Mathematics and Statistics 10(6): 1210-1217, 2022 http://www.hrpub.org
DOI: 10.13189/ms.2022.100607

On the Generalized Quadratic-Quartic Cauchy Functional Equation and its Stability over Non-Archimedean Normed Space

A. Ramachandran, S. Sangeetha*

Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, India

Received August 6, 2022; Revised September 29, 2022; Accepted October 23, 2022

Cite This Paper in the following Citation Styles

***(a)**: [1] A. Ramachandran, S. Sangeetha, "On the Generalized Quadratic-Quartic Cauchy Functional Equation and its Stability over Non-Archimedean Normed Space," Mathematics and Statistics, Vol.10, No.6, pp. 1210-1217, 2022. DOI: 10.13189/ms.2022.100607*

***(b)**: A. Ramachandran, S. Sangeetha (2022). On the Generalized Quadratic-Quartic Cauchy Functional Equation and its Stability over Non-Archimedean Normed Space. Mathematics and Statistics, 10(6), 1210-1217. DOI: 10.13189/ms.2022.100607*

Abstract Functional equation plays a very important and interesting role in the area of mathematics, which involves simple algebraic manipulations and through which one can arrive an in-teresting solution. The theory of functional equations is also used in the development of other areas such as analysis, al-gebra, Geometry etc., the new methods and techniques are applied in solving problem in Information theory, Finance, Geometry, wireless sensor networks etc., In recent decades, the study of various types of stability of a functional equa-tion such as HUS (Hyers-Ulam stability), HURS (Hyers-Ulam-Rassias stability) and generalized HUS of different types of functional equation and also for mixed type were discussed by many authors in various space. The problem of the sta-bility of different functional equations has been widely stud-ied by many authors, and more interesting results have been proved in the classical case (Archimedean). In recent years, the analogues results of the stability problem of these func-tional equations were investigated in non-Archimedean space. The aim of this study is to investigate the HUS of a mixed type of general Quadratic-Quartic Cauchy functional equation in non-Archimedean normed space. In this current article, we prove the generalized HUS for the following Quadratic-Quartic Cauchy functional equation over non-Archimedean Normed space.

$$\mathfrak{g}(\kappa x + y) + \mathfrak{g}(\kappa x - y) = \kappa^2\mathfrak{g}(x + y) + \kappa^2\mathfrak{g}(x - y) + 2\mathfrak{g}(\kappa x) - 2\kappa^2\mathfrak{g}(x) - 2(\kappa^2 - 1)\mathfrak{g}(y).$$

Keywords Hyers-Ulam Stability (HUS), Quadratic Function, Quartic Function, Non-Archimedean Normed (NAN) Space

1 Introduction

The stability of functional equations arose in 1940 from a question by Ulam [14] on the stability of group homomorphisms.

Given two groups H_1 and H_2 with the metric $d(.,.)$ on H_2 and for $\varepsilon > 0$, does there exist $\delta > 0$ such that if a mapping $G : H_1 \to H_2$ satisfies the inequality $d(G(a,b), G(a)G(b)) < \delta$ for all $a, b \in H_1$, then there exists a homomorphism $G' : H_1 \to H_2$ with $d(G(a), G'(a)) < \varepsilon$ for every $a \in H_1$?

Hyers [11] gave the headmost positive response to Ulam's query for Banach spaces such that

$$\|\mathfrak{g}(x + y) - \mathfrak{g}(x) - \mathfrak{g}(y)\| \leq \delta \qquad \forall x, y \in E \tag{1}$$

for some $\delta > 0$. Then there is a unique additive mapping $\top : E \to E'$ such that

$$\|\mathfrak{g}(x) - \top(x)\| \leq \delta \qquad \forall x \in E \tag{2}$$

Furthermore, if $\mathfrak{g}(tx)$ is continuous at $t \in R$ for any fixed $x \in E$, then $\top$ is linear(For intance [3]).

Aoki [1] stereotyped the Hyers theorem for additive mappings. Hyers theorem was generalized by Rassias [13] by allowing the Cauchy difference to be unbounded. Gajada [8] responded to the question for the case $p > 1$, posed by Rassias. Moslehian & Rassias [12] proved generalized HUS of the Cauchy functional equation and the Q_2F equation in NAN spaces.

The quadratic-quartic functional equation

$$\mathfrak{g}(2x+y)+\mathfrak{g}(2x-y)=4\mathfrak{g}(x+y)+4\mathfrak{g}(x-y) + 2\mathfrak{g}(2x)-8\mathfrak{g}(x)-6\mathfrak{g}(y) \quad (3)$$

was introduced by Young-Su Lee et al., [16] it is easy to show that the function $\mathfrak{g}(x) = ax^2 + bx^4$ is a solution of the functional equation(3), which is called a mixed type Q_2 Q_4F equation. For more detailed definitions of mixed type functional equations, we can refer to [4–7, 10].

Later Gordji et al., [9] introduced the following generalized quadratic-quartic mixed type functional equation.

$$\mathfrak{g}(\kappa x+y)+\mathfrak{g}(\kappa x-y)=\kappa^2\mathfrak{g}(x+y)+\kappa^2\mathfrak{g}(x-y) + 2\mathfrak{g}(\kappa x)-2\kappa^2\mathfrak{g}(x)-2(\kappa^2-1)\mathfrak{g}(y) \quad (4)$$

In this current article, to the best of our knowledge there is no discussion so far concerning the HUS of the following generalized Q_2 Q_4F equation.

$$Q\mathfrak{g}(x,y): \mathfrak{g}(\kappa x+y)+\mathfrak{g}(\kappa x-y)=\kappa^2\mathfrak{g}(x+y) + \kappa^2\mathfrak{g}(x-y)+2\mathfrak{g}(\kappa x)-2\kappa^2\mathfrak{g}(x)-2(\kappa^2-1)\mathfrak{g}(y) \quad (5)$$

Assume that G is a group and X is a complete non-Archimedean space throughout this article and A, B are vector space, the generalized quadratic-quartic functional equation is defined as below.

$$Q\mathfrak{g}(x,y): \mathfrak{g}(\kappa x+y)+\mathfrak{g}(\kappa x-y)=\kappa^2\mathfrak{g}(x+y) + \kappa^2\mathfrak{g}(x-y)+2\mathfrak{g}(\kappa x)-2\kappa^2\mathfrak{g}(x)-2(\kappa^2-1)\mathfrak{g}(y) \quad (6)$$

for every $x, y \in G$. Consider this functional inequality

$$\|Q\mathfrak{g}(x,y)\| \le \varphi(x,y) \quad (7)$$

for an upper bound function φ from $G^2 \to [0,\infty)$.

2 Preliminaries

Definition 2.1. *[2, 15] Let X be a vector space over a non-Archimedean field $\mathcal{K}$ with a non-trivial valuation $|.|$. A function $\|.\| : X \to \mathcal{R}$ is a non-Archimedean norm if it satisfies the following conditions*

1. $\|P\| \ge 0$ *and* $= 0$ *iff* $P = 0$.
2. $\|\alpha P\| = |\alpha|\|P\|, \alpha \in \mathcal{K}, P \in X$.
3. $\|P+S\| \le \max\{\|P\|, \|S\|\}$, $P, S \in X$.

Remark 2.2. *Because of this,*
$\|x_p - x_q\| \le \max\{\|x_{r+1}-x_r\| : q \le r \le p-1\} \quad (p > q)$

A sequence $\{x_p\}$ is Cauchy iff $\{x_{p+1}-x_p\} \to 0$, in a NAN space. By a complete NAN space every Cauchy sequence is convergent.

3 Main Results

Before proceeding to prove the main results in this section, we need the following lemma.

Lemma 3.1. *[5] If a function $\mathfrak{g}$ from A to B satisfies (4), then $\mathfrak{g}$ is quadratic-quartic.*

Theorem 3.2. *Let a function φ from $G^2 \to [0,\infty)$ be such that*

$$\lim_{\kappa\to\infty} \frac{1}{|2|^{2\kappa}}\varphi(2^{\kappa+1}x,\ 2^{\kappa+1}y) = 0 = \lim_{\kappa\to\infty} \frac{1}{|2|^{2\kappa}}\bar{\varphi}(2^{\kappa}x) \quad \forall\, x,y \in G. \quad (8)$$

Let $\mathfrak{g}$ be an even function from $G \to X$ that fulfils the inequality

$$\left\|Q\mathfrak{g}(x,y)\right\| \le \xi(x,y) \quad (9)$$

and $\mathfrak{g}(0) = 0$. Then, uniqueness of quadratic function $Q_2 : G \to X$ exists and

$$\left\|\mathfrak{g}(2x) - 16\mathfrak{g}(x) - Q_2(x)\right\| \le \frac{1}{|2|^2}\psi_{q_2}(x) \quad \forall\, x \in G. \quad (10)$$

Where

$$\psi_{q_2}(x) = \lim_{\kappa\to\infty} \max\left\{\frac{1}{|2|^{2\jmath}}\bar{\varphi}(2^{\jmath}x) : 0 \le \jmath < \kappa\right\} \quad (11)$$

$$\begin{aligned}\bar{\varphi}(x) = \max\Bigg\{ & 4\Bigg\{\xi(2x,x), \kappa^2 max\Big\{\xi(x,2x), \\ & \frac{\kappa^2}{\kappa^2-1}\xi(0,x)\Big\}, (4\kappa^2-2)\xi(x,x), \\ & 2max\Big\{\xi(x,\kappa x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-1)x)\Big\}, \\ & max\Big\{\max\Big\{\xi(x,(\kappa+1)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,\kappa x), \frac{1}{\kappa^2-1}\xi(0,x)\Big\}, \\ & \max\Big\{\xi(x,(\kappa-1)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-2)x)\Big\}\Big\}\Bigg\}, \\ & \xi(2x,2x), \kappa^2\max\Big\{\xi(x,3x), \frac{\kappa^2}{\kappa^2-1}\xi(0,2x)\Big\}, \\ & 2\max\Big\{\xi(x,\kappa x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-1)x)\Big\}, \\ & \kappa^2\xi(x,x), 2(\kappa^2-1)\max\Big\{\xi(x,2x), \frac{\kappa^2}{\kappa^2-1}\xi(0,x)\Big\}, \\ & \max\Big\{\xi(x,(\kappa+2)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa+1)x), \\ & \frac{1}{\kappa^2-1}\xi(0,2x)\Big\}\max\Big\{\xi(x,(\kappa-2)x), \\ & \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-3)x)\Big\}\Bigg\} \quad \forall\, x \in G. \end{aligned} \quad (12)$$

Proof. Substituting $x = 0$ then x with y in (9), we see that

$$\left\|(\kappa^2-1)\mathfrak{g}(x) - (\kappa^2-1)\mathfrak{g}(-x)\right\| \le \xi(0,x) \quad \forall x \in G. \quad (13)$$

Putting y by $x, 2x, \kappa x, (\kappa+1)x, (\kappa-1)x, (\kappa+2)x, (\kappa-2)x$ and $3x$ in (9), sequentially, we obtain

$$\Big\|\mathfrak{g}((\kappa+1)x)+\mathfrak{g}((\kappa-1)x)-\kappa^2\mathfrak{g}(2x)-2\mathfrak{g}(\kappa x) + 2k^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}(x)\Big\| \le \xi(x,x) \quad (14)$$

$$\Big\|\mathfrak{g}((\kappa+2)x)+\mathfrak{g}((\kappa-2)x)-\kappa^2\mathfrak{g}(3x)-\kappa^2\mathfrak{g}(-x) - 2\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}(2x)\Big\| \le \xi(x,2x) \quad (15)$$

$$\Big\|\mathfrak{g}(2\kappa x)-\kappa^2\mathfrak{g}((1+\kappa)x)-\kappa^2\mathfrak{g}((1-\kappa)x) + 2(\kappa^2-2)\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x)\Big\| \le \xi(x,\kappa x) \quad (16)$$

$$\Big\|\mathfrak{g}((2\kappa+1)x)+\mathfrak{g}(-x)-\kappa^2\mathfrak{g}((\kappa+2)x-\kappa^2\mathfrak{g}(-\kappa x) - 2\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}((\kappa+1)x)\Big\| \le \xi(x,(\kappa+1)x) \quad (17)$$

$$\Big\|\mathfrak{g}((2\kappa-1)x)+\mathfrak{g}(x)-\kappa^2\mathfrak{g}(-(\kappa-2)x-(\kappa^2+2)\mathfrak{g}(\kappa x) + 2\kappa^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}((\kappa-1)x)\Big\| \le \xi(x,(\kappa-1)x) \quad (18)$$

$$\Big\|\mathfrak{g}(2(\kappa+1)x)+\mathfrak{g}(-2x)-\kappa^2\mathfrak{g}((\kappa+3)x) - \kappa^2\mathfrak{g}(-(\kappa+1)x)-2\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x) + 2(\kappa^2-1)\mathfrak{g}((\kappa+2)x)\Big\| \le \xi(x,(\kappa+2)x) \quad (19)$$

$$\Big\|\mathfrak{g}(2(\kappa-1)x)+\mathfrak{g}(2x)-\kappa^2\mathfrak{g}((\kappa-1)x) - \kappa^2\mathfrak{g}(-(\kappa-3)x)-2\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x) + 2(\kappa^2-1)\mathfrak{g}((\kappa-2)x)\Big\| \le \xi(x,(\kappa-2)x) \quad (20)$$

$$\Big\|\mathfrak{g}((\kappa+3)x)+\mathfrak{g}((\kappa-3)x)-\kappa^2\mathfrak{g}(4x)-\kappa^2\mathfrak{g}(-2x) - 2\mathfrak{g}(\kappa x) + 2\kappa^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}(3x)\Big\| \le \xi(x,3x) \quad \forall\, x\in G. \quad (21)$$

Combining (13) and (15) to (21), sequentially, gives in the following inequalities

$$\Big\|\mathfrak{g}((\kappa+2)x)+\mathfrak{g}((\kappa-2)x)-\kappa^2\mathfrak{g}(3x)-\kappa^2\mathfrak{g}(x) - 2\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}(2x)\Big\| \le \max\left\{\xi(x,2x), \frac{\kappa^2}{\kappa^2-1}\xi(0,x)\right\} \quad (22)$$

$$\Big\|\mathfrak{g}(2\kappa x)-\kappa^2\mathfrak{g}((1+\kappa)x)-\kappa^2\mathfrak{g}((\kappa-1)x) + 2(\kappa^2-2)\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x)\Big\| \le \max\left\{\xi(x,\kappa x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-1)x)\right\} \quad (23)$$

$$\Big\|\mathfrak{g}((2\kappa+1)x)+\mathfrak{g}(x)-\kappa^2\mathfrak{g}((\kappa+2)x)-\kappa^2\mathfrak{g}(\kappa x) - 2\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}((\kappa+1)x)\Big\| \le \max\left\{\xi(x,(\kappa+1)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,\kappa x), \frac{1}{\kappa^2-1}\xi(0,x)\right\} \quad (24)$$

$$\Big\|\mathfrak{g}((2\kappa-1)x)+\mathfrak{g}(x)-\kappa^2\mathfrak{g}((\kappa-2)x)-(\kappa^2+2)\mathfrak{g}(\kappa x) + 2\kappa^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}((\kappa-1)x)\Big\| \le \max\left\{\xi(x,(\kappa-1)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-2)x)\right\} \quad (25)$$

$$\Big\|\mathfrak{g}(2(\kappa+1)x)+\mathfrak{g}(2x)-\kappa^2\mathfrak{g}((\kappa+3)x)-\kappa^2\mathfrak{g}((\kappa+1)x) - 2\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}((\kappa+2)x)\Big\| \le \max\left\{\xi(x,(\kappa+2)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa+1)x), \frac{1}{\kappa^2-1}\xi(0,2x)\right\} \quad (26)$$

$$\Big\|\mathfrak{g}(2(\kappa-1)x)+\mathfrak{g}(2x)-\kappa^2\mathfrak{g}((\kappa-1)x)-\kappa^2\mathfrak{g}((\kappa-3)x) - 2\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}((\kappa-2)x)\Big\| \le \max\left\{\xi(x,(\kappa-2)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-3)x)\right\} \quad (27)$$

$$\Big\|\mathfrak{g}((\kappa+3)x)+\mathfrak{g}((\kappa-3)x)-\kappa^2\mathfrak{g}(4x)-\kappa^2\mathfrak{g}(2x) - 2\mathfrak{g}(\kappa x)+2\kappa^2\mathfrak{g}(x)+2(\kappa^2-1)\mathfrak{g}(3x)\Big\| \le \max\left\{\xi(x,3x), \frac{\kappa^2}{\kappa^2-1}\xi(0,2x)\right\} \quad \forall\, x\in G. \quad (28)$$

Substituting x and y by $2x$ and x in (9), to arrive

$$\Big\|\mathfrak{g}((2\kappa+1)x)+\mathfrak{g}((2\kappa-1)x)-\kappa^2\mathfrak{g}(3x)-2\mathfrak{g}(2\kappa x) + 2\kappa^2\mathfrak{g}(2x)+(\kappa^2-2)\mathfrak{g}(x)\Big\| \le \xi(2x,x) \quad \forall\, x\in G. \quad (29)$$

Setting $2x$ and $2y$ in place of x and y in (9), sequentially, to yields

$$\Big\|\mathfrak{g}((2\kappa+1)x)+\mathfrak{g}((2\kappa-1)x)-\kappa^2\mathfrak{g}(4x)-2\mathfrak{g}(2\kappa x) + 2(\kappa^2-1)\mathfrak{g}(2x)\Big\| \leq \xi(2x,2x) \quad \forall\, x\in G. \tag{30}$$

It results from (14), (22), (23), (24), (25), and (29), that

$$\Big\|\mathfrak{g}(3x)-6\mathfrak{g}(2x)+15\mathfrak{g}(x)\Big\| \leq \frac{1}{(\kappa^2-\kappa^4)} max\Bigg\{\xi(2x,x), \kappa^2 max\Big\{\xi(x,2x), \frac{\kappa^2}{\kappa^2-1}\xi(0,x)\Big\}, (4\kappa^2-2)\xi(x,x), 2max\Big\{\xi(x,\kappa x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-1)x)\Big\}, max\Big\{\max\Big\{\xi(x,(\kappa+1)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,\kappa x), \frac{1}{\kappa^2-1}\xi(0,x)\Big\}, \max\Big\{\xi(x,(\kappa-1)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-2)x)\Big\}\Big\}\Bigg\}$$

$$\Big\|4\mathfrak{g}(3x)-24\mathfrak{g}(2x)+60\mathfrak{g}(x)\Big\| \leq \frac{1}{(\kappa^2-\kappa^4)} max\Big\{4\phi_1(x)\Big\} \quad \forall\, x\in G. \tag{31}$$

Where

$$\phi_1(x) = \xi(2x,x), \kappa^2 max\Big\{\xi(x,2x), \frac{\kappa^2}{\kappa^2-1}\xi(0,x)\Big\}, (4\kappa^2-2)\xi(x,x), 2max\Big\{\xi(x,\kappa x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-1)x)\Big\}, max\Big\{\max\Big\{\xi(x,(\kappa+1)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,\kappa x), \frac{1}{\kappa^2-1}\xi(0,x)\Big\}, \max\Big\{\xi(x,(\kappa-1)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-2)x)\Big\}\Big\}$$

Further from (14), (22), (23),(26), (27), (28), and (30), we conclude

$$\Big\|\mathfrak{g}(4x)-4\mathfrak{g}(3x)+4\mathfrak{g}(2x)+4\mathfrak{g}(x)\Big\| \leq \frac{1}{(\kappa^2-\kappa^4)} \max\Bigg\{\xi(2x,2x), \kappa^2\max\Big\{\xi(x,3x), \frac{\kappa^2}{\kappa^2-1}\xi(0,2x)\Big\}, 2\max\Big\{\xi(x,\kappa x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-1)x)\Big\}, \kappa^2\xi(x,x), 2(\kappa^2-1)\max\Big\{\xi(x,2x), \frac{\kappa^2}{\kappa^2-1}\xi(0,x)\Big\}, \max\Big\{\xi(x,(\kappa+2)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa+1)x), \frac{1}{\kappa^2-1}\xi(0,2x)\Big\} \max\Big\{\xi(x,(\kappa-2)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-3)x)\Big\}\Bigg\}$$

$$\Big\|\mathfrak{g}(4x)-4\mathfrak{g}(3x)+4\mathfrak{g}(2x)+4\mathfrak{g}(x)\Big\| \leq \frac{1}{(\kappa^2-\kappa^4)} \max\Big\{\phi_2(x)\Big\} \quad \forall\, x\in G. \tag{32}$$

Where

$$\phi_2(x) = \xi(2x,2x), \kappa^2\max\Big\{\xi(x,3x), \frac{\kappa^2}{\kappa^2-1}\xi(0,2x)\Big\}, 2\max\Big\{\xi(x,\kappa x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-1)x)\Big\}, \kappa^2\xi(x,x), 2(\kappa^2-1)\max\Big\{\xi(x,2x), \frac{\kappa^2}{\kappa^2-1}\xi(0,x)\Big\}, \max\Big\{\xi(x,(\kappa+2)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa+1)x), \frac{1}{\kappa^2-1}\xi(0,2x)\Big\} \max\Big\{\xi(x,(\kappa-2)x), \frac{\kappa^2}{\kappa^2-1}\xi(0,(\kappa-3)x)\Big\}$$

Using (31) and (32)

$$\Big\|\mathfrak{g}(4x)-20\mathfrak{g}(2x)+64\mathfrak{g}(x)\Big\| \leq \max\Big\{4\phi_1(x), \phi_2(x)\Big\}$$

$$\Big\|\mathfrak{g}(4x)-20\mathfrak{g}(2x)+64\mathfrak{g}(x)\Big\| \leq \bar{\varphi}(x) \quad \forall\, x\in G. \tag{33}$$

Where $\bar{\varphi}(x) = \max\Big\{4\phi_1(x), \phi_2(x)\Big\}$

Let $\mathfrak{h}_1$ be a mapping from G into X defined through $\mathfrak{h}_1(x) = \mathfrak{g}(2x)-16\mathfrak{g}(x)$ for each $x\in G$. We obtain

$$\|\mathfrak{h}_1(2x)-4\mathfrak{h}_1(x)\| \leq \bar{\varphi}(x) \qquad \forall x\in G. \tag{34}$$

Replacing x by $2^{\kappa-1}x$ in (34), we arrive

$$\left\|\frac{\mathfrak{h}_1(2^\kappa x)}{2^{2\kappa}} - \frac{\mathfrak{h}_1(2^{\kappa-1}x)}{2^{2(\kappa-1)}}\right\| \leq \frac{1}{|2|^{2\kappa}}\,\bar{\varphi}(2^{\kappa-1}x) \quad \forall x\in G. \tag{35}$$

It follows from (8) and (35) the sequence $\left\{\frac{\mathfrak{h}_1(2^\kappa x)}{2^{2\kappa}}\right\}$ is Cauchy. Since X is complete.
Therefore $\left\{\frac{\mathfrak{h}_1(2^\kappa x)}{2^{2\kappa}}\right\}$ is convergent.

$$Let\ Q_2(x) = \lim_{\kappa\to\infty} \frac{\mathfrak{h}_1(2^\kappa x)}{2^{2\kappa}} \quad \forall\, x \in G. \tag{36}$$

From (34) and (35) it follows by induction that

$$\left\|\frac{\mathfrak{h}_1(2^\kappa x)}{2^{2\kappa}} - \mathfrak{h}_1(x)\right\| \le \max\left\{\frac{1}{|2|^{2\kappa}}\bar{\varphi}(2^{\kappa-1}x) + \cdots + \frac{1}{|2|^2}\bar{\varphi}(2^0 x)\right\}$$

$$\left\|\frac{\mathfrak{h}_1(2^\kappa x)}{2^{2\kappa}} - \mathfrak{h}_1(x)\right\| \le \frac{1}{|2|^2}\max\left\{\frac{1}{|2|^{2\jmath}}\bar{\varphi}(2^\jmath x)\ : 0 \le \jmath < \kappa\right\} \tag{37}$$

$\forall\, \kappa \in N$ and for all $x \in G$. As $\kappa \to \infty$ in (37) and using (11), we get (10). Now to prove Q_2 is quadratic, it follows from (8), (35) and (36) we obtain

$$\begin{aligned}\|Q_2(2x) - 4Q_2(x)\| &= \lim_{\kappa\to\infty}\left\|\tfrac{\mathfrak{h}_1(2^\kappa 2x)}{2^{2\kappa}} - 2^2\tfrac{\mathfrak{h}_1(2^\kappa x)}{2^{2\kappa}}\right\| \\ &\le |2|^2 \lim_{\kappa\to\infty}\tfrac{1}{|2|^{2(\kappa+1)}}\ \bar{\varphi}(2^\kappa x) \\ &= \lim_{\kappa\to\infty}\tfrac{1}{|2|^{2\kappa}}\ \bar{\varphi}(2^\kappa x) \quad \forall\, x \in G. \\ &= 0\end{aligned}$$

Hence

$$Q_2(2x) = 4Q_2(x) \quad \forall\, x \in G. \tag{38}$$

On the other hand (9), (8) and (36) imply that

$$\begin{aligned}\left\|Q_{Q_2}(x,y)\right\| &= \left\|\lim_{\kappa\to\infty}\frac{Q\mathfrak{h}_1(2^\kappa x,\ 2^\kappa y)}{2^{2\kappa}}\right\| \\ &= \lim_{\kappa\to\infty}\frac{1}{|2|^{2\kappa}}\left\|Q\mathfrak{h}_1(2^\kappa x, 2^\kappa y)\right\| \\ &= \lim_{\kappa\to\infty}\frac{1}{|2|^{2\kappa}}\left\|Q\mathfrak{g}(2^{\kappa+1}x, 2^{\kappa+1}y) - 16Q\mathfrak{g}(2^\kappa x, 2^\kappa y)\right\| \\ &\le \lim_{\kappa\to\infty}\frac{1}{|2|^{2\kappa}}\max\left\{\varphi(2^{\kappa+1}x, 2^{\kappa+1}y), |16|\varphi(2^\kappa x, 2^\kappa y)\right\} \\ &= 0 \quad \forall\, x, y \in G.\end{aligned}$$

Hence Q_2 satisfies (4). Thus by Lemma (3.1). Hence $[Q_2(2x) - 16Q_2(x)]$ is quadratic.

Uniqueness: Let there exist another quadratic function Q_2'

$$\begin{aligned}\left\|Q_2(x) - Q_2'(x)\right\| &= \lim_{\iota\to\infty}\left\|\frac{Q_2(2^\iota x)}{2^{2\iota}} - \frac{Q_2'(2^\iota x)}{2^{2\iota}}\right\| \\ &\le \lim_{\iota\to\infty}\frac{1}{|2|^{2\iota}}\max\left\{\left\|Q_2(2^\iota x) - \mathfrak{h}_1(2^\iota x)\right\|,\ \left\|\mathfrak{h}_1(2^\iota x) - Q_2'(2^\iota x)\right\|\right\}\end{aligned}$$

$$\begin{aligned}\left\|Q_2(x) - Q_2'(x)\right\| &\le \frac{1}{|2|^2}\lim_{\iota\to\infty}\lim_{\kappa\to\infty}\max\left\{\frac{1}{|2|^{2\jmath}}\bar{\varphi}(2^\jmath x) : \iota \le \jmath < \kappa + \iota\right\} \\ &= 0 \quad \forall\, x \in G.\end{aligned}$$

Therefore $Q_2(x) = Q_2'(x)$.

□

Theorem 3.3. *Let a function φ from $G^2 \to [0, \infty)$ be such that*

$$\lim_{\kappa\to\infty}\frac{1}{|2|^{4\kappa}}\varphi(2^{\kappa+1}x,\ 2^{\kappa+1}y) = 0 = \lim_{\kappa\to\infty}\frac{1}{|2|^{4\kappa}}\bar{\varphi}(2^\kappa x) \quad \forall\, x, y \in G. \tag{39}$$

Let $\mathfrak{g}$ be an even function from $G \to X$ that fulfils the inequality

$$\left\|Q\mathfrak{g}(x,y)\right\| \le \xi(x,y) \tag{40}$$

and $\mathfrak{g}(0) = 0$. Then, uniqueness of quartic function $Q_4 : G \to X$ exists and

$$\left\|\mathfrak{g}(2x) - 4\mathfrak{g}(x) - Q_4(x)\right\| \le \frac{1}{|2|^4}\psi_{q_4}(x) \quad \forall\, x \in G. \tag{41}$$

Where

$$\psi_{q_4}(x) = \lim_{\kappa\to\infty}\max\left\{\frac{1}{|2|^{4\jmath}}\bar{\varphi}(2^\jmath x) : 0 \le \jmath < \kappa\right\} \tag{42}$$

$\bar{\varphi}(x)$ is defined in (12) $\forall\, x \in G$.

Proof. In such a way analogous to that of Theorem 3.2, we obtain,

$$\left\|\mathfrak{g}(4x) - 20\mathfrak{g}(2x) + 64\mathfrak{g}(x)\right\| \le \bar{\varphi}(x) \quad \forall\, x \in G. \tag{43}$$

Let $\mathfrak{h}_2$ be a mapping from G into X defined through $\mathfrak{h}_2(x) = \mathfrak{g}(2x) - 4\mathfrak{g}(x)$ for each $x \in G$. We obtain

$$\left\|\mathfrak{h}_2(2x) - 16\mathfrak{h}_2(x)\right\| \le \bar{\varphi}(x) \quad \forall\, x \in G. \tag{44}$$

Replacing x by $2^{\kappa-1}x$ in (44)

$$\left\|\frac{\mathfrak{h}_2(2^\kappa x)}{2^{4\kappa}} - \frac{\mathfrak{h}_2(2^{\kappa-1}x)}{2^{4(\kappa-1)}}\right\| \le \frac{1}{|2|^{4\kappa}}\ \bar{\varphi}(2^{\kappa-1}x) \quad \forall\, x \in G. \tag{45}$$

It follows from (39) and (45) the sequence $\left\{\frac{\mathfrak{h}_2(2^\kappa x)}{2^{4\kappa}}\right\}$ is Cauchy . Since X is complete. Therefore $\left\{\frac{\mathfrak{h}_2(2^\kappa x)}{2^{4\kappa}}\right\}$ is convergent.

$$Let\ Q_4(x) = \lim_{\kappa\to\infty} \frac{\mathfrak{h}_2(2^\kappa x)}{2^{4\kappa}} \quad \forall\, x \in G. \tag{46}$$

From (44) and (45) it follows by induction that

$$\begin{aligned}
&\left\|\frac{\mathfrak{h}_2(2^\kappa x)}{2^{4\kappa}} - \mathfrak{h}_2(x)\right\| \\
&\leq \max\left\{\frac{1}{|2|^{4\kappa}}\bar{\varphi}(2^{\kappa-1}x) + \cdots + \frac{1}{|2|^4}\bar{\varphi}(2^0 x)\right\} \\
&\leq \max\left\{\frac{1}{|2|^4(\jmath+1)}\bar{\varphi}(2^\jmath x) \ : 0 \leq \jmath < \kappa\right\} \\
&\leq \frac{1}{|2|^4}\max\left\{\frac{1}{|2|^{4\jmath}}\bar{\varphi}(2^\jmath x) \ : 0 \leq \jmath < \kappa\right\}
\end{aligned} \tag{47}$$

$\forall\, \kappa \in N$ and for all $x \in G$. As $\kappa \to \infty$ in (47) and using (42), we get (41). Now to prove Q_4 is quartic. It follows from (39), (45) and (46) we obtain

$$\begin{aligned}
\left\|Q_4(2x) - 16Q_4(x)\right\| &= \lim_{\kappa\to\infty}\left\|\frac{\mathfrak{h}_2(2^\kappa 2x)}{2^{4\kappa}} - 2^4\frac{\mathfrak{h}_2(2^\kappa x)}{2^{4\kappa}}\right\| \\
&\leq |2|^4 \lim_{\kappa\to\infty} \frac{1}{|2|^{4(\kappa+1)}}\ \bar{\varphi}(2^\kappa x) \\
&= \lim_{\kappa\to\infty} \frac{1}{|2|^{4\kappa}}\ \bar{\varphi}(2^\kappa x) \quad \forall\, x \in G. \\
&= 0
\end{aligned}$$

Hence

$$Q_4(2x) = 16Q_4(x) \quad \forall\, x \in G. \tag{48}$$

On the other hand (40), (39) and (46) imply that

$$\begin{aligned}
\|Q_{Q_4}(x,y)\| &= \left\|\lim_{\kappa\to\infty}\frac{Q\mathfrak{h}_2(2^\kappa x, 2^\kappa y)}{2^{4\kappa}}\right\| \\
&= \lim_{\kappa\to\infty}\frac{1}{|2|^{4\kappa}}\left\|Q\mathfrak{h}_2(2^\kappa x, 2^\kappa y)\right\| \\
&= \lim_{\kappa\to\infty}\frac{1}{|2|^{4\kappa}}\left\|Q\mathfrak{g}(2^{\kappa+1}x, 2^{\kappa+1}y) - 4Q\mathfrak{g}(2^\kappa x, 2^\kappa y)\right\| \\
&\leq \lim_{\kappa\to\infty}\frac{1}{|2|^{4\kappa}}\max\left\{\varphi(2^{\kappa+1}x, 2^{\kappa+1}y), |4|\varphi(2^\kappa x, 2^\kappa y)\right\} \\
&= 0 \quad \forall\, x, y \in G.
\end{aligned}$$

Therefore Q_4 satisfies (4). Thus by Lemma (3.1). Hence $[Q_2(2x) - 4Q_2(x)]$ is quartic.

Uniqueness: Let there exist another quadratic function Q'_4

$$\begin{aligned}
\|Q_4(x) - Q'_4(x)\| &= \lim_{\iota\to\infty}\left\|\frac{Q_4(2^\iota x)}{2^{4\iota}} - \frac{Q'_4(2^\iota x)}{2^{4\iota}}\right\| \\
&\leq \lim_{\iota\to\infty}\frac{1}{|2|^{4\iota}}\max\Big\{\left\|Q_4(2^\iota x) - \mathfrak{h}_2(2^\iota x)\right\|, \\
&\qquad \left\|\mathfrak{h}_2(2^\iota x) - Q'_4(2^\iota x)\right\|\Big\}
\end{aligned}$$

$$\begin{aligned}
&\|Q_4(x) - Q'_4(x)\| \\
&\leq \frac{1}{|2|^4}\lim_{\iota\to\infty}\lim_{\kappa\to\infty}\max\left\{\frac{1}{|2|^{4\jmath}}\bar{\varphi}(2^\kappa x) : \iota \leq \jmath < \kappa + \iota\right\} \\
&= 0 \quad \forall\, x \in G.
\end{aligned}$$

Therefore $Q_4(x) = Q'_4(x)$. □

Theorem 3.4. *Let a function φ from $G^2 \to [0,\infty)$ be such that*

$$\lim_{\kappa\to\infty}\frac{1}{|2|^{2\kappa}}\varphi(2^{\kappa+1}x, 2^{\kappa+1}y) = 0 = \lim_{\kappa\to\infty}\frac{1}{|2|^{2\kappa}}\bar{\varphi}(2^\kappa x) \quad \forall\, x, y \in G. \tag{49}$$

$$\lim_{\kappa\to\infty}\frac{1}{|2|^{2\kappa}}\varphi(2^{\kappa+1}x, 2^{\kappa+1}y) = 0 = \lim_{\kappa\to\infty}\frac{1}{|2|^{4\kappa}}\bar{\varphi}(2^\kappa x) \quad \forall\, x, y \in G. \tag{50}$$

Let $\mathfrak{g}$ be an even function from $G \to X$ that fulfils the inequality

$$\left\|Q\mathfrak{g}(x,y)\right\| \leq \xi(x,y) \tag{51}$$

and $\mathfrak{g}(0) = 0$. Then, uniqueness of quadratic function $Q_2 : G \to X$ and uniqueness of quartic function $Q_4 : G \to X$ exist and

$$\left\|\mathfrak{g}(x) - Q_2(x) - Q_4(x)\right\| \leq \frac{1}{|192|}\max\left\{\psi_{q_4}(x), |4|\psi_{q_2}\right\} \quad \forall x, y \in G. \tag{52}$$

Where $\psi_{q_2}(x)$ is defined in (11) & $\psi_{q_4}(x)$ is defined in (42).

Proof. By Theorems 3.2 & 3.3, there exists a quadratic function from G into X and quartic function from G into X such that

$$\left\|\mathfrak{g}(2x) - 16\mathfrak{g}(x) - Q_2(x)\right\| \leq \frac{1}{|2|^2}\psi_{q_2}(x) \tag{53}$$

$$\left\|\mathfrak{g}(2x) - 4\mathfrak{g}(x) - Q_4(x)\right\| \leq \frac{1}{|2|^4}\psi_{q_4}(x) \tag{54}$$

$$\left\|\mathfrak{g}(x) - \bar{Q}_4(x) - \bar{Q}_2(x)\right\| \leq \frac{1}{|192|}\max\left\{\psi_{q_4}(x), |4|\psi_{q_2}(x)\right\} \quad \forall\, x \in G.$$

So we get (52) by setting $\bar{Q}_4(x) = \frac{Q_4(x)}{12}$ and $\bar{Q}_2(x) = \frac{-Q_2(x)}{12}$ for each $x \in G$.

To show that Q_2 and Q_4 are unique. Let $Q_2^{'}$, $Q_4^{'}$: $G \to X$ be another quadratic and quartic functions respectively satisfying (52). Let $\bar{\bar{Q}}_2 = Q_2 - Q_2{'}$ and $\bar{\bar{Q}}_4 = Q_4 - Q_4{'}$.

Hence

$$\left\|\bar{\bar{Q}}_2(x) + \bar{\bar{Q}}_4(x)\right\| \leq \max\left\{\left\|\mathfrak{g}(x) - Q_2^{'}(x) - Q_4^{'}(x)\right\|, \left\|\mathfrak{g}(x) - Q_2(x) - Q_4(x)\right\|\right\}$$

$$\leq \frac{1}{|192|}\max\left\{\psi_{q_4}(x), |4|\psi_{q_2}(x)\right\} \quad \forall\, x \in G.$$

Since

$$\lim_{\iota\to\infty}\lim_{\kappa\to\infty}\max\left\{\frac{1}{|2|^{2(\jmath+1)}}\bar{\varphi}(2^{\jmath}x) : \iota \leq \jmath < \kappa+\iota\right\} = 0$$

$$\lim_{\iota\to\infty}\lim_{\kappa\to\infty}\max\left\{\frac{1}{|2|^{4(\jmath+1)}}\bar{\varphi}(2^{\jmath}x) : \iota \leq \jmath < \kappa+\iota\right\} = 0 \quad \forall x \in G. \tag{55}$$

$$\lim_{\kappa\to\infty}\frac{1}{|2|^{4\kappa}}\left\|\bar{\bar{Q}}_2(2^{\kappa}x) + \bar{\bar{Q}}_4(2^{\kappa}x)\right\| = 0 \qquad \forall\, x \in G.$$

Hence, we get $\bar{\bar{Q}}_2 = 0$ and $\bar{\bar{Q}}_4 = 0$ and the proof is complete. □

4 Conclusion

The problem of the stability of distinct types of functional equations has been widely investigated by many authors, and more interesting results have been proved in the classical case (Archimedean). In this article, we have discussed HUS for generalized quadratic-quartic Cauchy functional equation (4) in non-Archimedean normed space and also checked the uniqueness property. We hope that this research will represent a further improvement in the field of functional equations in non-Archimedean normed space.

REFERENCES

[1] Aoki T., "On the stability of the linear transformation in Banach spaces," J. Math. Soc. Japan, vol. 2, pp. 64–66, 1951. DOI: 10.2969/jmsj/00210064

[2] Bachman G., Introduction to p-adic numbers and valuation theory, Academic Press, 1964.

[3] Bourgin D.G., "Classes of transformations and bordering transformations," Bulletin of the American Mathematical Society, vol. 57, pp. 223–237, 1951. DOI: 10.1090/S0002-9904-1951-09511-7

[4] Ebadian A., Najati A., Eshaghi Gordji M., "On approximate additive–quartic and quadratic–cubic functional equations in two variables on abelian groups," Results Math., vol. 58, pp. 39-53, 2010. DOI:10.1007/s00025-010-0018-4 2010.

[5] Eshaghi Gordji M., Khodaei H., Khodabakhsh R., "General quartic–cubic–quadratic functional equation in non-Archimedean normed spaces," U.P.B. Sci. Bull. Ser. A., vol. 72, no. 3, pp. 69–84, 2010.

[6] Gordji M. E., Savadkouhi M. B., Park C., "Quadratic-quartic functional equations in RN-spaces," Journal of Inequalities and Applications, vol. 1, pp. 1-14, 2009. DOI:10.1155/2009/868423

[7] Eshaghi Gordji M., Bavand Savadkouhi M., "Stability of cubic and quartic functional equations in non-Archimedean spaces," Acta Appl. Math., vol. 110, pp. 1321–1329, 2010. DOI: 10.1007/s10440-009-9512-7

[8] Gajda Z., "On stability of additive mappings," Int. J. Math. Math. Sci., vol. 14, pp. 431–434, 1991. DOI: 10.1155/S016117129100056X

[9] Gordji M. E., Abbaszadeh, S., Park C., "On the stability of a generalized quadratic and quartic type functional equation in quasi-Banach spaces," Journal of Inequalities and Applications, vol. 2009, pp. 1-26, 2009. DOI: 10.1155/2009/153084

[10] Gordji M.E., Kaboli Gharetapeh S., Rassias J.M., Zolfaghari, S., "Solution and stability of a mixed type additive, quadratic and cubic functional equation," Adv. Differ. Equ., vol. 2009, pp. 1-17, 2009. DOI: 10.1155/2009/826130

[11] Hyers D.H., Isac G., and Rassias Th. M., "Stability of Functional Equations in Several Variables, Progress in Nonlinear Differential Equations and Their Applications," vol. 34, Birkh¨auser, Boston, Mass, USA, 1998.

[12] Moslehian M.S., Rassias, Th. M., "Stability of functional equations in non-Archimedean spaces," Appl.Anal. Discret. Math., vol. 1, pp. 325–334, 2007. DOI: 10.2298/AADM0702325M

[13] Rassias Th. M.,"On the stability of the linear mapping in Banach spaces," Proc. Am.Math. Soc., vol.72, pp. 297–300, 1978. DOI: 10.2307/2042795

[14] Ulam S.M., Problems in Modern Mathematics, chapter 6, JohnWiley Sons, New York, NY, USA, 1940.

[15] Van Rooij A C M., Non-Archimedean functional analysis, Monographs and Textbooks in Pure and Applied Math., vol. 51, Marcel Dekker, New York 1978.

[16] Young-Su Lee, Soomin Kim, and Chaewon Kim, "Stability for the Mixed Type of Quartic and Quadratic Functional Equations," Journal of Function Spaces, vol. 2014, pp. 1-10, 2014. DOI: 10.1155/2014/853743