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Some Fixed Point Results in Bicomplex Valued Metric Spaces

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Abstract Fixed points are also called as invariant points. Invariant point theorems are very essential tools in solving problems arising in different branches of mathematical analysis. In the present paper, we establish three unique common invariant point theorems using two self-mappings, four self-mappings and six self-mappings in the bicomplex valued metric space. In the first theorem, we generate a common invariant point theorem for four self-mappings by using weaker conditions such as weakly compatible, generalized contraction and (CLR_{AB}) property. Then, in the second theorem, we generate a common invariant point theorem for six self-mappings by using inclusion relation, generalized contraction, weakly compatible and commuting maps. Further, in the third theorem, we generate a common coupled invariant point for two self mappings using different contractions in the bicomplex valued metric space. The above results are the extention and generalization of the results of [11] in the Bicomplex metric space. Moreover, we provide an example which supports the results.

Keywords Bicomplex Valued Metric Space, Common Fixed Point, Coupled Fixed Point, CLR Property, Weakly Compatible Mappings

1 Introduction

The concepts of bicomplex numbers and tricomplex numbers were introduced in the year 1892 by Segre[1]. Complex valued metric spaces are introduced by Azam et al.[2], in the year 2011 and some results were studied for such spaces. Very recently, the bicomplex valued metric space was introduced by Cho et al.[5] and some fixed point results were obtained. In the year 2019, Jebril, Datta, Sarkar and Biswas [6] derived some fixed point outcomes using rational contractions in bicomplex valued metric space.

Imdad et al.[8] introduced a new notion, called CLR-property for self maps in 2012. Afterwards, by using it several mathematicians obtained some fixed point results ([3],[4],[9] and [10]). The main purpose of this work is to prove some invariant point outcomes using various contractions for four self mappings, six self mappings and coupled invariant point theorems using weakly compatibility, CLR_{AB} property and commuting maps in bicomplex valued metric spaces.

2 Preliminaries

We denote C_0 = R(Real numbers), C_1 = C(Complex numbers) and C_2 = Set of all bicomplex numbers.
Let $\varpi, \vartheta \in C_1$, then we define a partial order $\preceq$ on C_1 as:
$\varpi \preceq \vartheta \iff Re(\varpi) \leq Re(\vartheta)$ and $Im(\varpi) \leq Im(\vartheta)$.
Also $\varpi \prec \vartheta$ if $Re(\varpi) < Re(\vartheta)$ and $Im(\varpi) < Im(\vartheta)$.
Segre[1] defined the bicomplex number as:

$$\zeta = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2,$$

where $b_1, b_2, b_3, b_4 \in C_0$, and i_1, i_2 are the independent units such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$,
we defined C_2 as:

$$C_2 = \{\zeta : \zeta = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2, b_1, b_2, b_3, b_4 \in C_0\},$$

i.e.,

$$C_2 = \{\zeta : \zeta = \varpi + i_2 \vartheta, \varpi, \vartheta \in C_1\}$$

where $\varpi = b_1 + b_2 i_1 \in C_1$ and $\vartheta = b_3 + b_4 i_1 \in C_1$.
If $\zeta = \varpi + i_2\vartheta$ and $\gamma = u + i_2 v$ then $\zeta \pm \gamma = (\varpi + i_2\vartheta) \pm (u + i_2 v) = (\varpi \pm u) + i_2(\vartheta \pm v)$ and the product is $\zeta.\gamma = (\varpi + i_2\vartheta).(u + i_2 v) = (\varpi u - \vartheta v) + i_2(\varpi v + \vartheta u)$.

The norm $\|.\|: C_2 \to \mathbb{C}_0^+$ is
defined by
$\|\zeta\| = \|\varpi + i_2\vartheta\| = \{|\varpi|^2 + |\vartheta|^2\}^{\frac{1}{2}} = (b_1^2 + b_2^2 + b_3^2 + b_4^2)^{\frac{1}{2}}$
where $\zeta = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_1 i_2 = \varpi + i_2\vartheta \in C_2$

We define a partial order $\preceq_{i_2}$ On C_2 as:
For $\zeta = \varpi + i_2\vartheta$, $\gamma = u + i_2 v \in C_2$ then
$\zeta \preceq_{i_2} \gamma \iff$ if $\varpi \preceq u$ and $\vartheta \preceq v$.
that is, $\zeta \preceq_{i_2} \gamma$ if :
(1) $\varpi = u, \vartheta = v$ or
(2) $\varpi \prec u, \vartheta = v$ or
(3) $\varpi = u, \vartheta \prec v$ or
(4) $\varpi \prec u, \vartheta \prec v$.

For any two bicomplex numbers $\zeta, \gamma \in C_2$:
(i) $\zeta \preceq_{i_2} \gamma \implies \|\zeta\| \leq \|\gamma\|$
(ii) $\|\zeta + \gamma\| \leq \|\zeta\| + \|\gamma\|$

Definition2.1.[5]Let Ω be a nonempty set. Then the mapping $\partial : \Omega \times \Omega \to C_2$ is said to bicomplex-valued metric on Ω if
1. $0 \preceq_{i_2} \partial(\varpi, \vartheta)$ for all $\varpi, \vartheta \in \Omega$,
2. $\partial(\varpi, \vartheta) = 0 \iff \varpi = \vartheta$,
3. $\partial(\varpi, \vartheta) = \partial(\vartheta, \varpi)$ for all $\varpi, \vartheta \in \Omega$ and
4. $\partial(\varpi, \vartheta) \preceq_{i_2} \partial(\varpi, u) + \partial(u, \vartheta)$ for all $\varpi, \vartheta, u \in \Omega$.
Here (Ω, ∂) is called the bicomplex valued metric space.

Let (Ω, ∂) be a bicomplex valued metric space for the following definitions:
Definition 2.2.[5]
(1). A sequence $\{\varpi_n\}$ in Ω is said to be converges to ϖ if for each $0 \prec_{i_2}$ r $\in C_2$ $\exists$ $n_0 \in$ N such that $\partial(\varpi_n, \varpi) \prec_{i_2}$ r, $\forall$ $n > n_0$ and we write $\lim_{n\to\infty} \varpi_n = \varpi$.
(2). A sequence $\{\varpi_n\}$ in Ω is said to be a cauchy sequence if for each $0 \prec_{i_2}$ r $\in C_2$ $\exists$ $n_0 \in$ N such that $\partial(\varpi_n, \varpi_{n+m}) \prec_{i_2}$ r, $\forall$ m,n $\in$ N and $n > n_0$.
(3.) We say that (Ω, ∂) is complete if each cauchy sequence of Ω is convergent.

Definition 2.3. We say that two maps h, k : $\Omega \to \Omega$ are commutes if hk(ϖ) = kh(ϖ) for all $\varpi \in \Omega$.

Definition 2.4. We say that two maps h, k : $\Omega \to \Omega$ are compatible if $\lim_{n\to\infty} \partial(hk\varpi_n, kh\varpi_n) = 0$ whenever sequence $\{\varpi_n\}$ in Ω satisfies $\lim_{n\to\infty} h\varpi_n = \lim_{n\to\infty} k\varpi_n = \varpi$ for $\varpi \in \Omega$.

Definition 2.5. We say that two maps h, k : $\Omega \to \Omega$ are weakly compatible if hϖ = kϖ for some $\varpi \in \Omega$ implies hk(ϖ) = kh(ϖ).

Definition 2.6. Let h,k,A,B : $\Omega \to \Omega$ be four maps. We say that $\{h, A\}$ and $\{k, B\}$ are satisfy the CLR_{AB} property if we can find sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in Ω satisfies $\lim_{n\to\infty} h\varpi_n = \lim_{n\to\infty} A\varpi_n = \lim_{n\to\infty} k\vartheta_n = \lim_{n\to\infty} B\vartheta_n = \varpi$ for some $\varpi \in$ A(Ω)$\cap$B(Ω).

Definition 2.7. Let h : $\Omega \times \Omega \to \Omega$ be a function. Then we say an element $(\varpi, \vartheta) \in \Omega \times \Omega$ is coupled invariant point of h if $h(\varpi, \vartheta) = \varpi$ and $h(\vartheta, \varpi) = \vartheta$.

Lemma 2.1.[7] We say a sequence $\{w_n\}$ in Ω is converges to a point w $\iff \lim_{n\to\infty} \|\partial(w_n, w)\| = 0$.

3 Main Results

Theorem 3.1. Suppose (Ω, ∂) be a complete Bicomplex valued metric space and h,k,A and B are self mappings on Ω satisfying
(i) $\partial(h\varpi, k\vartheta) \preceq_{i_2} \tau_1\partial(A\varpi, B\vartheta) + \tau_2\partial(A\varpi, h\varpi) + \tau_3\partial(B\vartheta, k\vartheta), \forall\varpi, \vartheta \in \Omega$,
where τ_1,τ_2 and τ_3 be nonnegative real number such that $\tau_1 + \tau_2 + \tau_3 < 1$.
(ii) $\{B, k\}$ and $\{A, h\}$ be weakly compatible,
(iii) $\{B, k\}$ and $\{A, h\}$ satisfy CLR_{AB} property.
Then h,k,A and B have a one and only common invariant point.
Proof: Since $\{B, k\}$ and $\{A, h\}$ satisfy CLR_{AB} property, then there exists sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in Ω such that $\lim_{n\to\infty} h\varpi_n = \lim_{n\to\infty} A\varpi_n = \lim_{n\to\infty} k\vartheta_n = \lim_{n\to\infty} B\vartheta_n = \jmath$, for some $\jmath \in A\Omega \cap B\Omega$. Then $\jmath$ = Bη_1 = Aη_2, for some $\eta_1, \eta_2 \in \Omega$.
Now we prove that kη_1= Bη_1. For each n$\in \mathbb{N}$, we have
$\partial(h\varpi_n, k\eta_1) \preceq_{i_2} \tau_1\partial(A\varpi_n, B\eta_1) + \tau_2\partial(A\varpi_n, h\varpi_n) + \tau_3\partial(B\eta_1, k\eta_1)$
Letting n $\to \infty$, we get
$\partial(B\eta_1, k\eta_1) \preceq_{i_2} \tau_1\partial(B\eta_1, B\eta_1) + \tau_2\partial(B\eta_1, B\eta_1) + \tau_3\partial(B\eta_1, k\eta_1)$
i.e.,$\partial(B\eta_1, k\eta_1) \preceq_{i_2} \tau_3\partial(B\eta_1, k\eta_1)$
Therefore we have
$\|\partial(B\eta_1, k\eta_1)\| \leq \tau_3 \|\partial(B\eta_1, k\eta_1)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$
Therefore we get $\|\partial(B\eta_1, k\eta_1)\|$ = 0. Thus B$\eta_1 = k\eta_1$.
Now we prove that A$\eta_2 = h\eta_2$. For each n$\in \mathbb{N}$, we condider
$\partial(h\eta_2, k\vartheta_n) \preceq_{i_2} \tau_1\partial(A\eta_2, B\vartheta_n) + \tau_2\partial(A\eta_2, h\eta_2) + \tau_3\partial(B\vartheta_n, k\vartheta_n)$
Letting n $\to \infty$, we get
$\partial(h\eta_2, A\eta_2) \preceq_{i_2} \tau_1\partial(A\eta_2, A\eta_2) + \tau_2\partial(A\eta_2, h\eta_2) + \tau_3\partial(A\eta_2, A\eta_2)$
i.e., $\partial(h\eta_2, A\eta_2) \preceq_{i_2} \tau_2\partial(h\eta_2, A\eta_2)$
Therefore we have $\|\partial(h\eta_2, A\eta_2)\| \leq \tau_2\|\partial(h\eta_2, A\eta_2)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$
Therefore, we get $\|\partial(h\eta_2, A\eta_2)\|$ = 0. Thus $h\eta_2 = A\eta_2$.
Hence $B\eta_1$ = $k\eta_1$ = $h\eta_2$ = $A\eta_2$ = $\jmath$.
Given that $\{A, h\}$ is weakly compatible and hη_2 = Aη_2 then we get hAη_2 = Ahη_2. So, h$\jmath$=A$\jmath$.
Given that $\{B, k\}$ is weakly compatible and kη_1 = Bη_1 then we get kBη_1 = Bkη_1. So, k$\jmath$ = B$\jmath$.

Now we prove that h$\jmath$ = $\jmath$:
Consider $\partial(h\jmath,k\eta_1) \preceq_{i_2} \tau_1\partial(A\jmath,B\eta_1) + \tau_2\partial(A\jmath,h\jmath) + \tau_3\partial(B\eta_1,k\eta_1)$
i.e.,, $\partial(h\jmath,,\jmath) \preceq_{i_2} \tau_1\partial(h\jmath,\jmath) + \tau_2\partial(h\jmath,h\jmath) + \tau_3\partial(\jmath,\jmath)$
i.e.,, $\partial(h\jmath,,\jmath) \preceq_{i_2} \tau_1\partial(h\jmath,\jmath)$
Therefore we have $\|\partial(h\jmath,\jmath)\| \leq \tau_1 \|\partial(h\jmath,\jmath)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$
Therefore, we get $\|\partial(h\jmath,\jmath)\|$ = 0. Thus h$\jmath$ = $\jmath$. So, we have h$\jmath$ = $\jmath$ = A$\jmath$.
Now we prove that k$\jmath$ = $\jmath$:
Consider
$\partial(h\eta_2,k\jmath) \preceq_{i_2} \tau_1\partial(A\eta_2,B\jmath) + \tau_2\partial(A\eta_2,h\eta_2) + \tau_3\partial(B\jmath,k\jmath)$
i.e., $\partial(\jmath,k\jmath) \preceq_{i_2} \tau_1\partial(\jmath,k\jmath) + \tau_2\partial(\jmath,\jmath) + \tau_3\partial(k\jmath,k\jmath)$
Therefore we have $\|\partial(\jmath,k\jmath)\| \leq \tau_1\|\partial(\jmath,k\jmath)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$.
Therefore, we get $\|\partial(\jmath,k\jmath)\|$ = 0. Thus k$\jmath$ = $\jmath$. So, we have k$\jmath$ = $\jmath$ = B$\jmath$.
Hence h$\jmath$ = A$\jmath$ = $\jmath$ = k$\jmath$ = B$\jmath$.
Therefore $\jmath$ is common invariant point of A,h,k and B.
Now we prove $\jmath$ is unique:
For this, we consider δ is any other common invariant point of h,k,A and B.
Then hδ = kδ = Aδ = Bδ = δ.
Now, Consider
$\partial(\jmath,\delta) = \partial(h\jmath,k\delta) \preceq_{i_2} \tau_1\partial(A\jmath,B\delta) + \tau_2\partial(A\jmath,h\jmath) + \tau_3\partial(B\delta,k\delta)$
i.e., $\partial(\jmath,\delta) \preceq_{i_2} \tau_1\partial(\jmath,\delta) + \tau_2\partial(\jmath,\jmath) + \tau_3\partial(\delta,\delta)$
i.e., $\partial(\jmath,\delta) \preceq_{i_2} \tau_1\ \partial(\jmath,\delta)$
Therefore we have $\|\partial(\jmath,\delta)\| \leq \tau_1\|\partial(\jmath,\delta)\|$
which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$
Hence, we get $\|\partial(\jmath, \delta)\| = 0$.Thus $\jmath$ = δ.
Hence, $\jmath$ is the one and only one common invariant point of h,k,A and B.

Example 3.1. Consider Ω = [0,1] and define $\partial\colon \Omega \times \Omega \to C_2$ by

$$\partial(\varpi,\vartheta) = \begin{cases} 0, & \text{for } \varpi = \vartheta \quad and \\ i_2max\{\varpi,\vartheta\}, & otherwise \end{cases}$$

for all $\varpi, \vartheta \in \Omega$.
Define h,k,A and B be self maps on Ω defined as:
For $\varpi \in \Omega, h(\varpi) = \frac{\varpi}{3}, k(\varpi) = \frac{\varpi}{3}, A(\varpi) = \varpi$ and $B(\varpi) = \varpi$.
Case(i): We show that {h,A} and {k,B} satisfy CLR_{AB} property. For this , we choose $\varpi_n = \frac{1}{2n}$ and $\vartheta_n = \frac{1}{3n+1}$ for n$\in$ **N**. Clearly, $< \varpi_n >$ and $< \vartheta_n >$ are in Ω. Then $\partial(A\varpi_n, 0) = \partial(\frac{1}{2n}, 0)$ converges to 0 as $n \to \infty$. Also, $\partial(h\varpi_n, 0) = \partial(\frac{1}{6n}, 0)$ converges to 0 as $n \to \infty$. Similarly, we get $\partial(k\vartheta_n, 0) = \partial(\frac{1}{9n+1}, 0) \to 0$ as $n \to \infty$. and $\partial(B\vartheta_n, 0) = \partial(\frac{1}{3n+1}, 0) \to 0$ as $n \to \infty$.
Since A0 = 0 = B0, So, we have $0 \in A\Omega \cap B\Omega$. Therefore, we have sequences $\{\varpi_n\}$ and $\{\vartheta_n\}$ in Ω so that $\lim_{n\to\infty} h\varpi_n = \lim_{n\to\infty} A\varpi_n = \lim_{n\to\infty} k\vartheta_n = \lim_{n\to\infty} B\vartheta_n = 0$. Thus {h,A} and {k,B} satisfies CLR_{AB} property.
case(ii): we show that {h,A} and {k,B} are weakly compatible. Now, $h\varpi = A\varpi \implies \frac{\varpi}{3} = \varpi \implies \varpi = 0$ and hA(0) = h(0) = 0 and Ah(0) = A(0) = 0. Thus $hA(\varpi) = Ah(\varpi)$, whenever $h\varpi = A\varpi$, for all $\varpi \in \Omega$. Hence {h,A} is weakly compatible in Ω.
Also, $k\varpi = B\varpi \implies \frac{\varpi}{3} = \varpi \implies \varpi = 0$ and kB(0) = Bk(0). Thus, $kB(\varpi) = Bk(\varpi)$, whenever $k\varpi = B\varpi$ for all $\varpi \in \Omega$. Hence, {k,B} is weakly compatible in Ω.
case(iii): Now, $\partial(h\varpi, k\vartheta) = \partial(\frac{\varpi}{3}, \frac{\vartheta}{3}) = i_2max\{\frac{\varpi}{3}, \frac{\vartheta}{3}\}$,
$\partial(A\varpi, B\vartheta) = \partial(\varpi, \vartheta) = i_2max\{\varpi, \vartheta\}$,
$\partial(A\varpi, h\varpi) = \partial(\varpi, \frac{\varpi}{3}) = i_2\max\{\varpi, \frac{\varpi}{3}\} = i_2\varpi$,
$\partial(B\vartheta, k\vartheta) = \partial(\vartheta, \frac{\vartheta}{3}) = i_2max\{\vartheta, \frac{\vartheta}{3}\} = i_2\vartheta$.
subcase(i) if $\varpi > \vartheta$ then
$\partial(h\varpi, k\vartheta) = i_2max\{\frac{\varpi}{3}, \frac{\vartheta}{3}\} = i_2\frac{\varpi}{3}$,
$\partial(A\varpi, B\vartheta) = i_2max\{\varpi, \vartheta\} = i_2\varpi$,
$\partial(A\varpi, h\varpi) = i_2max\{\varpi, \frac{\varpi}{3}\} = i_2\varpi$, $\partial(B\vartheta, k\vartheta) = i_2\vartheta$.
Now, $\partial(h\varpi, k\vartheta) = i_2\frac{\varpi}{3} \preceq_{i_2} \frac{1}{4}[i_2\varpi] + \frac{1}{4}[i_2\varpi] + \frac{1}{4}[i_2\vartheta]$
i.e., $\partial(h\varpi, k\vartheta) \preceq_{i_2} \frac{1}{4}\partial(A\varpi, B\vartheta) + \frac{1}{4}\partial(A\varpi, h\varpi) + \frac{1}{4}\partial(B\vartheta, k\vartheta)$
By choosing $\tau_1 = \frac{1}{4}$, $\tau_2 = \frac{1}{4}$, $\tau_3 = \frac{1}{4}$, Here τ_1,τ_2,τ_3 be nonnegative real numbers such that $\tau_1 + \tau_2 + \tau_3 < 1$. Hence
$d(h\varpi, k\vartheta) \preceq_{i_2} \tau_1 d(A\varpi, B\vartheta) + \tau_2 d(A\varpi, h\varpi) + \tau_3 d(B\vartheta, k\vartheta)$.
subcase(ii) if $\varpi < \vartheta$ then
$\partial(h\varpi, k\vartheta) = i_2\max\{\frac{\varpi}{3}, \frac{\vartheta}{3}\} = i_2\frac{\vartheta}{3}$,
$\partial(A\varpi, B\vartheta) = i_2max\{\varpi, \vartheta\} = i_2\vartheta$,
$\partial(A\varpi, h\varpi) = i_2max\{\varpi, \frac{\varpi}{3}\} = i_2\varpi, \partial(B\vartheta, k\vartheta) = i_2\vartheta$.
Now, $\partial(h\varpi, k\vartheta) = i_2\frac{\vartheta}{3} \preceq_{i_2} \frac{1}{4}[i_2\vartheta] + \frac{1}{4}[i_2\varpi] + \frac{1}{4}[i_2\vartheta]$
i.e., $\partial(h\varpi, k\vartheta) \preceq_{i_2} \frac{1}{4}\partial(A\varpi, B\vartheta) + \frac{1}{4}\partial(A\varpi, h\varpi) + \frac{1}{4}\partial(B\vartheta, k\vartheta)$
By choosing $\tau_1 = \frac{1}{4}$, $\tau_2 = \frac{1}{4}$, $\tau_3 = \frac{1}{4}$,
Here τ_1,τ_2,τ_3 be nonnegative real numbers such that $\tau_1 + \tau_2 + \tau_3 < 1$. Hence
$\partial(h\varpi, k\vartheta) \preceq_{i_2} \tau_1\partial(A\varpi, B\vartheta) + \tau_2\partial(A\varpi, h\varpi) + \tau_3\partial(B\vartheta, k\vartheta)$.

Corollary 3.1. Suppose (Ω, ∂) be a complete Bicomplex valued metric space and h,k and A be self mappings on Ω satisfies
(i) $\partial(hz, kw) \preceq_{i_2} \tau_1\partial(Az, Aw) + \tau_2\partial(Az, hz) + \tau_3\partial(Aw, kw)$, for all z,w $\in$ Ω, where τ_1,τ_2 and τ_3 be nonnegative real number such that $\tau_1 + \tau_2 + \tau_3 < 1$.
(ii) $\{h, A\}$ and $\{k, A\}$ are weakly compatible,
(iii) $\{h, A\}$ and $\{k, A\}$ satisfy CLR_A property.
Then h,k and A have a unique common invariant point.
Proof: We can prove this results easily by substituting B = A in the Theorem 3.1.

Theorem 3.2. Suppose (Ω, ∂) be a complete Bicomplex valued metric space and H,I,C,P,Q,R be the self mappings on Ω satisfies (i) H(Ω) $\supseteq$ QR(Ω) and I(Ω) $\supseteq$ CP(Ω) (ii) $\partial(CP\varpi, QR\vartheta) \preceq_{i_2} \tau_1\partial(H\varpi, I\vartheta) + \tau_2\partial(H\varpi, CP\varpi) + \tau_3\partial(I\vartheta, QR\vartheta) + \tau_4\partial(H\varpi, QR\vartheta)$ for all $\varpi, \vartheta \in \Omega$, where τ_1,τ_2,τ_3 and τ_4 be nonnegative real number such that $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$. (iii) Suppose (QR,I) and (CP,H) be weakly compatible. (Q,R), (Q,I) (R,I),(C,P),(C,H) and (P,H) are pairs of commuting maps. Then Q,R,C,P,I and H contains one and only one common invariant point in Ω.
Proof: Let $\varpi_0 \in \Omega$. Since $H(\Omega) \supseteq$ QR(Ω) and I(Ω) $\supseteq$ CP(Ω) then we can find a sequence $\{\varpi'_n\}$ in Ω such that
CPϖ_{2l} = I$\varpi_{2l+1} = \varpi'_{2l}$ and QRϖ_{2l+1} = H$\varpi_{2l+2} = \varpi'_{2l+1}$ for l=0,1,2,....
Consider $\partial(\varpi'_{2l}, \varpi'_{2l+1}) = \partial(CP\varpi_{2l}, QR\varpi_{2l+1})$
$\preceq_{i_2} \tau_1\partial(H\varpi_{2l}, I\varpi_{2l+1}) + \tau_2\partial(H\varpi_{2l}, CP\varpi_{2l})$
$+ \tau_3\partial(I\varpi_{2l+1}, QR\varpi_{2l+1}) + \tau_4\partial(H\varpi_{2l}, QR\varpi_{2l+1})$

= $\tau_1\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$ + $\tau_2\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$ + $\tau_3\ \partial(\varpi'_{2l}, \varpi'_{2l+1})$
+ $\tau_4\ \partial(\varpi'_{2l-1}, \varpi'_{2l+1})$
= $\tau_1\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$ + $\tau_2\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$ + $\tau_3\ \partial(\varpi'_{2l}, \varpi'_{2l+1})$
+ $\tau_4\ [\partial(\varpi'_{2l-1}, \varpi'_{2l})$+ $\partial(\varpi'_{2l}, \varpi'_{2l+1})]$
i.e., $(1 - \tau_3 - \tau_4)\ \partial(\varpi'_{2l}, \varpi'_{2l+1}) \preceq_{i_2} (\tau_1 + \tau_2 + \tau_4)$ $\partial(\varpi'_{2l-1}, \varpi'_{2l})$
i.e., $\partial(\varpi'_{2l}, \varpi'_{2l+1}) \preceq_{i_2} (\frac{\tau_1+\tau_2+\tau_4}{1-\tau_3-\tau_4})\ \partial(\varpi'_{2l-1}, \varpi'_{2l})$
Similarly, we consider
$\partial(\varpi'_{2l+1}, \varpi'_{2l+2})$ = $\partial(QR\varpi_{2l+1}$, CP$\varpi_{2l+2})$
= $\partial(CP\varpi_{2l+2}$, QR$\varpi_{2l+1})$
$\preceq_{i_2} \tau_1\ \partial(H\varpi_{2l+2}$, I$\varpi_{2l+1})$ + $\tau_2\ \partial(H\varpi_{2l+2}$, CP$\varpi_{2l+2})$
+ $\tau_3\ \partial(I\varpi_{2l+1}$, QR$\varpi_{2l+1})$ + $\tau_4\ \partial(H\varpi_{2l+2}$, QR$\varpi_{2l+1})$
= $\tau_1\ \partial(\varpi'_{2l+1}, \varpi'_{2l})$ + $\tau_2\ \partial(\varpi'_{2l+1}, \varpi'_{2l+2})$
+ $\tau_3\ \partial(\varpi'_{2l}, \varpi'_{2l+1})$ + $\tau_4\ \partial(\varpi'_{2l+1}, \varpi'_{2l+1})$
i.e.,(1-τ_2) $\partial(\varpi'_{2l+1}, \varpi'_{2l+2}) \preceq_{i_2} (\tau_1 + \tau_3)\partial(\varpi'_{2l}, \varpi'_{2l+1})$
i.e.,$\partial(\varpi'_{2l+1}, \varpi'_{2l+2}) \preceq_{i_2} (\frac{\tau_1+\tau_3}{1-\tau_2})\partial(\varpi'_{2l}, \varpi'_{2l+1})$
Let us consider σ = max $\{\frac{\tau_1+\tau_2+\tau_4}{1-\tau_3-\tau_4}, \frac{\tau_1+\tau_3}{1-\tau_2}\}$
then σ <1, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Now, for m,l $\in$ N and l < m, we consider
$\partial(\varpi'_l, \varpi'_m) \preceq_{i_2} \partial(\varpi'_l, \varpi'_{l+1})$ + $\partial(\varpi'_{l+1}, \varpi'_{l+2})$ +.....
+ $\partial(\varpi'_{m-1}, \varpi'_m)$
$\preceq_{i_2} (\sigma^l + \sigma^{l+1} + ... + \sigma^{m-1})\partial(\varpi'_0, \varpi'_1)$
i.e., $\partial(\varpi'_l, \varpi'_m) \preceq_{i_2} (\frac{\sigma^l}{1-\sigma})\partial(\varpi'_0, \varpi'_1)$
Therefore we obtain
$\|\partial(\varpi'_l, \varpi'_m)\| \preceq_{i_2} (\frac{\sigma^l}{1-\sigma})\|\partial(\varpi'_0, \varpi'_1)\|$
Since $\sigma < 1$, as n,m $\to \infty$, we get $\|\partial(\varpi'_l, \varpi'_m)\| \to 0$
Hence $\{\varpi'_n\}$ be a cauuchy sequence in complete space Ω, then $\exists\ \jmath \in \Omega$ such that
$\lim_{n\to\infty} CP\varpi_{2n}$ = $\lim_{n\to\infty} I\varpi_{2n+1}$ = $\lim_{n\to\infty} QR\varpi_{2n+1}$ = $\lim_{n\to\infty} P\varpi_{2n+2}$ = $\jmath$.
Since QR(Ω) $\subseteq$ H(Ω), then $\exists$ z $\in \Omega$ such that Hz = $\jmath$.
Now we consider
$\partial(CPz, \jmath) \preceq_{i_2} \partial(CPz, QR\varpi_{2n+1}) + \partial(QR\varpi_{2n+1}, \jmath)$
$\preceq_{i_2} \tau_1\partial(Hz, I\varpi_{2n+1})$ + $\tau_2\ \partial(Hz, CPz)$
$+ \tau_3\ \partial(I\varpi_{2n+1}, QR\varpi_{2n+1})$
+ $\tau_4\ \partial(Hz, QR\varpi_{2n+1}) + \partial(QR\varpi_{2n+1}, \jmath)$
Letting n $\to \infty$, we have
(CPz, $\jmath$) $\preceq_{i_2} \tau_1\partial(\jmath, \jmath) + \tau_2\partial(\jmath, CPz) + \tau_3\partial(\jmath, \eta)$
+ $\tau_4\partial(\jmath, \jmath) + \partial(\jmath, \jmath)$
Therefore we get
$\|\partial(CPz, \jmath)\| \leq \tau_2\|\partial(CPz, \jmath)\|$
which is a contradiction, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore we get, $\|\partial(CPz, \jmath)\|$ = 0.
Hence CPz = Hz = $\jmath$.
Again Since, CP(Ω) $\subseteq$ I(Ω), so there exists w $\in \Omega$ with Iw = $\jmath$.
Now we consider,
$\partial(\jmath, QRw) = \partial(CPz, QRw)$
$\preceq_{i_2} \tau_1\partial(Hz, Iw) + \tau_2\partial(Hz, CPz) + \tau_3\partial(Iw, QRw)$
+ $\tau_4\partial(Hz, QRw)$
i.e., $\partial(\jmath, QRw) \preceq_{i_2} (\tau_3 + \tau_4)\partial(\jmath, QRw)$
i.e., $\|\partial(\jmath, QRw)\| \preceq_{i_2} (\tau_3 + \tau_4)\|\partial(\jmath, QRw)\|$
which is a contradiction, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore we get,$\|\partial(\jmath, QRw)\|$ = 0.
Hence QRw = $\jmath$ = Iw.
Thus we get CPz = Hz = QRw = Iw = $\jmath$.

Since I and QR are weakly compatible, then I(QR)w = QR(I)w implies I$\jmath$ = QR$\jmath$.
Since CP and H are weakly compatible, then (CP)Hz = H(CP)z implies CP$\jmath$ = H$\jmath$.
Now we show that CP$\jmath$ = H$\jmath$ = $\jmath$:
We now consider
$\partial(CP\jmath, \jmath) = \partial(CP\jmath, QRw)$
$\preceq_{i_2} \tau_1\partial(H\jmath, IIw) + \tau_2\partial(H\jmath, CP\jmath) + \tau_3\partial(Iw, QRw)$
+ $\tau_4\partial(H\jmath, QRw)$
= $\tau_1\partial(CP\jmath, \jmath) + \tau_2\partial(H\jmath, H\jmath) + \tau_3\partial(Iw, Iw)$
+ $\tau_4\partial(CP\jmath, \jmath)$
i.e., $\partial(CP\jmath, \jmath) \preceq_{i_2} (\tau_1 + \tau_4)\partial(CP\jmath, \jmath)$
i.e., $\|\partial(CP\jmath, \jmath)\| \leq (\tau_1 + \tau_4)\|\partial(CP\jmath, \jmath)\|$
which is a contradiction, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore, we get $\|\partial(CP\jmath, \jmath)\|$ = 0.
Hence $CP\jmath = \jmath = H\jmath$.
Now, we show that $QR\jmath = \jmath$:
We now consider
$\partial(\jmath, QR\jmath) = \partial(CP\jmath, QR\jmath)$
$\preceq_{i_2} \tau_1\partial(H\jmath, I\jmath) + \tau_2\partial(H\jmath, CP\jmath) + \tau_3\partial(I\jmath, QR\jmath)$
+ $\tau_4\partial(H\eta, QR\eta)$
= $\tau_1\partial(\jmath, QR\jmath) + \tau_2\partial(H\jmath, H\jmath) + \tau_3\partial(I\jmath, I\jmath)$
+ $\tau_4\partial(\jmath, QR\jmath)$
i.e., $\partial(\jmath, QR\jmath) \preceq_{i_2} (\tau_1 + \tau_4)\partial(\jmath, QR\jmath)$
i.e., $\|\partial(\jmath, QR\jmath)\| \leq (\tau_1 + \tau_4)\|\partial(\jmath, QR\jmath)\|$
which is a contradiction, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore, we get $\|\partial(\jmath, QR\jmath)\|$ = 0.
Hence, $QR\jmath = \jmath = I\jmath$.
Thus, we get $CP\jmath = H\jmath = QR\jmath = I\jmath = \jmath$.
So, $\jmath$ be a common invariant point of H,I,CP and QR.
Since we have commuting conditions of pairs, we get
$Q\jmath = Q(QR\jmath) = Q(RQ\jmath) = (QR)Q\jmath$ and
$Q\jmath = Q(H\jmath) = H(Q\jmath)$;
$R\jmath = R(H\jmath) = HR\jmath$ and
$R\jmath = R(QR\jmath) = (RQ)R\jmath = (QR)R\jmath$.
Thus Q$\jmath$ and R$\jmath$ are common invariant points of (QR,H).
Therefore, we get Q$\jmath$ = $\jmath$ = R$\jmath$ = H$\jmath$ = QR$\jmath$.
Simiarly, we can easily prove, C$\jmath$ = $\jmath$ = P$\jmath$ = I$\jmath$ = CP$\jmath$.
Thus, $\jmath$ be a common invariant point of H,I,C,P,Q and R.
Now we prove $\jmath$ is unique. Suppose γ be common invariant point of H,I,C,P,Q and R other than $\jmath$.
Now we consider,
$\partial(\jmath, \gamma) = \partial(CP\jmath, QR\gamma)$
$\preceq_{i_2} \tau_1\partial(H\jmath, I\gamma) + \tau_2\partial(H\jmath, CP\eta) + \tau_3\partial(I\gamma, QR\gamma)$
+ $\tau_4\partial(H\jmath, QR\gamma)$
i.e., $\partial(\jmath, \gamma) \preceq_{i_2} (\tau_1 + \tau_4)\partial(\jmath, \gamma)$
i.e.,$\|\partial(\jmath, \gamma)\| \preceq_{i_2} (\tau_1 + \tau_4)\|\partial(\jmath, \gamma)\|$
which is a contradiction, Since $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$.
Therefore, we get $\|\partial(\jmath, \gamma)\|$ = 0.
Hence, we get $\jmath = \gamma$.
Thus $\jmath$ is the one and only common invariant point of H,I,C,P,Q and R.

Corollary 3.2. Suppose (Ω, ∂) be a complete Bicomplex valued metric space and H,C,P,Q,R be the self mappings on Ω satisfies (i) $H(\Omega) \supseteq QR(\Omega)$ and $H(\Omega) \supseteq CP(\Omega)$ (ii) $\partial(CP\varpi, QR\vartheta) \preceq_{i_2} \tau_1\partial(H\varpi, H\vartheta) + \tau_2\partial(H\varpi, CP\varpi) + \tau_3\partial(H\vartheta, QR\vartheta) + \tau_4\partial(H\varpi, QR\vartheta)$ for all $\varpi, \vartheta \in \Omega$, where τ_1,τ_2,τ_3 and τ_4 be nonnegative real number such that $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$. (iii) Suppose that (QR,H) and (CP,H) be weakly compatible. (Q,R), (Q,H) (R,H),(C,P),(C,H) and (P,H) are pairs of commuting maps. Then Q,R,C,P and H have a unique common invariant point in Ω.

Proof: This results can be prove easily by substituting I = H in the above theorem.

Theorem 3.3. Suppose (Ω, ∂) be a complete Bicomplex metric space and h,k:$\Omega \times \Omega \to \Omega$ be two functions satisfies

$\partial(h(\varpi,\jmath),k(\rho,\sigma)) \preceq_{i_2} \tau_1 \frac{\partial(\varpi,\rho)+\partial(\jmath,\sigma)}{2}$
+ $\tau_2 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\rho,\varpi)}{2}$ + $\tau_3 \frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\rho,k(\rho,\sigma))}{2}$

where $\varpi,\jmath,\rho,\sigma \in \Omega$ and τ_1,τ_2 and τ_3 are nonnegative integers such that $1 > \tau_1 + \tau_2 + \tau_3$. Then h and k contains one and only common coupled invariant point in $\Omega \times \Omega$.

Proof: Consider two arbitrary elements $\varpi_0, \jmath_0 \in \Omega$. We define two sequences $\{\varpi_n\},\{\jmath_n\}$ such that $\varpi_{2l+1} = h(\varpi_{2l},\jmath_{2l}), \varpi_{2l+2} = k(\varpi_{2l+1},\jmath_{2l+1})$, $\jmath_{2l+1} = h(\jmath_{2l},\varpi_{2l}), \jmath_{2l+2} = k(\jmath_{2l+1},\varpi_{2l+1})$, for l=0,1,2...

Now we consider,

$\partial(\varpi_{2l+1},\varpi_{2l+2}) = \partial(h(\varpi_{2l},\jmath_{2l}),k(\varpi_{2l+1},\jmath_{2l+1}))$
$\preceq_{i_2} \tau_1\frac{\partial(\varpi_{2l},\varpi_{2l+1})+\partial(\jmath_{2l},\jmath_{2l+1})}{2}$
$+ \tau_2\frac{\partial(\varpi_{2l},h(\varpi_{2l},\jmath_{2l}))+\partial(\varpi_{2l+1},\varpi_{2l})}{2}$
+ $\tau_3\frac{\partial(\varpi_{2l},h(\varpi_{2l},\jmath_{2l}))+\partial(\varpi_{2l+1},k(\varpi_{2l+1},\jmath_{2l+1}))}{2}$
= $\tau_1 \frac{\partial(\varpi_{2l},\varpi_{2l+1})+\partial(\jmath_{2l},\jmath_{2l+1})}{2}$ + $\tau_2 \frac{\partial(\varpi_{2l},\varpi_{2l+1})+\partial(\varpi_{2l+1},\varpi_{2l})}{2}$ + $\tau_3 \frac{\partial(\varpi_{2l},\varpi_{2l+1})+\partial(\varpi_{2l+1},\varpi_{2l+2})}{2}$
= $(\frac{\tau_1+2\tau_2+\tau_3}{2})\partial(\varpi_{2l},\varpi_{2l+1})$ + $(\frac{\tau_1}{2})$ $\partial(\jmath_{2l},\jmath_{2l+1})$
+ $(\frac{\tau_3}{2})\partial(\varpi_{2l+1},\varpi_{2l+2})$

i.e.,

$(2-\tau_3)\partial(\varpi_{2l+1},\varpi_{2l+2}) \preceq_{i_2} (\tau_1+2\tau_2+\tau_3)\partial(\varpi_{2l},\varpi_{2l+1})$
+ (τ_1) $\partial(\jmath_{2l},\jmath_{2l+1})$–(3.1)

Again, we consider

$\partial(\jmath_{2l+1},\jmath_{2l+2}) = \partial(h(\jmath_{2l},\varpi_{2l}),k(\jmath_{2l+1},\varpi_{2l+1}))$
$\preceq_{i_2} \tau_1\frac{\partial(\jmath'_{2l},\jmath_{2l+1})+\partial(\varpi_{2l},\varpi_{2l+1})}{2}$
+ τ_2 $\frac{\partial(\jmath_{2l},h(\jmath_{2l},\varpi_{2l}))+\partial(\jmath_{2l+1},\jmath_{2l})}{2}$
+ τ_3 $\frac{\partial(\jmath_{2l},h(\jmath_{2l},\varpi_{2l}))+\partial(\jmath_{2l+1},k(\jmath_{2l+1},\varpi_{2l+1}))}{2}$
= τ_1 $\frac{\partial(\jmath_{2l},\jmath_{2l+1})+\partial(\varpi_{2l},\varpi_{2l+1})}{2}$ + τ_2 $\frac{\partial(\jmath_{2l},\jmath_{2l+1})+\partial(\jmath_{2l+1},\jmath_{2l})}{2}$
+ τ_3 $\frac{\partial(\jmath_{2l},\jmath_{2l+1})+\partial(\jmath_{2l+1},\jmath_{2l+2})}{2}$
= $(\frac{\tau_1+2\tau_2+\tau_3}{2})\partial(\jmath_{2l},\jmath_{2l+1})$ + $(\frac{\tau_1}{2})$ $\partial(\varpi_{2l},\varpi_{2l+1})$
+ $(\frac{\tau_3}{2})\partial(\jmath_{2l+1},\jmath_{2l+2})$

i.e.,

$(2-\tau_3)\partial(\jmath_{2l+1},\jmath_{2l+2}) \preceq_{i_2} (\tau_1+2\tau_2+\tau_3)\partial(\jmath_{2l},\jmath_{2l+1})$
+ (τ_1) $\partial(\varpi_{2l},\varpi_{2l+1})$ –(3.2)

By adding the equations (3.1) and (3.2) we get

$\partial(\varpi_{2l+1},\varpi_{2l+2})$ $+\partial(\jmath_{2l+1},\jmath_{2l+2})$ $\preceq_{i_2}$ $\eta[\partial(\varpi_{2l},\varpi_{2l+1})$
$+\partial(\jmath_{2l},\jmath_{2l+1})]$

where $\eta = \frac{2\tau_1+2\tau_2+\tau_3}{2-\tau_3}$ and $0 \leq \eta < 1$, Since $1 > \tau_1 + \tau_2 + \tau_3$.

Similarly, we can easily show that

$\partial(\varpi_{2l+2},\varpi_{2l+3})$ $+\partial(\jmath_{2l+2},\jmath_{2l+3})$ $\preceq_{i_2}$ $\eta[\partial(\varpi_{2l+1},\varpi_{2l+2})$
$+\partial(\jmath_{2l+1},\jmath_{2l+2})]$

Then, for any l $\in$ N, we get

$\partial(\varpi_{l+2},\varpi_{l+1})$ $+\partial(\jmath_{l+2},\jmath_{l+1}) \preceq_{i_2} \eta[\partial(\varpi_{l+1},\varpi_l)$
$+\partial(\jmath_{l+1},\jmath_l)]$
$\preceq_{i_2} \eta^2[\partial(\varpi_l,\varpi_{l-1})$ $+\partial(\jmath_l,\jmath_{l-1})]$
....
$\preceq_{i_2} \eta^{l+1}[\partial(\varpi_1,\varpi_0)$ $+\partial(\jmath_1,\jmath_0)]$

Now, we consider $m, l \in$ N and m >l , we get

$\partial(\varpi_m,\varpi_l) + \partial(\jmath_m,\jmath_l) \preceq_{i_2} [\partial(\varpi_l,\varpi_{l+1})$+ $\partial(\jmath_l,\jmath_{l+1})]$
$+ [\partial(\varpi_{l+1},\varpi_m) + \partial(\jmath_{l+1},\jmath_m)]$
$\preceq_{i_2} [\partial(\varpi_l,\varpi_{l+1})$+$\partial(\jmath_l,\jmath_{l+1})]$+ $[\partial(\varpi_{l+1},\varpi_{l+2})$
$+ \partial(\jmath_{l+1},\jmath_{l+2})] + ... + [\partial(\varpi_{m-1},\varpi_m) + \partial(\jmath_{m-1},\jmath_m)]$
$\preceq_{i_2} [\eta^l + \eta^{l+1} + \eta^{l+2} + ... + \eta^{m-1}][\partial(\varpi_1,\varpi_0) + \partial(\jmath_1,\jmath_0)]$
$\preceq_{i_2} (\frac{\eta^l}{1-\eta})$ $[\partial(\varpi_1,\varpi_0) + \partial(\jmath_1,\jmath_0)]$

Since $0 \leq \eta < 1$, Then $\partial(\varpi_m,\varpi_l) \to 0$ & $\partial(\jmath_m,\jmath_l) \to 0$, as l,m $\to \infty$.

Hence $\{\varpi_n\}$ and $\{\jmath_n\}$ be two cauchy sequences in X and there exists $\varpi,\jmath \in$ X such that $(\varpi_n) \to \varpi$ and $(\jmath_n) \to \jmath$ as n$\to \infty$.

Now we consider

$\partial(h(\varpi,\jmath),\varpi) \preceq_{i_2} \partial(h(\varpi,\jmath),\varpi_{2l+2}) + \partial(\varpi_{2l+2},\varpi)$
= $\partial(h(\varpi,\jmath),k(\varpi_{2l+1},s_{2l+1}) + \partial(\varpi_{2l+2},\varpi)$
$\preceq_{i_2} \tau_1$ $\frac{\partial(\varpi,\varpi_{2l+1})+\partial(\jmath,\jmath_{2l+1})}{2}$ + τ_2 $\frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi_{2l+1},\varpi)}{2}$
+ τ_3 $\frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi_{2l+1},k(\varpi_{2l+1},\jmath_{2l+1}))}{2}$ $+\partial(\varpi_{2l+2},\varpi)$
= τ_1 $\frac{\partial(\varpi,\varpi_{2l+1})+\partial(\jmath,\jmath_{2l+1})}{2}$ + τ_2 $\frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi_{2l+1},\varpi)}{2}$
+ τ_3 $\frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi_{2l+1},\varpi_{2l+2})}{2}$ $+\partial(\varpi_{2l+2},\varpi)$

Letting the limit as l $\to \infty$, then we get

$\|\partial(h(\varpi,\jmath),\varpi)\| \leq (\frac{\tau_2+\tau_3}{2})\|\partial(h(\varpi,\jmath),\varpi)\|$

which is a contradiction, since $\tau_1 + \tau_2 + \tau_3 < 1$.

Therefore, we get $\|\partial(h(\varpi,\jmath),\varpi)\|$ = 0.

Hence, $h(\varpi,\jmath) = \varpi$. Similarly it can easily show that $h(\jmath,\varpi) = \jmath$.

Now we consider,

$\partial(\varpi,k(\varpi,\jmath)) = \partial(h(\varpi,\jmath),k(\varpi,\jmath))$
$\preceq_{i_2} \tau_1$ $\frac{\partial(\varpi,\varpi)+d(\jmath,\jmath)}{2}$ + τ_2 $\frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi,\varpi)}{2}$
+ τ_3 $\frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\varpi,k(\varpi,\jmath))}{2}$

i.e., $\partial(\varpi,k(\varpi,\jmath)) \preceq_{i_2} \frac{\tau_3}{2}$ $\partial(\varpi,k(\varpi,\jmath))$

i.e., $\|\partial(\varpi,k(\varpi,\jmath))\| \leq \frac{\tau_3}{2}$ $\|\partial(\varpi,k(\varpi,\jmath))\|$

i.e., $(1-\frac{\tau_3}{2})$ $\|\partial(\varpi,k(\varpi,\jmath))\| \leq 0$.

Since $1 > \tau_1 + \tau_2 + \tau_3$.

Therefore, we get $\|\partial(\varpi,k(\varpi,\jmath))\|$ = 0. Hence $k(\varpi,\jmath) = \varpi$

Similarly, we can easily show that $k(\jmath,\varpi) = \varpi$.

Thus, $(\varpi,\jmath)$ is a common coupled invariant point of h and k.

Now we prove $(\varpi,\jmath)$ is unique.

Let (ℓ,υ) be any other common coupled invariant point of h and k. Then $h(\ell,\upsilon) = k(\ell,\upsilon) = \ell$ and $h(\upsilon,\ell) = k(\upsilon,\ell) = \upsilon$.

Now we consider,

$\partial(\varpi,\ell) = \partial(h(\varpi,\jmath),k(\ell,\upsilon))$
$\preceq_{i_2} \tau_1$ $\frac{\partial(\varpi,\ell)+\partial(\jmath,\upsilon)}{2}$ + τ_2 $\frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\ell,\varpi)}{2}$
+ τ_3 $\frac{\partial(\varpi,h(\varpi,\jmath))+\partial(\ell,k(\ell,\upsilon))}{2}$
= τ_1 $\frac{\partial(\varpi,\ell)+\partial(\jmath,\upsilon)}{2}$ + τ_2 $\frac{\partial(\varpi,\varpi)+\partial(\ell,\varpi)}{2}$ + τ_3 $\frac{\partial(\varpi,\varpi)+\partial(\ell,\ell)}{2}$
= $(\frac{\tau_1+\tau_2}{2})\partial(\varpi,\ell)$ + $\frac{\tau_1}{2}\partial(\jmath,\upsilon)$ —(3.3)

Similarly, we can show that

$\partial(\jmath,\upsilon) \preceq_{i_2} (\frac{\tau_1+\tau_2}{2})\partial(\jmath,\upsilon)$+ $\frac{\tau_1}{2}\partial(\varpi,\ell)$ —(3.4)

By adding the equations (3.3) and (3.4), we get

$\partial(\varpi,\ell) + \partial(\jmath,\upsilon) \preceq_{i_2} (\frac{2\tau_1+\tau_2}{2})[\partial(\varpi,\ell) + \partial(\jmath,\upsilon)]$

i.e., $(1-\frac{2\tau_1+\tau_2}{2})[\partial(\varpi,\ell) + \partial(\jmath,\upsilon)] \preceq_{i_2} 0$.

Since $1 > \tau_1 + \tau_2 + \tau_3$,

Therefore, we get $\|\partial(\varpi,\ell) + \partial(\jmath,\upsilon)\| \leq 0$.

Then we get $\partial(\varpi,\ell) + \partial(\jmath,\upsilon) = 0$.

Hence, $\varpi = \ell$ and $\jmath = \upsilon$. i.e., $(\varpi,\jmath) = (\ell,\upsilon)$.

Hence $(\varpi,\jmath)$ is the one and only one common coupled invariant point of h and k.

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