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Central Automorphisms in n-abelian Groups

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Abstract The study of $\mathrm{Aut}(G)$, the group of automorphisms of G, has been undertaken by various authors. One way to facilitate this study is to investigate the structure of $\mathrm{Aut}_c(G)$, the subgroup of central automorphisms. For some classes of groups, algebraic properties like solvability, nilpotency, abelian and nilpotency relative to an automorphism can be deduced through the study of the subgroups $\mathrm{Aut}_c(G)$ and $\mathrm{Aut}_{c^*}(G)$ where $\mathrm{Aut}_{c^*}(G)$ is the group of central automorphisms that fix $Z(G)$ point-wise. For instance, [6], if $\mathrm{Aut}_c(G) = \mathrm{Aut}(G)$ then G is nilpotent of class 2 and if G is f-nilpotent for $f \in \mathrm{Aut}_{c^*}(G)$, then for a group G, the notions of relative nilpotency and nilpotency coincide [8]. The group is abelian if G is identity nilpotent only [8]. For an arbitrary group G, the subgroups $\mathrm{Aut}_c(G)$ and $\mathrm{Aut}_{c^*}(G)$ are trivial, but for the case when G is a p-group, $\mathrm{Aut}_c(G)$ is non-trivial and the structure of $\mathrm{Aut}_{c^*}(G)$ have been described [4]. The study of the influence of types of subgroups on the structure of G is a powerful technique, thus, one can investigate the influence of maximal invariant subgroups of G on the structure of $\mathrm{Aut}_{c^*}(G)$. We shall consider a class of finite, non-commutative, n-abelian groups that are not necessarily p-groups. Here, $n = 2l + 1$ is a positive integer and l is an odd integer. The purpose of this paper is to explicitly describe the central automorphisms of $G = G_l$ that fix the center element-wise and consequently the algebraic structure of $\mathrm{Aut}_{c^*}(G)$. For this goal, we will study the invariant normal subgroups M of G such that $G' \subseteq M$ and M is maximal in G. It suffices to study $\mathrm{Hom}(G/M, Z(G))$, the group of homomorphisms from the quotient G/M to the center $Z(G)$. We explore the central automorphism group of pullbacks involving groups of the form G_l. We extend our study to central automorphisms in this class of groups G_l, in which the mapping $x \mapsto x^n$ is an automorphism. For such groups, $\mathrm{Aut}_{c^*}(G)$ can be described through $\mathrm{Hom}(G/M, Z(G))$, where M is normal and a maximal subgroup in G such that the quotient group G/M is abelian. We show that $\mathrm{Hom}(G/M, Z(G)) \cong \mathrm{Aut}_{c^*}(G)$ and $\mathrm{Aut}_{c^*}(G)$ is isomorphic to the cyclic group of order a prime p. The class of groups studied in our paper falls under a bigger class of groups which have a special characterization that their non normal subgroups are contranormal. The results of this paper can be generalized to this bigger class of groups.

Keywords Central Automorphism, n-abelian, Pullback, Group Action, Maximal Subgroup

1 Introduction

In this paper, the groups will be finite and the center of a group G will be denoted by $Z(G)$. An automorphism f of G is called a central automorphism if $x^{-1}f(x) \in Z(G)$ for all $x \in G$. Let $\mathrm{Aut}_c(G)$ be the group of central automorphisms of G. We will denote by $\mathrm{Aut}_{c^*}(G)$, the group of central automorphisms of G that fix the center element-wise. The study of central automorphisms of a group G facilitates the study of the group of automorphisms, $\mathrm{Aut}(G)$.
Many authors have studied the group of central automorphisms, see, for example, [4], [7], [9], [11]. The central automorphisms that fix the center of G point-wise have been studied in [3] for G a finite p-group. Non-abelian p-groups in which $\mathrm{Aut}_c(G) = \mathrm{Aut}(G)$ have been studied by Curran and McCaughan, see [7].

A group G that has no non trivial abelian factor is called purely non-abelian. Jafari and Jamali, in their paper, [10], have investigated the group structure of central automorphisms for $G = K \times H$, where K is purely non-abelian and H an abelian subgroup of G. Adney and Yen [1], have shown that, for a purely non abelian group G, $|\mathrm{Aut}_c(G)| = |\mathrm{Hom}(G/G', Z(G)|$ where G' is the derived subgroup of G. In their paper [12], Mousavi and Shomali, have gone on to show that $|\mathrm{Aut}_c(G)| = |\mathrm{Hom}(G), Z(G)|$. The authors gave the group of central automomorphisms of G, where $G = K \rtimes H$, the semidirect product of K and H and $(|H|, |K|) = 1$.

In the quest to explicitly describe the algebraic structure of the group $\mathrm{Aut}_c(G)$, one would need to study the invariant normal subgroups M such that $G' \subseteq M$ and M is maximal in G. It suffices to study the homomorphisms in the group $\mathrm{Hom}(G/M, Z(G))$.

Let $n = 2l + 1$ be a positive integer, where l is an odd integer and let G_l be a non-abelian group that is n-abelian and G not necessarily a p- group. We describe the central automorphism maps in $\mathrm{Aut}_c(G_l)$ and explicitly describe $\mathrm{Aut}_{c^*}(G_l)$. For groups G such that the map $x \mapsto x^n$ is an automorphism, we study the group of homomorphisms from the quotient G/M onto the center $Z(G)$ where, M is normal in G and the subgroup M is maximal with respect to the characterization that G/M is abelian.

In section **2**, we fix notation and in section **3** we recall the Adney-Yen map and adapt it to describe the group $\mathrm{Hom}(G/M, Z(G))$.

2 Materials and Methods

Definition 1. *Let G be a group and let $f \in \mathrm{Aut}(G)$. Then, f is a central automorphism if f commutes with all inner automorphisms of G. Equivalently, if $x^{-1}f(x) \in Z(G),\ \forall x \in G$.*

Definition 2. *[5] A group G is called n-abelian if $(xy)^n = x^n y^n$ for all $x,\ y \in G$ and some fixed integer n.*

We give a proposition that summarizes some properties that hold in groups in which maps of the form $x \mapsto x^n;\ n \geq 1$ are automorphisms. We will use the convention $[x, y] = x^{-1}y^{-1}xy$ for all x and y in G.

Proposition 3. *Let G be a group. If the map $f_n : x \mapsto x^n,\ n \geq 1$ is an automorphism of G, then $x^{n-1} \in Z(G)\ \forall x \in G$.*

Proof
Since G is n-abelian, then, for any $x, y \in G$, $(xy)^n = x^n y^n$, also, $(xy)^{n-1} = y^{n-1}x^{n-1}$. Now, $G' \cap \mathrm{Ker}(f_n) = \{1\}$ where G' is the derived subgroup of G, so, by [2], G is $n-1$ abelian. Then, $x^{n-1}y^{n-1} = (xy)^{n-1} = y^{n-1}x^{n-1}$ and from the fact that $[x^{n-1}, y^n] = 1$, we have, $[x^{n-1}, y] = x^{1-n}y^{-1}x^{n-1}y = x^{1-n}y^{-1}(y^n x^{n-1} y^{-n})y$.
Since, $x^{n-1}y^{n-1} = y^{n-1}x^{n-1}$, then, $[x^{n-1}, y] = x^{1-n}y^{n-1}x^{n-1}y^{1-n} = 1$ and $x^{n-1} \in Z(G)$.

2.1 Central automorphisms in some class of n-abelian groups

Proposition 4. *Let $n \geq 1$ be a positive integer and let G be a group such that the map $f_n : x \mapsto x^n$ is an automorphism of G. Then there are exact sequences*

(i) $1 \longrightarrow Z(G) \longrightarrow G \longrightarrow G/Z(G) \longrightarrow 1$, where the map f_n is identity in $G/Z(G)$.

(ii) $1 \longrightarrow M \longrightarrow G \longrightarrow A \longrightarrow 1$, where the map f_n is identity in M, A is an abelian subgroup and the subgroup M is maximal with respect to this characterization.

Proof

(i) The map $f_n : x \mapsto x^n$ is an automorphism so, we have that, $x^{n-1} \in Z(G)$ for all $x \in G$. Then, $\overline{x}^{\,n-1} = \overline{1}$ for all $\overline{x} \in G/Z(G)$ and $\overline{x}^{\,n} = \overline{x}^{n-1}\,\overline{x} = \overline{x}$. Thus, f_n is identity in $G/Z(G)$.

(ii) The map $x \mapsto x^n$ is an endomorphism and G is n-abelian. Let $M = \{x \in G \,|\, x^n = x\}$. Clearly, M is normal in G. Now, G is $n-1$-abelian so $[x, y]^{n-1} = [x^{n-1}, y^{n-1}] = 1$ and $x^{n-1} \in Z(G)$ for all $x \in G$. Then, $[x, y]^n = [x, y]^{n-1}[x, y] = [x, y]$. Therefore, $G' \subseteq M$ and $A = G/M$ is abelian.

We will consider the following class of groups:
Let l be a positive odd integer and let G_l be a group given by

$$G_l = \langle x, y \,|\, x^l = y^2 = (xy)^2,\ x^{2l} = 1\rangle.$$

Note that, $y^{2l} = y^{2(2r+1)} = y^2$, for some $r \in \mathbb{Z}^+$ since l is an odd integer.

Lemma 5. *The group G_l is n-abelian, where $n = 2l + 1$.*

Proof
Let $n = 2l + 1$ where l is an odd positive integer. Then, $x^n = x^{2l+1} = x^{2l}x = x$ and $y^n = y^{2l+1} = y^3 = y^{-1}$. Also, $(xy)^n = (xy)^{2l}xy = y^{2l}xy = y^2xy = xyxyxy = xy^3 = xy^{-1}$.

Lemma 6. *The map $f_n : G_l \longrightarrow G_l\,;\ f(g) = g^n$ where $n = 2l + 1$ is an automorphism of G_l.*

Proof
On one hand, $x^n = x^{2l+1} = x^{2l}x = x$ and $y^n = y^{-1}$ and on the other hand, $(xy)^n = xy^{-1}$. Therefore, $f_n(xy) = f_n(x)f_n(y)$.
The map f_n is a group homomorphism defined by $x \mapsto x,\ y \mapsto y^{-1}$. Thus, f_n is one-to one and onto.

Lemma 7. *The center is given by $Z(G_l) = \{1, y^2\}$, and the map $\sigma : G_l \longrightarrow Z(G_l)$ given by $\sigma(g) = g^{2l}$ is a surjective group homomorphism.*

Proof
By Proposition 3, $g^{2l} \in Z(G_l)\ \forall g \in G_l$. Now, $y^{2l} = y^{2(2r+1)} = y^2$, $x^{2l} = 1$ and $y^2x = x^lx = xx^l = xy^2$. Also, $\sigma(xy) = (xy)^{2l} = ((xy)^2)^l = y^{2l} = y^{2(2r+1)} = y^2$. Thus, $\sigma(xy) = \sigma(x)\sigma(y)$ and σ is a surjective group homomorphism.

Lemma 8. *The map f is a central automorphism that fixes $Z(G_l)$ element-wise.*

Proof

Clearly, $1 = x^{-1}f_n(x) \in Z(G_l)$ and $y^{-1}f_n(y) = y^{-1}y^{-1} = y^2 \in Z(G_l)$.

Theorem 9. *Let l be a positive odd integer and let $n = 2l + 1$. Consider the group $G_l = \langle x, y \,|\, x^l = y^2 = (xy)^2\rangle$ and let $f_n \in \mathrm{Aut}_c(G_l)$. Let L be an arbitrary group, $f : G_l \longrightarrow L$ a*

group homomorphism, and let $\omega : L \longrightarrow \mathrm{Aut}(K)$ *be an action of the group* L *on a given group* K.
Assume that f_n *is identity in* $K \rtimes L$. *If* $f_n \in \mathrm{Aut}_{c^*}(K \rtimes G_l)$ *then* $f_n \in \mathrm{Aut}_{c^*}(G_l)$.

Proof
The group G_l acts on the group K through the action $\omega f : G_l \longrightarrow \mathrm{Aut}(K)$ and the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & K \rtimes G_l & \xrightarrow{p_1} & G_l & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \psi & & \downarrow f & & \\ 1 & \longrightarrow & K & \longrightarrow & K \rtimes L & \xrightarrow{p_2} & L & \longrightarrow & 1. \end{array}$$

The right square is a pullback. Define the pullback B as follows: $B = \{(u,(k,v)) \in G_l \times (K \rtimes L)\,;\, f(u) = v\}$. The homomorphisms ψ and the projection p_1 induce a homomorphism $\tau : K \rtimes G_l \longrightarrow B$ defined by $\tau(k,u) = (u, \psi(k,u)) = (u,(k,f(u)))$. We can define a morphism $\gamma : B \longrightarrow K \rtimes G_l$; $\gamma(u,(k,f(u))) = (k,u)$ and $\tau\gamma = \mathrm{I}_B$, $\gamma\tau = \mathrm{I}_{K \rtimes G_l}$. We have that τ is a monomorphism and an epimorphism. Suppose that f_n is identity in $K \rtimes L$.
Then $\tau(f_n(k,u)) = \tau((k,u)^n) = (\tau(k,u))^n = (u,(k,f(u)))^n = (u^n,(k,f(u))^n) = (u^n,(k,f(u))) = \tau(k,u^n)$. Hence, $(k,u)^n = (k,u^n)$ since τ is a monomorphism. By the definition of f_n on $K \rtimes G_l$, that (k,u) is mapped to (k,u^n) for all $k \in K$ and all $u \in G_l$, we conclude that if $f_n \in \mathrm{Aut}_{c^*}(K \rtimes G_l)$ then $f_n \in \mathrm{Aut}_{c^*}(G_l)$.

Corollary 10. *Let* H *be an abelian group and let* N *be a finite group. Let* $\omega : H \longrightarrow \mathrm{Aut}(N)$ *be an action of* H *on* N. *If* $f \in \mathrm{Aut}_{c^*}(N \rtimes H)$ *then,* $f \in \mathrm{Aut}_{c^*}(H)$. *In particular, if* $Z(H)$ *acts trivially on N, then,* $\mathrm{Aut}_{c^*}(N \rtimes H) = \mathrm{Aut}_{c^*}(H)$.

Proof

The map f is identity in $N \rtimes \mathrm{Ker}(\omega)$. Apply Theorem 9 to

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & N \rtimes H & \xrightarrow{p_1} & H & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \psi & & \downarrow \sigma & & \\ 1 & \longrightarrow & N & \longrightarrow & N \rtimes \mathrm{Ker}(\omega) & \xrightarrow{p_2} & \mathrm{Ker}(\omega) & \longrightarrow & 1. \end{array}$$

If $Z(H)$ acts trivially on N then, $Z(H) \leq Z(G)$ where, $G = N \rtimes H$ and, for any $f \in \mathrm{Aut}_{c^*}(G)$, $f_H \in \mathrm{Aut}_{c^*}(H)$. Then, $\mathrm{Aut}_{c^*}(G) = \mathrm{Aut}_{c^*}(H) \times \mathrm{Aut}_{c^*}(N) \cong \mathrm{Aut}_{c^*}(H)$ since f is identity in N.

A non-abelian group G is called purely non-abelian if G has no non-trivial abelian direct factors.

We recall the Adney-Yen map [1]. If $f \in \mathrm{Aut}_c(G)$, define a map $\varphi : \mathrm{Aut}_c(G) \longrightarrow \mathrm{Hom}(G, Z(G))$ by $\varphi(f) = \sigma_f$, where $\sigma_f(g) = g^{-1}f(g)$ for all $g \in G$. The authors, in [1], proved the following theorem

Theorem 11. *Let* $\varphi : \mathrm{Aut}_c(G) \longrightarrow \mathrm{Hom}(G, Z(G))$. *Then*

(i) φ *is always one-to-one.*

(ii) φ *is onto if and only if G is purely non-abelian.*

Proposition 12. *Let* $G = N \rtimes H$ *be a purely non-abelian group, where* H *is finite and abelian,* N *a finite group and* $\omega : H \longrightarrow \mathrm{Aut}(N)\,;\, \omega(h)(x) = hxh^{-1}$ *is the action of* H *on* N. *Let* $f_n \in \mathrm{Aut}(G)$ *be a map defined by* $f_n : x \mapsto x^n$.
If r *is the cardinality of* $N \rtimes H/\mathrm{Ker}(\omega)$ *then* $f_n \in \mathrm{Aut}(G)$ *where* $f_n : g \mapsto g^n$ *with* $n = r + 1$.

Proof
If r is the order of $N \rtimes H/\mathrm{Ker}(\omega)$, then $g^r \in \mathrm{Ker}(\omega)$ for all $g \in G$ and $f_r \in \mathrm{Hom}(G, Z(G))$, where $f_r : g \mapsto g^r$. The Adney-Yen map is onto, therefore, $f_r = \varphi(f_n) = \sigma_{f_n}$ for some $f_n \in \mathrm{Aut}_c(G)$. Then, $f_r(g) = \sigma_{f_n}(g) \Leftrightarrow g^r = g^{-1}f_n(g) \Leftrightarrow f_n(g) = g^{r+1}$.
Note that, for $n = r + 1$, the map $f_n \in \mathrm{Aut}_c(G)$ and f_n moves $N \rtimes H/\mathrm{Ker}(\omega)$.

Theorem 13. *Let G be a group and let* $f_n : G \longrightarrow G\,;\, g \mapsto g^n$ *be a central automorphism that fixes the center element-wise. Then* $\mathrm{Aut}_{c^*}(G) \cong \mathrm{Hom}(G/M, Z(G))$ *where* $M = \{g \in G \,|\, g^n = g\}$ *is a maximal subgroup of G such that* G/M *is abelian.*

Proof
There is an exact sequence $1 \longrightarrow M \longrightarrow G \longrightarrow A \longrightarrow 1$, where the map f_n is identity in M, and the subgroup M is a maximal subgroup such that $G/M \cong A$ is abelian. We have the extension G of G' by H where $G/G' \cong H$ is abelian. Now, $G' \cap H = \{1\}$. Similarly, there is an extension G of M by A where $G/M \cong A$ is abelian, $A \leq H$ and $G' \cap A = \{1\}$. The automorphism f_n fixes G' and $Z(G)$ point-wise. By Proposition 4 (ii), $G' \subseteq M$ and $M = G'Z(G)$. Also, by maximality of M and it follows that $Z(G) \subseteq M$ with $G' \cap Z(G) = \{1\}$. Define a map $\overline{\varphi} : \mathrm{Aut}_{c^*}(G) \longrightarrow \mathrm{Hom}(G/M, Z(G))$ by $\overline{\varphi}(f_n) = \sigma_{\overline{f_n}}$, where $\sigma_{\overline{f_n}}(gM) = \sigma_{f_n}(g) = g^{-1}f_n(g)$. $\overline{\varphi}$ is a well defined homomorphism. If $f_{n_1} = f_{n_2}$, then $g^{-1}f_{n_1}(g) = g^{-1}f_{n_2}(g)$ for all $f_{n_1}, f_{n_1} \in \mathrm{Aut}_{c^*}(G)$ and all $g \in G$. $\overline{\varphi}$ is a group homomorphism since $\sigma_{f_{n_1}f_{n_2}}(g) = g^{-1}f_{n_1}(f_{n_2}(g)) = g^{-1}f_{n_1}(gg^{-1}f_{n_2}(g)) = g^{-1}f_{n_1}(g)g^{-1}f_{n_2}(g) = \sigma_{f_{n_1}}(g)\sigma_{f_{n_2}}(g)$ and the result follows. It is easy to see that $\overline{\varphi}$ is one-to-one. We show that $\overline{\varphi}$ is onto.
For any $\delta \in \mathrm{Hom}(G/M, Z(G))$, define $(f_n)_\delta : G \longrightarrow G$ by $(f_n)_\delta(g) = g\delta(\overline{g})$ for all $g \in G$, where $\overline{g} = gM$. We have, $(f_n)_\delta(g_1g_2) = g_1g_2\delta(\overline{g_1g_2}) = g_1g_2\delta(\overline{g_1})\delta(\overline{g_2}) = (f_n)_\delta(g_1)(f_n)_\delta(g_2)$ since $\delta(\overline{g_1}) \in Z(G)$. Hence, $(f_n)_\delta$ is an endomorphism of G. If $g \in \mathrm{Ker}((f_n)_\delta)$, then $1 = (f_n)_\delta(g) = g\delta(\overline{g}) \Leftrightarrow g = \delta(\overline{g})^{-1} \in Z(G)$. Therefore, $(f_n)_\delta$ is one-to-one. Then, $\overline{g} = 1$ and $1 = (f_n)_\delta(g) = g\delta(\overline{g}) = g$. $(f_n)_\delta$ is a central automorphism, since, for any $g, x \in G$, $\tau_x(f_n)_\delta(g) = x(g\delta(\overline{g}))x^{-1} = xgx^{-1}\delta(\overline{g}) = xgx^{-1}\delta(\overline{xgx^{-1}}) = (f_n)_\delta\tau_x(g)$. $(f_n)_\delta$ fixes the center element-wise, since, for all $z \in Z(G)$, we have, $(f_n)_\delta(z) = z\delta(\overline{z}) = z$. Finally, $\overline{\varphi}((f_n)_\delta) = \delta_{\overline{f_n}}$.

Theorem 14. *Let G be a group and let* $f_n : G \longrightarrow G\,;\, x \mapsto x^n$ *be a central automorphism of G that fixes the center element-wise. Then* $\mathrm{Aut}_{c^*}(G)$ *is cyclic and its order is a prime number* p.

Proof
From Proposition 4, $M = \{g \in G \mid g^n = g\}$ is a normal subgroup. Also, M is a maximal subgroup such that G/M is abelian. The group G/M is a cyclic of prime order p, thus, G/M is a simple group and any homomorphism from G/M onto $Z(G)$ is either trivial or injective. Now, from Theorem 13, $\mathrm{Aut}_{c^*}(G) \cong \mathrm{Hom}(G/M, Z(G)) \cong \mathbb{Z}_p$.

We will illustrate the result in Theorem 14 through the following example:

Example 15. *Consider the group* $G_l = \langle x, y \mid x^l = y^2 = (xy)^2 \rangle$, *where* l *is an odd positive integer. Put* $z = xy^2$, *then,* $G_m = \langle x, y \mid z^l = y^4 = 1,\ yzy^{-1} = z^{-1} \rangle$, $G_l' = \langle x^2 \rangle$, $Z(G_l) = \langle y^2 \rangle = \{1, y^2\}$, $H = \langle y \rangle$ *and* $G_l \cong G_l' \rtimes H$ *with the action* $\omega : H \longrightarrow \mathrm{Aut}(G_l')$ *given by* $\omega(y)(z) = yzy^{-1} = z^{-1}$.
Let $l = 3$ *then,* $n = 7$ *and* $f_n : g \mapsto g^n = g^7$ *is a central automorphism of* G_l, *given by* $x \mapsto x,\ y \mapsto y^{-1}$. *The automorphism* f_n *fixes* $Z(G_l)$ *element-wise. Note that* $G_l' \cap Z(G_l) = \{1\}$. $M = G_l'Z(G_l) = \{1, y^2, x^2, x^4, x^2y^2, x^4y^2\} = \langle x \rangle$ ($x^4y^2 = x,\ y^2 = x^3,\ x^2y^2 = x^5$). $G_l - M = \{y, xy, x^2y, xy^3, x^2y^3, y^3\}$.
For $f_n \in \mathrm{Aut}_{c^*}(G_l)$, $\sigma_{\overline{f_n}} \in \mathrm{Hom}(G_l/M, Z(G_l))$. *Now,* $\sigma_{\overline{f_n}}(\overline{g}) = \sigma_{f_n}(g) = g^{-1}f_n(g) = 1$ *for all* g *in* M *and* $\sigma_{\overline{f_n}}(\overline{g}) = \sigma_{f_n}(g) = g^{-1}f_n(g) = y^2$ *for all* g *in* $G_l - M$. *We have that* $\mathrm{Aut}_{c^*}(G_l) \cong \mathrm{Hom}(G_l/M, Z(G_l)) \cong \mathbb{Z}_2$.

3 Conclusions

The study and explicit description of the group $\mathrm{Aut}_{c^*}(G)$ is of importance to the study of nilpotency of a group relative to an automorphism. It has been shown that, if f is a central automorphism that fixes the center of G point-wise, then for the group G, the notion f-nilpotent and nilpotent are equivalent. Let G be an n-abelian group in which the map $f : x \mapsto x^n$ is an automorphism. Can one use the information on the structure of $\mathrm{Aut}_{c^*}(G)$ to study the f-nilpotence of the group G?

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(2) Informed Consent Statement: Not applicable.

(3) Data Availability Statement: Not applicable.

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