta_1 t}\left[h\cosh(\Delta_2 t) + \left(\frac{2c + h(d - a)}{2\Delta_2}\right)\sinh(\Delta_2 t)\right], \tag{44}$$

and this completes the proof. □

4 Closed-form series solution

As indicated in section 2, the solution of the present problem depends on the solution of the system (16)-(17). To find such solution, we just compare between the system (16)-(17) and the system (18)-(19) to assign the values of a, b, c, and d. By this, we have

$$a = \omega - VD\gamma_n^2, \quad b = \lambda V, \quad c = \alpha, \quad d = -\lambda. \tag{45}$$

Also, the quantities θ_1 and θ_2 are determined by comparing the ICs in (13) and (20), this gives

$$\theta_1 = \frac{4\phi_0}{\gamma_n L}, \qquad \theta_2 = \frac{4h\phi_0}{\gamma_n L}, \quad \text{where} \quad \theta_2 = h\theta_1. \tag{46}$$

Using the above values we find

$$\Delta_1 = \frac{1}{2}(a - \lambda), \tag{47}$$

$$\Delta_2 = \frac{1}{2}\sqrt{(\lambda + a)^2 + 4\alpha\lambda V}, \tag{48}$$

$$\frac{a + hb - \Delta_1}{\Delta_2} = \frac{2\alpha V + \lambda + a}{\sqrt{(\lambda + a)^2 + 4\alpha\lambda V}}. \tag{49}$$

Accordingly, we obtain $z_1(t)$ and $z_2(t)$ in the following final form

$$\begin{aligned} z_1(t) =& \frac{4\phi_0}{\gamma_n L} e^{\frac{1}{2}(a-\lambda)t} \times \\ & \left[\cosh\left(\frac{1}{2}\sqrt{(\lambda + a)^2 + 4\alpha\lambda V t}\right) + \frac{2\alpha V + \lambda + a}{\sqrt{(\lambda + a)^2 + 4\alpha\lambda V}}\sinh\left(\frac{1}{2}\sqrt{(\lambda + a)^2 + 4\alpha\lambda V t}\right)\right], \end{aligned} \tag{50}$$

and

$$\begin{aligned} z_2(t) =&\frac{4h\phi_0}{\gamma_n L} e^{\frac{1}{2}(a-\lambda)t} \times \\ &\left[\cosh\left(\frac{1}{2}\sqrt{(\lambda+a)^2+4\alpha\lambda V t}\right) + \frac{\lambda - a}{\sqrt{(\lambda+a)^2+4\alpha\lambda V}} \sinh\left(\frac{1}{2}\sqrt{(\lambda+a)^2+4\alpha\lambda V t}\right)\right]. \end{aligned} \tag{51}$$

Therefore, in view of Eqs. (9) and (10) we obtain the following solutions for $\phi(x,t)$ and $C(x,t)$

$$\begin{aligned} \phi(x,t) =&4\phi_0 e^{-\frac{1}{2}\lambda t} \sum_{n=0}^{\infty} \frac{\sin(\gamma_n x)}{\gamma_n L} e^{\frac{1}{2}at} \times \\ &\left[\cosh\left(\frac{1}{2}\sqrt{(\lambda+a)^2+4\alpha\lambda V t}\right) + \frac{2\alpha V + \lambda + a}{\sqrt{(\lambda+a)^2+4\alpha\lambda V}} \sinh\left(\frac{1}{2}\sqrt{(\lambda+a)^2+4\alpha\lambda V t}\right)\right], \end{aligned} \tag{52}$$

and

$$\begin{aligned} C(x,t) =&4h\phi_0 e^{-\frac{1}{2}\lambda t} \sum_{n=0}^{\infty} \frac{\sin(\gamma_n x)}{\gamma_n L} e^{\frac{1}{2}at} \times \\ &\left[\cosh\left(\frac{1}{2}\sqrt{(\lambda+a)^2+4\alpha\lambda V t}\right) + \frac{\lambda - a}{\sqrt{(\lambda+a)^2+4\alpha\lambda V}} \sinh\left(\frac{1}{2}\sqrt{(\lambda+a)^2+4\alpha\lambda V t}\right)\right], \end{aligned} \tag{53}$$

respectively. It can be shown that the present solution for the system (1)-(5) is in full agreement with the corresponding one in Ref. [2] which can be accomplished via additional simplifications for the involved quantities in Ref. [2].

5 Advantage of the proposed method

The main advantages of the proposed approach can be summarized as follows

- It transforms the given PDEs to a system of linear ordinary differential equations (ODEs).
- It facilitates the way to get the solution of the transformed system of ODEs through a simple analytical procedure.
- It avoids the use of any natural transformations such as the LT [2].
- It gives the solution in a direct manner, hence, the complex computational work of other approaches can be avoided.
- It is a straightforward approach and can be further extended/applied to solve other kinds of diffusion kinetic problems.
- It obtains the analytic solution for the current system which is optimal if compared with computer-oriented numerical methods.

6 Conclusion

A system of two coupled partial differential equations (PDEs) is analyzed in this paper. These PDEs under the present physical conditions explain the diffusion kinetic problem with one delayed neutron precursor concentration in Cartesian geometry. A new direct approach was developed in this work and accordingly the closed-form solution was determined. The current method overcome the difficulties in literature via avoiding the complexity of the LT-method [2]. In comparison with numerical/analytical approaches in the relevant literature, our approach is much simpler and direct. Similar diffusion kinetic problems can be treated in future utilizing the current method.

Declarations

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Competing interests: The author declares that there is no competing interests.

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